THE

SCIENTIFIC PAPERS

OF

JOHN COUCH ADAMS.

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IN THE UNIVERSITY OF CAMBRIDGE.

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PART I.

EXTRACTS

FROM

UNPUBLISHED MANUSCRIPTS

EDITED BY

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PART II.

TERRESTRIAL MAGNETISM

EDITED BY

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PREFACE TO PART I.

The Manuscripts left by Professor John Couch Adams were a mass of notes, studies, and rough work,—the accumulation of his lifetime. What they contained was not known, but anyone familiar with his published work could recall more than one occasion when chance drew from him an unexpected description of results or researches of which it seemed we might otherwise never have heard. Hence it seemed right to examine the store from which these were drawn so that fresh matter of interest should not needlessly be lost.

The result of this search is contained in the following pages, of which lecture courses and elucidations of his published work form the greater part.

It is clear that he had not kept unpublished any completed work of great importance; yet these extracts may be read with interest for many reasons: some he had himself promised to publish at a convenient time; others contain the methods of investigations of which he has merely stated the results; and others, again, though fragmentary, are significant, because they indicate the plan he approved for attacking certain large problems.

The papers, as they reached me, and indeed as Adams left them, were almost devoid of arrangement, except that they were folded in parcels of a few pages each, the product of a day's or a few days' sitting; each parcel was generally very clear in itself, but carried no indication of its purpose or relations to others. It would have been a hopeless task to discover whether such a mass contained matter of value had not almost every page been dated. This permitted reference to a diary, which was sometimes very useful, though it was often silent at his most active seasons.

As a guide in the lecture courses I had notes taken by Mr A. Graham of Cambridge Observatory, by the late Rev. A. Freeman, by myself, and by others, which were of considerable value. But in most cases the purpose

of each new set of papers was little more than a guess until all the writings in any one subject were collected. When this was done it was possible to decide upon their bearing and importance, and then the work of transcription was generally straightforward, though here and there it became anything but easy. For example, in his lectures on the Lunar Theory there were in most cases many drafts of each lecture, differing substantially, and these had to be united; or again, it was often very hard to find the source of some formula or number quoted without reference.

At first it seemed that there was a danger that some considerable work might be overlooked altogether, but I am confident that such is not the case; and that no lacuna worth mention is to be found in the following pages is the best proof that I can offer. One which I was obliged to pass at first is mentioned on p. 127, but, as will be seen on p. 237, I was able to fill it up after the earlier sheets were printed off.

Very few, probably, have written their studies in a form so finished as Adams; "he never blotted a line;" it is impossible to exaggerate the impression left by a study of these unrevised papers of his absolute mastery of every detail of this most intricate and difficult subject, of freedom and ease in handling it from any standpoint, and of certainty and exactness in his operations, seeming indeed to symbolize as well as to calculate the motions of the stars. But it will easily be understood that from such material it was impossible in all cases to reproduce his own words and order; to do so would have done unnecessary violence to the matter, burdening it with any crudity that chanced to accompany its conception; and such a course would have been most repugnant to Adams's own fastidious care. It was in fact necessary that I should rewrite the papers. In doing so I have tried not to disturb what was characteristic, and have added nothing to the matter but an occasional explanatory sentence, and this is enclosed in brackets. Where a paper proved incomplete, it either appears with its defects, or is suppressed altogether.

R. A. SAMPSON.

DURHAM,
2 *May*, 1900.

PREFACE TO PART II.

I PROPOSE to give in this Preface a short account of Professor John Couch Adams's Theory of Terrestrial Magnetism, and of his determination of the Gaussian magnetic constants. This work was first taken in hand by him just fifty years ago, not long after the discovery of the planet Neptune. I find from his papers that the earliest work which he did on this subject was begun in the year 1849, and that he was led to it by the study of the translation of Gauss's Memoir on the Theory of Terrestrial Magnetism given in Taylor's *Scientific Memoirs* which was published in 1841. Gauss himself says in that memoir that he was stimulated to undertake the work on the publication of Sabine's map of the total intensity in the seventh Report of the British Association (*i.e.* in 1837), but that the data were very scanty for the accurate determination of the magnetic constants. For their accurate determination data should be supplied from accurate observations of magnetic declination, horizontal intensity, and dip, taken at stations uniformly distributed, as in a network, over the surface of the Earth.

Not only fifty years ago, when Gauss wrote, but even to the present day, the progress made in the theory of terrestrial magnetism has suffered from the lack of data derived from observations, because even now there are very few magnetic Observatories in existence, and those few are for the most part grouped very close together, leaving other parts of the Earth, and especially the southern hemisphere, almost entirely wanting in the facts of observation without which all theories can be but visionary.

In his calculations on the magnetic potential of the Earth and on the theoretical expression of the magnetic components X, Y and Z, to the north, to the west, and vertically downwards respectively, Gauss expressed

them for any point of the Earth's surface in series consisting of quantities to which he gave the name of magnetic constants, with coefficients involving Legendre's coefficients, and which are functions of the colatitude of the point.

From the very imperfect data which he possessed, Gauss determined the numerical values of the magnetic constants by his equations up to terms of the fourth order—*i.e.* he determined the values of the first twenty-four magnetic constants, *i.e.* three of the first order, five of the second, seven of the third, and nine of the fourth order.

No one could be more conscious of the fact than Gauss himself was that his data were so meagre and so insufficient that he could by no means rely on the values derived from them, and I fear that even now, at the end of this nineteenth century, we must say with him that the observed facts are far too scanty and that our stock of observations is still too small to enable us to get out trustworthy values of the magnetic potential and the magnetic elements for a given epoch. For this purpose the observations should be strictly contemporaneous, and so we require more Observatories where continuous records are taken.

For Gauss's method, which was also the method followed in practice by my brother, it is important for the accuracy and trustworthiness of the resulting values of the magnetic constants that the observations shall be taken from stations distributed as uniformly as possible over the Earth's surface; whereas we see that in the northern hemisphere the Observatories which exist are very unequally distributed, and that in the southern hemisphere there are only three first-class magnetic Observatories where continuous records are taken, viz. those of Batavia, Mauritius, and Melbourne.

This work on 'Terrestrial Magnetism' has been arranged under eight Sections. The first two sections treat of and establish simple and convenient relations between successive Legendre's coefficients and their derived differential coefficients regarded as functions of the colatitude $\theta = \cos^{-1}\mu$.

Taking P_n to represent Legendre's coefficient and Q_n^m to denote the value of

$$\frac{d^m P_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}},$$

certain simple and useful relations are found between successive values of Q for different values of n and m.

The symbol G_n^m is taken to represent the Gaussian function

$$\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} + \&c.,$$

and the symbol H_n^m is taken to represent $G_n^m (1-\mu^2)^{\frac{m}{2}}$.

Very simple relations are found between successive values of G for different values of n and m, and the numerical values of these functions are determined (1) for every degree of latitude on a sphere, and (2) for every degree of the geographical colatitude on a spheroid of eccentricity equal to that of the Earth itself. Very simple relations are also obtained between successive values of H and its differential coefficients for different values of n and m, and the magnetic potential V and the magnetic forces X, Y and Z are expressed in terms of these symbols H_n^m. The values of these functions H_n^m are determined for belts of latitude 5° apart (1) on a sphere, and (2) on a spheroid whose eccentricity equals that of the Earth's surface. The numerical values of G_n^m and also of H_n^m have been determined for all values of n and m from 0 to 10. Two distinct schemes of calculation were employed, and the calculations were made by different people and compared so as to ensure the accuracy of the results.

In the case of the spheroid, the functions G_n^m and H_n^m are regarded as functions of the *geographical* colatitude θ, and $\mu = \cos\theta$; and the symbols G'^m_n and H'^m_n are the same functions of the *geocentric* colatitude θ' of the same point, where $\mu' = \cos\theta'$.

A new theorem giving the values of $G' - G$ for different values of n and m is established, by means of which the accuracy of the calculated values of G and G' may readily be tested.

Section III. treats of the definite integrals of the product of two Legendre's coefficients, which enter largely into the Theory of Terrestrial Magnetism, and in Section IV. the product of any two Laplace's coefficients is similarly dealt with. Section V. treats of the Theory of Terrestrial Magnetism for the Earth regarded as a sphere, and contains new and useful relations between the definite integrals of the products of the expressions of the magnetic forces, which simplify the determination of the magnetic constants.

Taking V to represent the potential of the Earth's magnetic field, where λ is the longitude, θ the colatitude of a point on its surface, and r the

distance from the Earth's centre, X, Y and Z the magnetic forces in three directions at right angles to one another, X being the force towards the north perpendicular to the Earth's radius, Y the force perpendicular to the geographical meridian towards the west, and Z the force towards the Earth's centre; also taking $\cos\theta = \mu$, we have

$$X = -\frac{dV}{rd\theta} = \frac{\sin\theta}{r}\frac{dV}{d\mu} = \frac{(1-\mu^2)^{\frac{1}{2}}}{r}\cdot\frac{dV}{d\mu},$$

$$Y = -\frac{dV}{r\sin\theta d\lambda} = -\frac{(1-\mu^2)^{-\frac{1}{2}}}{r}\cdot\frac{dV}{d\lambda},$$

$$Z = -\frac{dV}{dr},$$

if east longitudes be considered positive.

There are two systems of values of V corresponding to magnetic forces whose origin is situated inside and outside the Earth's surface respectively, and by a convenient notation we may readily distinguish these two systems of values.

Making use of the functions denoted by H_n^m which I have above defined, and taking g_n^m and h_n^m to represent the Gaussian magnetic constants, g_n^m and h_n^m are coefficients of $\cos m\lambda$ and $\sin m\lambda$ respectively in the series of terms representing the magnetic potential.

The value of the magnetic potential V for magnetic forces whose origin is situated in the interior of the Earth is expressed by a series of terms of the form

$$\frac{1}{r^{n+1}}[H_n^m(g_n^m\cos m\lambda + h_n^m\sin m\lambda)].$$

Taking g_{-n}^m and h_{-n}^m to represent the values of the magnetic constants corresponding to this term of the series for forces situated outside the Earth's surface, the corresponding term in the magnetic potential will be

$$r^n[H_n^m(g_{-n}^m\cos m\lambda + h_{-n}^m\sin m\lambda)].$$

Hence

$$V = \Sigma\frac{1}{r^{n+1}}[H_n^m(g_n^m\cos m\lambda + h_n^m\sin m\lambda)] + \Sigma r^n[H_n^m(g_{-n}^m\cos m\lambda + h_{-n}^m\sin m\lambda)].$$

In the values of X, Y and Z there will be terms arising from each of these series of terms for V, and we may conveniently express them by modifying the notation in the same sense by using n subscript to refer to

internal forces, and $-n$ subscript to refer to external magnetic forces, whose origin is outside the Earth's surface, *i.e.* forces corresponding to negative powers of $\left(\frac{1}{r}\right)$.

The corresponding terms are:
in the value of X,

$$\frac{1}{r^{n+2}}(1-\mu^2)^{\frac{1}{2}}\frac{dH_n^m}{d\mu}(g_n^m \cos m\lambda + h_n^m \sin m\lambda)$$

and
$$r^{n-1}(1-\mu^2)^{\frac{1}{2}}\frac{dH_n^m}{d\mu}(g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda);$$

in the value of Y,

$$\frac{1}{r^{n+2}}(1-\mu^2)^{-\frac{1}{2}}\, mH_n^m\,(g_n^m \sin m\lambda - h_n^m \cos m\lambda)$$

and
$$r^{n-1}(1-\mu^2)^{-\frac{1}{2}}\, mH_n^m\,(g_{-n}^m \sin m\lambda - h_{-n}^m \cos m\lambda);$$

in the value of Z,

$$\frac{n+1}{r^{n+2}}H_n^m(g_n^m \cos m\lambda + h_n^m \sin m\lambda) \text{ and } -nr^{n-1}H_n^m(g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda).$$

It is also proved that

$$(1-\mu^2)^{\frac{1}{2}}\frac{dH_n^m}{d\mu} = (n-m)\,H_n^{m+1} - m\mu\,(1-\mu^2)^{-\frac{1}{2}}\,H_n^m$$

and
$$(1-\mu^2)^{\frac{1}{2}}\frac{dH_n^m}{d\mu} = \tfrac{1}{2}(n-m)\,H_n^{m+1} - \tfrac{1}{2}(n+m)\,H_n^{m-1};$$

and these relations are often useful in expressing the terms in the value of X.

It is found convenient to employ the notation with n and $-n$ subscript more generally to refer to internal and external forces respectively, and in this sense the following notation is employed:

Let

$$V_n^m = \frac{1}{r^{n+1}}H_n^m \text{ and } V_{-n}^m = r^n H_n^m,$$

and let

$$X_n^m = \frac{1}{r^{n+2}}\left[\tfrac{1}{2}(n-m)\,H_n^{m+1} - \tfrac{1}{2}(n+m)\,H_n^{m-1}\right]$$

be the coefficient of $(g_n^m \cos m\lambda + h_n^m \sin m\lambda)$ in the expression for X, the

force towards the north, and let X^m_{-n} be the corresponding coefficient of $(g^m_{-n}\cos m\lambda + h^m_{-n}\sin m\lambda)$ in the expression for X arising from forces outside the Earth's surface.

Then $$X^m_{-n} = r^{n-1}\left[\tfrac{1}{2}(n-m)\,H^{m+1}_n - \tfrac{1}{2}(n+m)\,H^{m-1}_n\right].$$

Using the notation Y^m_n and Y^m_{-n}, and also Z^m_n and Z^m_{-n} in the same way for the forces Y and Z, we have the potential

$$\begin{aligned}
V &= \Sigma\left[V^m_n\,(g^m_n\cos m\lambda + h^m_n\sin m\lambda)\right] + \Sigma\left[V^m_{-n}\,(g^m_{-n}\cos m\lambda + h^m_{-n}\sin m\lambda)\right]\\
X &= \Sigma\left[X^m_n\,(g^m_n\cos m\lambda + h^m_n\sin m\lambda)\right] + \Sigma\left[X^m_{-n}\,(g^m_{-n}\cos m\lambda + h^m_{-n}\sin m\lambda)\right]\\
Y &= \Sigma\left[Y^m_n\,(g^m_n\sin m\lambda - h^m_n\cos m\lambda)\right] + \Sigma\left[Y^m_{-n}\,(g^m_{-n}\sin m\lambda - h^m_{-n}\cos m\lambda)\right]\\
Z &= \Sigma\left[Z^m_n\,(g^m_n\cos m\lambda + h^m_n\sin m\lambda)\right] + \Sigma\left[Z^m_{-n}\,(g^m_{-n}\cos m\lambda + h^m_{-n}\sin m\lambda)\right].
\end{aligned}$$

Collecting coefficients of $\cos m\lambda$ and $\sin m\lambda$ in the values of V, X, Y and Z respectively:

$$\begin{array}{llll}
\text{The coefficient of } \cos m\lambda \text{ in} & V & \text{is} & \Sigma\,(V^m_n g^m_n + V^m_{-n} g^m_{-n}),\\
\quad ,, \qquad\qquad ,, & X & ,, & \Sigma\,(X^m_n g^m_n + X^m_{-n} g^m_{-n}),\\
\quad ,, \qquad\qquad ,, & Y & ,, & \Sigma\,(Y^m_n h^m_n + Y^m_{-n} h^m_{-n}),\\
\quad ,, \qquad\qquad ,, & Z & ,, & \Sigma\,(Z^m_n g^m_n + Z^m_{-n} g^m_{-n}).\\
\text{The coefficient of } \sin m\lambda \text{ in} & V & \text{is} & \Sigma\,(V^m_n h^m_n + V^m_{-n} h^m_{-n}),\\
\quad ,, \qquad\qquad ,, & X & ,, & \Sigma\,(X^m_n h^m_n + X^m_{-n} h^m_{-n}),\\
\quad ,, \qquad\qquad ,, & Y & ,, & \Sigma\,(Y^m_n g^m_n + Y^m_{-n} g^m_{-n}),\\
\quad ,, \qquad\qquad ,, & Z & ,, & \Sigma\,(Z^m_n h^m_n + Z^m_{-n} h^m_{-n}),
\end{array}$$

in which n takes all integral values for a given value of m.

In a portion of his work, in which he treats of the definite integral of the product of two Legendre's coefficients, Professor Adams proves the well-known formulæ that when n and n_1 are different from one another

$$\int_{-1}^{1} P_n P_{n_1}\,d\mu = 0,$$

and that when $n_1 = n$,

$$\int_{-1}^{1} (P_n)^2\,d\mu = \frac{2}{2n+1}.$$

He then proves that if

$$Q_n^m = (1-\mu^2)^{\frac{1}{2}} \cdot \frac{d^m P_n}{d\mu^m},$$

$$\int_{-1}^{1} Q_n^m Q_{n_1}^m \, d\mu = \frac{(n+m)!}{(n-m)!} \int_{-1}^{1} P_n P_{n_1} d\mu.$$

Hence if n and n_1 are not equal

$$\int_{-1}^{1} Q_n^m Q_{n_1}^m \, d\mu = 0.$$

But if $n_1 = n$, then

$$\int_{-1}^{1} (Q_n^m)^2 \, d\mu = \frac{(n+m)!}{(n-m)!} \cdot \frac{2}{2n+1}.$$

Hence if

$$\Pi_n^m = \left[\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} Q_n^m \quad \text{and} \quad \Pi_{n_1}^m = \left[\frac{(n_1-m)!}{(n_1+m)!}\right]^{\frac{1}{2}} Q_{n_1}^m$$

it follows that

$$\int_{-1}^{1} \Pi_n^m \Pi_{n_1}^m d\mu = 0,$$

and, when $n = n_1$, we have

$$\int_{-1}^{1} (\Pi_n^m)^2 \, d\mu = \frac{(n-m)!}{(n+m)!} \int_{-1}^{1} (Q_n^m)^2 \, d\mu = \frac{2}{2n+1} = \int_{-1}^{1} (P_n)^2 \, d\mu.$$

It is also shewn that

$$H_n^m = \frac{(n-m)!}{1 \,.\, 3 \,.\, 5 \ldots (2n-1)} Q_n^m.$$

And therefore, when n and n_1 are not equal, we have

$$\int_{-1}^{1} H_n^m \, H_{n_1}^m d\mu = 0,$$

and, when $n_1 = n$, we have

$$\int_{-1}^{1} (H_n^m)^2 \, d\mu = \frac{(n-m)! \, (n+m)!}{[1 \,.\, 3 \,.\, 5 \ldots (2n-1)]^2} \cdot \frac{2}{2n+1}.$$

We have seen that, on a sphere of radius unity,

$$X_n^m = (n-m) H_n^{m+1} - m\mu (1-\mu^2)^{-\frac{1}{2}} H_n^m = (1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu}$$

$$= m\mu (1-\mu^2)^{-\frac{1}{2}} H_n^m - (n+m) H_n^{m-1},$$

also $$Y_n^m = m (1-\mu^2)^{-\frac{1}{2}} H_n^m \text{ and } Z_n^m = (n+1) H_n^m.$$

Hence $$\mu Y_n^m - X_n^m = (n+m) H_n^{m-1},$$

and $$\mu Y_n^m + X_n^m = (n-m) H_n^{m+1},$$

also $$(1-\mu^2)^{\frac{1}{2}} Y_n^m = m H_n^m.$$

From these formulæ we find

$$\int_{-1}^{1} (Y_n^m)^2 d\mu + \int_{-1}^{1} (X_n^m)^2 d\mu = \int_{-1}^{1} (1-\mu^2) \left(\frac{dH_n^m}{d\mu}\right)^2 d\mu + \int_{-1}^{1} \frac{m^2}{1-\mu^2} (H_n^m)^2 d\mu,$$

and also

$$= \tfrac{1}{2} (n+m)^2 \int_{-1}^{1} (H_n^{m-1})^2 d\mu + \tfrac{1}{2} (n-m)^2 \int_{-1}^{1} (H_n^{m+1})^2 d\mu + m^2 \int_{-1}^{1} (H_n^m)^2 d\mu.$$

These definite integrals reduce to

$$n(n+1) \int_{-1}^{1} (H_n^m)^2 d\mu.$$

Hence since $Z_n^m = (n+1) H_n^m$, we have

$$\int_{-1}^{1} (X_n^m)^2 d\mu + \int_{-1}^{1} (Y_n^m)^2 d\mu + \int_{-1}^{1} (Z_n^m)^2 d\mu = (n+1)(2n+1) \int_{-1}^{1} (H_n^m)^2 d\mu$$

$$= 2 \frac{(n-m)!\ (n+m)!}{[1 \cdot 3 \cdot 5 \dots (2n-1)]^2} (n+1).$$

Putting n_1 for n in the above equations we get

$$\mu Y_{n_1}^m - X_{n_1}^m = (n_1+m) H_{n_1}^{m-1},$$

$$\mu Y_{n_1}^m + X_{n_1}^m = (n_1-m) H_{n_1}^{m+1},$$

and $$(1-\mu^2)^{\frac{1}{2}} Y_{n_1}^m = m H_{n_1}^m.$$

Hence

$$\frac{1}{2} (\mu Y_n^m - X_n^m)(\mu Y_{n_1}^m - X_{n_1}^m) + \frac{1}{2} (\mu Y_n^m + X_n^m)(\mu Y_{n_1}^m + X_{n_1}^m) + (1-\mu^2) Y_n^m Y_{n_1}^m$$

$$= X_n^m X_{n_1}^m + Y_n^m Y_{n_1}^m$$

$$= \frac{1}{2} (n+m)(n_1+m) H_n^{m-1} H_{n_1}^{m-1} + \frac{1}{2} (n-m)(n_1-m) H_n^{m+1} H_{n_1}^{m+1} + m^2 H_n^m H_{n_1}^m;$$

hence $$\int_{-1}^{1} X_n^m X_{n_1}^m d\mu + \int_{-1}^{1} Y_n^m Y_{n_1}^m d\mu = 0,$$

since we have seen that for any value of m and different values of n and n_1, the value of

$$\int_{-1}^{1} H_n^m H_{n_1}^m d\mu = 0.$$

For the same reason

$$\int_{-1}^{1} Z_n^m Z_{n_1}^m d\mu = 0.$$

Now let us consider the application of these formulæ to the determination of the numerical values of the magnetic constants of terrestrial magnetism. For a given value of μ (*i.e.* for a given latitude) we have a series of terms forming the coefficients of $\cos m\lambda$ and $\sin m\lambda$, in the values of the magnetic potential and of the magnetic forces X, Y, and Z, which are of the forms

$$\begin{gathered} \alpha_n H_n^m + \alpha_{n_1} H_{n_1}^m + \text{\&c.} \\ \alpha_n X_n^m + \alpha_{n_1} X_{n_1}^m + \text{\&c.} \\ \alpha_n Y_n^m + \alpha_{n_1} Y_{n_1}^m + \text{\&c.} \\ \alpha_n Z_n^m + \alpha_{n_1} Z_{n_1}^m + \text{\&c.} \end{gathered}$$

where α_n, α_{n_1}, &c., are the magnetic constants to be determined.

The numerical values of H_n^m, X_n^m, Y_n^m, and Z_n^m for different values of n and m must be calculated, and in any belt of latitude of breadth corresponding to the numerical value taken for $\delta\mu$, these coefficients must be equated to the values of the forces as derived from the magnetic observations taken in that belt of latitude.

The values of the magnetic forces X, Y, and Z are derived for every 10° of longitude and every 5° of latitude from the declination (δ), the dip (ι), and the horizontal force (ω), as given in the charts from which the observations are obtained. These values of the forces X, Y, and Z are analysed for belts of latitude 5° in breadth around the Earth's surface by a formula of the type $a_0 + a_1 \cos\lambda + b_1 \sin\lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda + \text{\&c.}$

If we take x_m to represent the coefficient of $\cos m\lambda$ in the expansion of the value of the force X for a given belt of latitude corresponding to the colatitude $$\theta = \cos^{-1}\mu :$$

then $$\alpha_n X_n^m + \alpha_{n_1} X_{n_1}^m + \alpha_{n_2} X_{n_2}^m + \text{\&c.} = x_m,$$

where x_m is derived from the observations. Similar equations, involving on one side the magnetic constants α_n, α_{n_1}, &c., and on the other the values derived from the observations, must be formed for all the successive different belts of latitude from the north pole to the south pole—*i.e.*, for all values of μ between 1 and -1.

The numerical values of X_n^m, $X_{n_1}^m$, &c., as well as the values of H_n^m (as above defined), have been determined for every degree of latitude and recorded for future use, but, in the actual determinations of the magnetic constants which have been made, belts of latitude 5° in breadth have been taken, or $\delta\theta$ has been taken as 5°, and the area of the belt is proportional to $\delta\mu$.

Supposing the observations equally distributed over the surface of the globe, or supposing the weight of any determination proportional to the surface of the corresponding element about the point of observation, then the weight of each of the above equations is proportional to $\delta\mu$, and multiplying the equation in X for each value of μ by X_n^m, and summing up the separate equations for the whole surface of the Earth, we get the final equation—

$$\alpha_n \int_{-1}^{1} (X_n^m)^2 \, d\mu + \alpha_{n_1} \int_{-1}^{1} X_n^m X_{n_1}^m \, d\mu + \&c. = \int_{-1}^{1} X_n^m x_m \, d\mu.$$

Similarly, the final equation for α_{n_1} is found by multiplying the above equations by $X_{n_1}^m$, $Y_{n_1}^m$, and $Z_{n_1}^m$ respectively, and we get

$$\alpha_n \int_{-1}^{1} X_n^m X_{n_1}^m d\mu + \alpha_{n_1} \int_{-1}^{1} (X_{n_1}^m)^2 \, d\mu + \&c. = \int_{-1}^{1} X_{n_1}^m x_m \, d\mu.$$

Similarly, if y_m denote the coefficient of $\sin m\lambda$ or $-\cos m\lambda$ in the value of the force Y as derived from observations, we have

$$\Sigma \, (\alpha_n Y_n) = y_m,$$

and the final equations for finding α_n and α_{n_1} respectively will be

$$\alpha_n \int_{-1}^{1} (Y_n^m)^2 \, d\mu + \alpha_{n_1} \int_{-1}^{1} Y_n^m Y_{n_1}^m d\mu + \&c. = \int_{-1}^{1} Y_n^m y_m d\mu,$$

and
$$\alpha_n \int_{-1}^{1} Y_n^m Y_{n_1}^m d\mu + \alpha_{n_1} \int_{-1}^{1} (Y_{n_1}^m)^2 \, d\mu + \&c. = \int_{-1}^{1} Y_{n_1}^m y_m d\mu.$$

Combining the final equations for α_n from X and Y together, we have

$$\alpha_n \int_{-1}^{1} [(X_n^m)^2 + (Y_n^m)^2] \, d\mu = \int_{-1}^{1} X_n^m x_m \, d\mu + \int_{-1}^{1} Y_n^m y_m \, d\mu,$$

since the coefficients of α_{n_1} and all the other terms on the left-hand side of this equation vanish when the integration is taken all over the Earth's surface.

Hence $$\alpha_n \,.\, n(n+1)\int_{-1}^{1}(H_n^m)^2\,d\mu = \int_{-1}^{1} X_n^m x_m\,d\mu + \int_{-1}^{1} Y_n^m y_m\,d\mu$$

i.e. $$\alpha_n \times 2n(n+1)\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2(2n+1)}$$

$$= \int_{-1}^{1} X_n^m x_m\,d\mu + \int_{-1}^{1} Y_n^m y_m\,d\mu.$$

Similarly, by putting n_1 for n, we may get the value of α_{n_1}.

In the same way the final equation for finding α_n from the equations for Z would give us

$$\alpha_n\int_{-1}^{1}(Z_n^m)^2\,d\mu + \alpha_{n_1}\int_{-1}^{1} Z_n^m Z_{n_1}^m\,d\mu + \text{\&c.} = \int_{-1}^{1} Z_n^m z_m\,d\mu\,;$$

or $$\alpha_n(n+1)^2\int_{-1}^{1}(H_n^m)^2\,d\mu = \int_{-1}^{1} Z_n^m z_m\,d\mu,$$

since $$\int_{-1}^{1} Z_n^m Z_{n_1}^m\,d\mu = 0\,;$$

i.e. $$\alpha_n\,2(n+1)^2\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2(2n+1)} = \int_{-1}^{1} Z_n^m z_m\,d\mu.$$

If we take into account separately the parts of the magnetic force at a point due to the internal and external centres of magnetic force, the general terms of the coefficient of $\cos m\lambda$ in the potential function will be of the form

$$\left(\frac{\alpha_n}{r^{n+1}} + \beta_n r^n\right) H_n^m,$$

and the corresponding coefficients in X, Y, and Z will be—

$$\text{in } X = \left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right)\left[\frac{1}{2}(n-m)\,H_n^{m+1} - \frac{1}{2}(n+m)\,H_n^{m-1}\right];$$

$$\text{in } (1-\mu^2)^{\frac{1}{2}}\,Y = \left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) mH_n^m\,;$$

$$\text{in } Z = \left[\frac{(n+1)\,\alpha_n}{r^{n+2}} - n\beta_n r^{n-1}\right] H_n^m.$$

If then, as before, we put $r=1$, we shall have the final equation for α_n as follows:

$$\alpha_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu+(n+1)^2\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$+\beta_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu-n(n+1)\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu+(n+1)\int_{-1}^{1}H_n^m z_m\,d\mu,$$

where the coefficient of $\beta_n=0$.

And
$$\alpha_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu-n(n+1)\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$+\beta_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu+n^2\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu-n\int_{-1}^{1}H_n^m z_m\,d\mu,$$

where the coefficient of $\alpha_n=0$.

Hence α_n and β_n are separately determined from the equations

$$2\alpha_n(n+1)\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2}$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu+(n+1)\int_{-1}^{1}H_n^m z_m\,d\mu,$$

and
$$2\beta_n\,.\,n\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2}$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu-n\int_{-1}^{1}H_n^m z_m\,d\mu.$$

Thus generally from the values of X and Y we derive

$$(\alpha_n+\beta_n)\,2n(n+1)\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2}$$

$$=(2n+1)\left[\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu\right],$$

and from the values of Z we derive

$$[(n+1)\,\alpha_n-n\beta_n]\int_{-1}^{1}(H_n^m)^2\,d\mu=\int_{-1}^{1}H_n^m z_m\,d\mu.$$

The above theory assumes that the integration is taken over the whole surface of the Earth, and that the observations are uniformly distributed over the Earth's surface, otherwise the coefficients of the neglected terms on the left-hand side of these equations will not vanish, and each equation may have other terms which are too important to be neglected, and so it will not be so easy to separate the magnetic constants from one another.

Section VI. contains the Theory of Terrestrial Magnetism for the Earth regarded as a spheroid and gives the theory of the determination of the magnetic constants. Let r, θ', λ be the polar coordinates of a point on the spheroidal surface referred to the Earth's centre as origin and axis of figure as initial line; let θ be the geographical colatitude (the angle which the normal makes with the axis) and let $\mu = \cos\theta$ and $\mu' = \cos\theta'$.

The angle of the vertical $\psi = \theta' - \theta$.

The values of the sines and cosines of these angles for values of θ differing by 1° from 0° to 90° have been computed, the eccentricity e of the elliptic section in the plane of the meridian being derived from Bessel's dimensions of the Earth as given in Encke's tables in the *Berliner Jahrbuch*, 1852.

The expressions for the magnetic potential and for the magnetic forces X, Y, and Z, in terms of the Gaussian magnetic constants g_n^m, h_n^m, will be of the same form as those given above for the sphere.

Where X is the total force towards the north perpendicular to the Earth's radius, Y the total force perpendicular to the geographical meridian towards the west, Z the force towards the Earth's centre, where $X = -\frac{dV}{rd\theta'}$, $Y = -\frac{1}{r\sin\theta'} \cdot \frac{dV}{d\lambda}$, and $Z = -\frac{dV}{dr}$ (east longitudes being considered positive).

If X' be the horizontal force in the meridian towards the north,

Y' the horizontal force perpendicular to the meridian towards the west,

Z' the vertical force on the spheroidal surface of the Earth,

then

$$X' = X\cos\psi + Z\sin\psi,$$

$$Y' = Y,$$

$$Z' = -X\sin\psi + Z\cos\psi.$$

The values of the coefficients of $g_n^m \cos m\lambda$ and $h_n^m \sin m\lambda$ in the potential function and in the forces X', Y', and Z' are denoted by the symbols V'^m_n, X'^m_n, Y'^m_n, and Z'^m_n respectively.

If r be the radius vector, $\mu = \cos\theta$ and $\mu' = \cos\theta'$.

Then $$V'^m_n = \frac{1}{r^{n+1}} H'^m_n, \text{ and } V'^m_{-n} = r^n H'^m_{-n},$$

H'^m_n being the same function of μ' that H^m_n is of μ.

The expressions for the magnetic forces on the spheroidal surface of the Earth will be as follows:—

Taking α_n and β_n to represent magnetic constants depending on internal and external sources of magnetic force respectively, the coefficient of $\cos m\lambda$ in the general term of the potential function V is

$$\left(\frac{\alpha_n}{r^{n+1}} + \beta_n r^n\right) H'^m_n.$$

The coefficients of $\cos m\lambda$ in the general terms of the forces X, Y, and Z are—

For X, $$\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right)(1-\mu'^2)^{\frac{1}{2}} \frac{dH'^m_n}{d\mu'}.$$

For Y, $$\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) m (1-\mu'^2)^{-\frac{1}{2}} H'^m_n.$$

For Z, $$\left(\frac{\alpha_n (n+1)}{r^{n+2}} - \beta_n \,.\, n \,.\, r^{n-1}\right) H'^m_n.$$

Taking the equatorial radius $=1$, δS an element of the Earth's surface and e the eccentricity, and taking into account only the terms to the order e^2, we have $\frac{1}{r^2} = 1 + e^2\mu^2$, $\sin\psi = e^2\mu(1-\mu^2)^{\frac{1}{2}}$ to the order e^2,

$$\mu' = \cos\theta - \sin\theta\sin\psi = \mu - e^2\mu(1-\mu^2)\ \frac{d\mu'}{d\mu} = 1 - e^2(1-3\mu^2),$$

and $$\frac{dS}{d\mu'} = -2\pi(1-e^2\mu^2);$$

also $$\frac{1}{r^{n+2}} = 1 + \frac{n+2}{2} e^2\mu^2,$$

and $$r^{n-1} = 1 - \frac{n-1}{2} e^2\mu^2.$$

Regarding H_n' and $\frac{dH_n'}{d\mu'}$, &c., as functions of μ', we have by Taylor's theorem—

$$H_n' = H_n - e^2\mu(1-\mu^2)\frac{dH_n}{d\mu} \text{ to the order } e^2,$$

and
$$\frac{dH_n'}{d\mu'} = \frac{dH_n}{d\mu} - e^2\mu(1-\mu^2)\frac{d^2H_n}{d\mu^2},$$

from which we derive the value of X_n for the spheroidal surface—

$$X_n = (1-\mu'^2)^{\frac{1}{2}}\frac{dH_n'}{d\mu'}$$

$$= (1-\mu^2)^{\frac{1}{2}}\frac{dH_n}{d\mu}(1-e^2\mu^2) + e^2\mu(1-\mu^2)^{\frac{1}{2}}\left[n(n+1) - \frac{m^2}{1-\mu^2}\right]H_n.$$

If now we substitute the values of X, Y, and Z in terms of H_n', $\frac{dH_n'}{d\mu'}$, &c., in the equations for X', Y', Z', the expressions for the magnetic forces become—

$$X' = \left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right)\frac{dH_n'}{d\mu'}(1-\mu^2)^{\frac{1}{2}}\cos\psi$$
$$+ \left[\frac{(n+1)\alpha_n}{r^{n+2}} - n\beta_n r^{n-1}\right]H_n'\sin\psi + \text{similar terms},$$

$$Y' = \left(\frac{\alpha}{r^{n+2}} + \beta_n r^{n-1}\right)mH_n'(1-\mu^2)^{-\frac{1}{2}} + \text{similar terms},$$

$$Z' = -\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right)\frac{dH_n'}{d\mu'}(1-\mu^2)^{\frac{1}{2}}\sin\psi$$
$$+ \left[\frac{(n+1)\alpha_n}{r^{n+2}} - n\beta_n r^{n-1}\right]H_n'\cos\psi + \text{similar terms}.$$

In these expressions for the magnetic forces the values of H_n', $\frac{dH_n'}{d\mu'}$, &c., in terms of H_n, $\frac{dH_n}{d\mu}$, &c., are substituted for each belt of latitude, and these theoretical expressions derived from the potential function for a given belt of latitude, and containing the magnetic constants, are equated to the corresponding coefficients derived from the magnetic observations taken in that belt of latitude.

In the case of the spheroid, as in the case of the sphere, the values of the forces X, Y, and Z derived for every $10°$ of longitude from the

observations of declination, inclination, and horizontal force are analysed for belts of latitude 5° in breadth around the Earth's surface by a formula of the type

$$a_0 + a_1 \cos\lambda + b_1 \sin\lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda + , \ \&c.,$$

and the coefficients of $\cos m\lambda$, $\sin m\lambda$, in this expansion are equated respectively to the coefficients of $\cos m\lambda$ and $\sin m\lambda$ in the expansion in terms of the potential function and magnetic constants as given above: thus for the force X, if α_n, α_{n_1}, α_{n_2}, &c., stand for the magnetic constants, and if x_m' be the coefficient of $\cos m\lambda$ as derived directly from the observations, then

$$\alpha_n X'^m_n + \alpha_{n_1} X'^m_{n_1} + \alpha_{n_2} X'^m_{n_2} + , \ \&c. = x_m',$$

and similar equations are obtained from the expressions for the forces Y' and Z'.

The values X'^m_n, Y'^m_n, and Z'^m_n, taken in these equations, are the values derived for the spheroidal surface of the Earth from the potential function, and these equations include not only the magnetic constants which were determined by Gauss, of the class indicated by α in the above equation, but they also include magnetic constants which may be spoken of for distinction as the β class (*i.e.* including, *e.g.*, the class answering to forces of external origin), those forces which depend upon sources outside the surface of the Earth.

The full values, then, of the coefficients of the magnetic constants will be of this form:

For the α class—

$$X'^m_n = \frac{1}{r^{n+2}} \left[\frac{1}{2}(n-m) H'^{m+1}_n - \frac{1}{2}(n+m) H'^{m-1}_n \right] \cos\psi + \frac{n+1}{r^{n+2}} H'^m_n \sin\psi,$$

$$Y'^m_n = \frac{1}{r^{n+2}} \left[m(1-\mu'^2)^{-\frac{1}{2}} H'^m_n \right],$$

$$Z'^m_n = -\frac{1}{r^{n+2}} \left[\frac{1}{2}(n-m) H'^{m+1}_n - \frac{1}{2}(n+m) H'^{m-1}_n \right] \sin\psi + \frac{n+1}{r^{n+2}} H'^m_n \cos\psi.$$

For the β class, which may be denoted by X'^m_{-n}, Y'^m_{-n}, and Z'^m_{-n}—

$$X'^m_{-n} = r^{n-1} \left[\frac{1}{2}(n-m) H'^{m+1}_n - \frac{1}{2}(n+m) H'^{m-1}_n \right] \cos\psi - nr^{n-1} H'^m_n \sin\psi,$$

$$Y'^m_{-n} = r^{n-1} \left[m(1-\mu'^2)^{-\frac{1}{2}} H'^m_n \right],$$

$$Z'^m_{-n} = -r^{n-1} \left[\frac{1}{2}(n-m) H'^{m+1}_n - \frac{1}{2}(n+m) H'^{m-1}_n \right] \sin\psi - nr^{n-1} H'^m_n \cos\psi.$$

In Section VII. is given the numerical calculation of the coefficients of the magnetic constants for the Earth's spheroidal surface.

The numerical values of these expressions for all values of m from 0 to 10, and for all values of n from 1 to 10, for the spheroidal surface of the Earth, have been calculated from the values of μ for every 5° of colatitude, and form the coefficients of the magnetic constants g_n^m, h_n^m, and g_{-n}^m, h_{-n}^m of the α and β class respectively in the equations for the determination of these constants.

The number of magnetic constants contained in these equations which have been completely formed is thus 120 of each class, or 240 magnetic constants in all, in place of the 24 constants of the α class which were previously determined by Gauss.

Regarding the Earth as a spheroid of revolution, the values of $\mu' = \cos\theta'$, where θ' is the geocentric colatitude, have been determined for every 5° of geographical colatitude. Also the values of $\cos\psi$, $\sin\psi$, $\frac{a}{r}$, G'^m_n, and H'^m_n have been calculated for every 5° of geographical colatitude (*i.e.* for the above values of μ') for all values of n and m from 0 to 10.

The weights of the observations of the magnetic elements for these belts of latitude have also been determined on the assumption that the weight is proportional to the area of the corresponding portion of the Earth's surface.

The values of H'^m_n as a function of the geocentric colatitude having been determined for every 5° of geographical colatitude on the spheroid, we next proceed to determine from them the values of

$$Y'^m_n,\ Y'^m_{-n},\ X'^m_n\ (= X^m_n \cos\psi + Z^m_n \sin\psi),\ X'^m_{-n},$$
$$Z'^m_n\ (= -X^m_n \sin\psi + Z^m_n \cos\psi) \text{ and } Z'^m_{-n},$$

the resolved parts of the expressions for the horizontal and vertical forces in the plane of the meridian on the spheroid.

These values are required in the formation of the *equations of condition*, and their numerical values are calculated for every 5° of geographical colatitude as well as for the Equator and the Poles. These values of X'^m_n, &c., have been calculated and recorded in tables for all values of n and m from 0 to 10, and have been employed as the theoretical coefficients of the magnetic constants g_n^m, h_n^m, &c., in the *equations of condition*.

Section VIII. treats of the mode of formation of the *Equations of Condition* and the *Final Equations* for determining the magnetic constants, the solution of the equations and the discussion of the results.

Formation of the Equations of Condition.

When $n-m$ is even, the value of X_n^m contains only odd powers of μ, and the values of Y_n^m and Z_n^m only even powers, and similarly when $n-m$ is odd, the value of X_n^m contains only even powers of μ, and the values of Y_n^m and Z_n^m only odd powers. Hence, if the coefficient of $\cos m\lambda$ in either of the quantities X, Y, Z be denoted by a_m and the coefficient of $\sin m\lambda$ by b_m for a given north latitude, and if a_m', b_m' denote the similar quantities for the corresponding south latitude, then we have, when $n-m$ is even,

$$\Sigma\,(X_n^m g_n^m + X_{-n}^m g_{-n}^m) = \frac{1}{2}(a_m - a_m') \quad \text{and} \quad \Sigma\,(X_n^m h_n^m + X_{-n}^m h_{-n}^m) = \frac{1}{2}(b_m - b_m'),$$

$$\Sigma\,(Y_n^m g_n^m + Y_{-n}^m g_{-n}^m) = \frac{1}{2}(b_m + b_m') \quad \text{and} \quad \Sigma\,(Y_n^m h_n^m + Y_{-n}^m h_{-n}^m) = -\frac{1}{2}(a_m + a_m'),$$

$$\Sigma\,(Z_n^m g_n^m + Z_{-n}^m g_{-n}^m) = \frac{1}{2}(a_m + a_m') \quad \text{and} \quad \Sigma\,(Z_n^m h_n^m + Z_{-n}^m h_{-n}^m) = \frac{1}{2}(b_m + b_m');$$

and when $n-m$ is odd,

$$\Sigma\,(X_n^m g_n^m + X_{-n}^m g_{-n}^m) = \frac{1}{2}(a_m + a_m') \quad \text{and} \quad \Sigma\,(X_n^m h_n^m + X_{-n}^m h_{-n}^m) = \frac{1}{2}(b_m + b_m'),$$

$$\Sigma\,(Y_n^m g_n^m + Y_{-n}^m g_{-n}^m) = \frac{1}{2}(b_m - b_m') \quad \text{and} \quad \Sigma\,(Y_n^m h_n^m + Y_{-n}^m h_{-n}^m) = -\frac{1}{2}(a_m - a_m'),$$

$$\Sigma\,(Z_n^m g_n^m + Z_{-n}^m g_{-n}^m) = \frac{1}{2}(a_m - a_m') \quad \text{and} \quad \Sigma\,(Z_n^m h_n^m + Z_{-n}^m h_{-n}^m) = \frac{1}{2}(b_m - b_m').$$

Hence the equations for the quantities h_n^m and h_{-n}^m will be found from the equations for g_n^m and g_{-n}^m, when $n-m$ is even, by substituting

$$\frac{1}{2}(b_m - b_m') \text{ for } \frac{1}{2}(a_m - a_m') \text{ in the equations for } X,$$

$$-\frac{1}{2}(a_m + a_m') \text{ for } \frac{1}{2}(b_m + b_m') \text{ in the equations for } Y,$$

and

$$\frac{1}{2}(b_m + b_m') \text{ for } \frac{1}{2}(a_m + a_m') \text{ in the equations for } Z.$$

And similarly the equations for h_n^m and h_{-n}^m will be found from the equations for g_n^m and g_{-n}^m, when $n-m$ is odd, by substituting

$$\frac{1}{2}(b_m+b_m') \text{ for } \frac{1}{2}(a_m+a_m') \text{ in the equations for } X,$$

$$-\frac{1}{2}(a_m-a_m') \text{ for } \frac{1}{2}(b_m-b_m') \text{ in the equations for } Y,$$

and
$$\frac{1}{2}(b_m-b_m') \text{ for } \frac{1}{2}(a_m-a_m') \text{ in the equations for } Z.$$

Thus each equation for the determination of magnetic constants is separated into two equations, each of which contains only one-half the number of magnetic constants to be determined.

In the first solution of the equations, the *absolute terms* (*i.e.* the terms derived from the *observed* values of the magnetic elements) are taken from Sabine's magnetic charts for the period about 1845, as published in the *Philosophical Transactions of the Royal Society*. In the second solution, the observed values of the magnetic elements are taken from the Admiralty charts for 1880 prepared by Captain Creak, kindly lent by the Lords of the Admiralty.

The values of X, Y and Z are calculated for every 10° of longitude and every 5° of latitude from the declination (δ), the dip (ι) and the horizontal force (ω) as given in the charts. Then the values of X, Y and Z are analysed for belts of latitude 5° in breadth around the earth by the formula

$$a_0+a_1\cos\lambda+b_1\sin\lambda+a_2\cos 2\lambda+b_2\sin 2\lambda+, \text{ \&c.}$$

The values of these coefficients for the different belts of latitude were obtained and tabulated. Then if a_m and b_m denote the values of two of these coefficients for a given northern latitude, and a_m', b_m' the corresponding values for an equal southern latitude, then the values of $\frac{1}{2}(a_m+a_m')$, $\frac{1}{2}(a_m-a_m')$, $\frac{1}{2}(b_m+b_m')$, and $\frac{1}{2}(b_m-b_m')$ and of their logarithms are determined. The values of these quantities are determined for each of the periods for which the magnetic constants are required.

The numerous tables contained in this work have been calculated with the utmost care, under Professor Adams' minute supervision and instructions, by Mr Graham, who has done a great part of the work, and by Mr Todd

and other Assistants at the Observatory of Cambridge; these tables have also been calculated by Mr Roberts and by Mr T. Wright and other Assistants at the *Nautical Almanack* Office and have been compared and tested most carefully in a variety of ways for their accuracy.

A short account of the method and the results of the investigation was given by Professor Adams in the Mathematical Section of the British Association at the Manchester Meeting in 1887, but unfortunately no record was made of this communication and no account of any part of the work was at any time written or published. Under these circumstances it has been no easy task to piece together into this connected whole the detached portions of the work which were delivered into my hands, without any explanation as to any part of them or as to their connection with one another, during my brother's last illness, when he was no longer able to give me any hints as to his theory or mode of treatment.

To the unravelling and the publication of this work, which was begun in 1849, and no part of which was published until 1887, I have devoted whatever leisure I could command during the past eight years,—as a tribute to the memory of one whose accuracy of work has probably never been surpassed.

I desire to acknowledge the very great assistance which I have received from Dr C. Chree, the Superintendent of the Kew Observatory, whom I have consulted on various parts of the work.

My best thanks are due to the Lords of the Admiralty and to Captain Creak for the permission to use and to reproduce the Charts of the magnetic elements for 1880; from which the observations for that period are taken. There is found to be a remarkable agreement on comparing the theoretical results for the mean vertical and horizontal forces over the polar areas at the north and south poles with the values deduced by Captain Creak from the observations and given in the polar Charts at the end of this volume.

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CONTENTS.

PART I. ASTRONOMY.

PART II. TERRESTRIAL MAGNETISM.

LECTURES ON THE LUNAR THEORY.

[LECTURES on the Lunar Theory were given by Adams from 1860 with few intermissions until 1889. Originally their aim was to illustrate geometrically the analytical processes and thereby render them more comprehensible, and they included some elegant theorems on the geometry of conics which have since become common property; but every year several lectures were rewritten, and thus the whole fabric gradually changed into the form in which it is here presented,—the form, practically, in which he gave them last.

Perhaps it is superfluous to say that these Lectures stand upon a different footing to treatises that are intended to form the basis of Tables. With such, completeness is the first object and manner of presentation is secondary. Immense as is the labour of forming a treatise of this description, there exist several that leave little to desire in respect to fulness of detail. Indeed it may be suspected that their very perfection in the quality they profess has stifled to some degree the proper development of the subject, because at first sight it suggests that there is little left to do in the Lunar Theory, unless one is prepared to track down the inconsiderable errors that have eluded his Masters. This seems a mistake; the methods most suitable for the whole task adapt themselves comparatively ill to each detail of it, and there seems much that remains to be done in respect to inventing methods suitable for attacking separately, as far as they permit of separate attack, the many difficulties into which the theory divides at the outset, and thence perhaps approximating to a more adequate knowledge than we now possess of the relative motion of Three Bodies. So far, with the notable exception of Dr G. W. Hill and those that have followed him, we have seen comparatively little effort in this direction.

This was the cardinal feature of Adams's plan, and his lectures shew the methods he had gradually elaborated to accomplish it. They separate the inequalities from one another as far as possible, and are content with indicating the manner in which these separate inequalities afterwards combine. To shew that, with so slight an apparatus and within so small a compass, the result is no mere sketch, we need but set side by side the coefficients of longitude found in these Lectures and the corresponding terms in Delaunay's *Théorie*.

		Adams.	Delaunay.
Variation, coeff. of	$\sin 2D$	$2106''{\cdot}4$	$2106''{\cdot}25$
	$\sin 4D$	8·74	8·75
Parallactic inequality,	$\sin D$	$-124{\cdot}90$*	$-127{\cdot}62$
	$\sin 3D$	0·73	0·84
	$\sin 5D$	0·01	0·01
Annual equation,	$\sin l'$	$-658{\cdot}9$	$-659{\cdot}23$
	$\sin (2D-l')$	152·09	152·11
	$\sin (2D+l')$	$-21{\cdot}57$	$-21{\cdot}63$
Evection,	$\sin (2D-l)$	4596·6	4607·77
	$\sin (2D+l)$	175·1	174·87
Further,			
Motion of Apse,	$1-c$	·008554	·008572
Motion of Node,	$g-1$	·003997	·003999

For those to whom the difficulties of the Lunar Theory are known, these numbers need no comment.

No Manuscript exists of Lecture I. It is taken substantially from my own notes of 1889.]

* With Delaunay's value of the Sun's Parallax, viz. $8''{\cdot}75$.

LECTURE I.

HISTORICAL SKETCH.

[The Lunar Theory may be said to have had its commencement with Newton. Many irregularities in the Moon's motion were known before his time, but it was he that first explained the cause of those irregularities and calculated their amounts from theory.

Of the inequalities which are due to the action of the Sun, the first, —which is called the Evection,—was discovered by Ptolemy, who lived at Alexandria in the first half of the second century of our era, under the reigns of Hadrian and Antoninus Pius. At a very early period the relative distance of the Moon at different times could be told from the angle it subtended, and its orbit could thus be mapped out. By such means Ptolemy found that its form was not the same from month to month, and that the longer axis moved continually though not uniformly in one direction. He represented this change by a motion of the centre of the ellipse, as we would put it, in an epicycle round the focus, obtaining thus a variable motion for the longer axis and a variable eccentricity.

The representation of position by means of epicycles is intimately related to the modern method of developing the coordinates in harmonic series; thus if we have

$$x = A_1 \cos(n_1 t + \alpha_1) + A_2 \cos(n_2 t + \alpha_2) + \dots$$
$$y = A_1 \sin(n_1 t + \alpha_1) + A_2 \sin(n_2 t + \alpha_2) + \dots$$

the motion of the point (x, y) is that on a circle of radius A_1 with angular velocity n_1, around a centre which moves on a circle of radius A_2 with angular velocity n_2, and so on; and if, more generally, we have

$$x = A_1 \cos(n_1 t + \alpha_1) + \dots$$
$$y = B_1 \sin(n_1 t + \alpha_1) + \dots$$

we may reduce this case to the former by rewriting

$$x = \frac{1}{2}(A_1 + B_1)\cos(n_1 t + \alpha_1) + \frac{1}{2}(A_1 - B_1)\cos(-n_1 t - \alpha_1) + \ldots,$$

$$y = \frac{1}{2}(A_1 + B_1)\sin(n_1 t + \alpha_1) + \frac{1}{2}(A_1 - B_1)\sin(-n_1 t - \alpha_1) + \ldots.$$

Probably we have here the reason why circular motions and epicycles were first employed.

Tycho Brahe (1546—1601) discovered the existence of another inequality in the Moon's Longitude quite different from the Elliptic Inequality and the Evection. He found it bore reference to the position of the Sun with regard to the Moon; so that when the Sun and the Moon were in conjunction or opposition or quadratures the position of the Moon was quite well represented by the existing theory, but from conjunction to the quadrature following, her position was more advanced than the place assigned to it, reaching a maximum of some 35′ about half-way; and in the second quadrant it was just as much behind. This inequality he called the Variation; it was the first that Newton accounted for theoretically, and if we were to suppose the Moon and Sun to move, except for mutual disturbance, in pure circles in the same plane, it is the only one that would present itself.

The next significant step was made by Horrox (1619—1641) who represented the Evection geometrically by motion in a variable ellipse, and gave very approximately the law of variation of the eccentricity and the motion of the apse. He supposed the focus of the orbit to move in an epicycle about its mean place.

Newton's *Principia* did not profess to be and was not intended for a complete exposition of the Lunar Theory. It was fragmentary; its object was to shew that the more prominent irregularities admitted of explanation on his newly discovered theory of universal gravitation. He explained the Variation completely, and traced its effects in Radius Vector as well as in Longitude; and he also saw clearly that the change of eccentricity and motion of the apse that constitute the Evection could be explained on his principles, but he did not give the investigation in the *Principia*, even to the extent to which he had actually carried it. The approximations are more difficult in this case than in that of the Variation, and require to be carried further in order to furnish results of the same accuracy as had already been obtained by Horrox from observation. He was more

successful in dealing with the motion of the node and the law of change of inclination. He shewed that when Sun and Node were in conjunction, then for nearly a month the Moon moved in a plane very approximately, and that the inclination of the orbit then reached its maximum, namely, $5° 17'$ about; but as the Sun moved away from the Node the latter also began to move, attaining its greatest rate when the separation was a quadrant, and that at this instant the inclination was $5°$ very nearly. He also assigned the law for intermediate positions. The fact that there was no motion when the Sun was at the Node, that is, in the plane of the Moon's orbit, confirmed his theory that these inequalities were due to the Sun's action.

When we spoke of Newton's results as fragmentary and incomplete, let it not be understood that he gave only very rude approximations to the truth. His results are far more accurate than those arrived at in elementary works of the present day upon the subject.

After Newton, Clairaut (1713—1765) treated the Lunar Theory analytically. He readily found the Variation and many other inequalities, but met with a difficulty in determining the motion of the apse. At first he made its mean motion only about one-half of the observed value, and supposed that this indicated a failure of Newton's law of the inverse square of the distance; but soon he recognized an error, caused by omission of terms which he had imagined would not affect the result. When these were included the calculated amount was nearly doubled.

The first Tables of the Moon which were sufficiently accurate for use in determining longitudes at sea by observation of Lunar Distances were those of Mayer. They obtained a prize offered by our Board of Longitude, and were published in 1770 by Maskelyne, the Astronomer Royal.

The first Theories which gave the Moon's place with an accuracy equal to that of observation were those of Damoiseau and Plana. The former was published in 1827, preceded in 1824 by Tables; the latter was published in 1832.

Hansen's Tables, which are those now used, were constructed from theory and were published in 1857 at the expense of the British Government.]

LECTURE II.

ACCELERATIONS OF THE MOON RELATIVE TO THE EARTH.

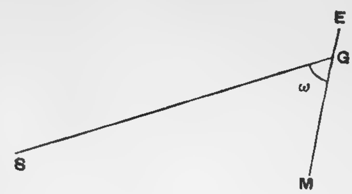
When three bodies move under their mutual attraction, their motions are unknown to us except in the cases when they are approximately elliptical; but this restriction includes almost all the most important cases in the Solar System.

If one body of the system is greatly predominant and if the lesser bodies are not close together, the centre of gravity of the greater body may be taken as a common focus around which the others move in approximate ellipses. Or again, if two bodies lie close together, their relative motion may be approximately the same as though they were isolated, although the system contains a third greatly predominant body; for their relative motion is affected by the difference of the attractions of the central body upon them and not by the absolute value of those attractions.

The Sun and Planets are an example of the first kind; the Earth, Moon and Sun of the second. The Earth and Moon describe orbits round the Sun which are approximately ellipses, and the Moon might be regarded as one of the planets; but this point of view would not be a simple one; the disturbing action of the Earth would be too great, though it is never so great as the direct attraction of the Sun, that is to say, never great enough to make the Moon's path convex to the Sun. The more convenient method is to refer the motion of the Moon to the Earth, and counting only the difference of the attractions of the Sun upon the Earth and upon the Moon, to find how this distorts the otherwise elliptical relative orbit. This is the method of the Lunar Theory.

The position of the Sun must be referred to the same origin; but since the Earth describes an ellipse about the Sun which is disturbed by

the action of the Moon, if we choose as origin the Earth's centre, we must allow for the disturbance of the Sun's position by the Moon. This correction may be evaded by choosing as origin, not the Earth's centre, but the centre of gravity of the Earth and Moon, with respect to which the Sun describes a curve so closely elliptical that no allowance is required. For, if S, E, M denote respectively the Sun, Earth, and Moon, and G the centre of gravity of E and M, the accelerating forces of S are



$$\text{on } E \quad S/SE^2 \text{ in } ES,$$

$$\text{on } M \quad S/SM^2 \text{ in } MS;$$

and these imply accelerations of G of amount

$$\frac{E}{E+M}\,\frac{S}{SE^2} \text{ parallel to } ES,$$

$$\frac{M}{E+M}\,\frac{S}{SM^2} \text{ parallel to } MS;$$

now the accelerations of S are

$$E/SE^2 \text{ in } SE,$$

$$M/SM^2 \text{ in } SM;$$

hence the acceleration of G relative to S is

$$\frac{S+E+M}{E+M}\,\frac{E}{SE^2} \text{ parallel to } ES,$$

$$\frac{S+E+M}{E+M}\,\frac{M}{SM^2} \text{ parallel to } MS;$$

or
$$\frac{S+E+M}{E+M}\left(E\,.\,\frac{GE}{SE^3}-M\,.\,\frac{GM}{SM^3}\right) \text{ in } GM,$$

$$\frac{S+E+M}{E+M}\left(E\,.\,\frac{SG}{SE^3}+M\,.\,\frac{SG}{SM^3}\right) \text{ in } GS.$$

Let
$$EM=r,\ SG=r',\ SGM=\omega; \text{ then}$$

$$GM=\frac{E}{E+M}\,r, \qquad GE=\frac{M}{E+M}\,r.$$

Hence

$$\frac{1}{SM^3}=\frac{1}{r'^3}\left[1+\frac{E}{E+M}\frac{r}{r'}3\cos\omega+\left(\frac{E}{E+M}\frac{r}{r'}\right)^2\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right)+\ldots\ldots\right],$$

$$\frac{1}{SE^3}=\frac{1}{r'^3}\left[1-\frac{M}{E+M}\frac{r}{r'}3\cos\omega+\left(\frac{M}{E+M}\frac{r}{r'}\right)^2\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right)+\ldots\ldots\right];$$

and the accelerations of G are

$$\frac{S+E+M}{r'^2}\left[\quad-\frac{EM}{(E+M)^2}\frac{r^2}{r'^2}3\cos\omega\qquad+\ldots\ldots\right]\text{ in }GM.$$

$$\frac{S+E+M}{r'^2}\left[1+\frac{EM}{(E+M)^2}\frac{r^2}{r'^2}\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right)+\ldots\ldots\right]\text{ in }GS.$$

Now r/r' is approximately $\frac{1}{400}$; neglecting the square of this quantity, we see that S moves about G in a pure ellipse.

Consider now the accelerations of the Moon relative to the Earth; subtracting the accelerations of the Earth from those of the Moon, we find

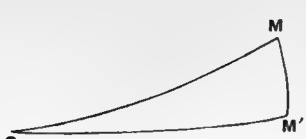
$$\frac{E+M}{ME^2}+S\left(\frac{MG}{SM^3}+\frac{EG}{SE^3}\right)\text{ in }MG,$$

$$S\left(\frac{SG}{SM^3}-\frac{SG}{SE^3}\right)\text{ parallel to }GS;$$

let $E+M=\mu$, $S=m'$; then these become

$$\frac{\mu}{r^2}+\frac{m'r}{r'^3}\left[1\qquad+\frac{E-M}{E+M}\frac{r}{r'}3\cos\omega\qquad+\ldots\ldots\right]\text{ in }ME,$$

$$\frac{m'r}{r'^3}\left[3\cos\omega+\frac{E-M}{E+M}\frac{r}{r'}\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right)+\ldots\ldots\right]\text{ parallel to }GS.$$



In the accompanying spherical triangle, let G be the centre of the sphere, SM' the ecliptic, and M' the projection of M.

Let $1/u$ be the projection of ME on the plane of the ecliptic;

θ the longitude of the Moon as seen from the Earth,

θ' the longitude of the Sun as seen from G,

s the tangent of the Moon's latitude MM'.

Then

$$SM=\omega, \qquad SM'=\theta-\theta',$$

$$r=(1+s^2)^{\frac{1}{2}}u^{-1}, \qquad \cos\omega=\cos(\theta-\theta')(1+s^2)^{-\frac{1}{2}},$$

and the accelerations of M relative to E are

$$\frac{\mu u^2}{1+s^2}+\frac{m'(1+s^2)^{\frac{1}{2}}}{r'^3u}\left[1+\frac{E-M}{E+M}\frac{1}{r'u}3\cos(\theta-\theta')+\ldots\ldots\right] \text{ in } ME,$$

$$\frac{m'}{r'^3u}\left[3\cos(\theta-\theta')+\frac{E-M}{E+M}\frac{1}{r'u}\left(-\frac{3}{2}(1+s^2)+\frac{15}{2}\cos^2(\theta-\theta')\right)+\ldots\ldots\right]$$

parallel to GS.

Call these quantities U and V respectively; then if we resolve parallel to $M'G$, perpendicular to $M'G$ in the plane of the ecliptic, and perpendicular to the plane of the ecliptic, we have the following quantities which we call P, T, S; viz.:—

$$P=\quad U(1+s^2)^{-\frac{1}{2}}-V\cos(\theta-\theta'),$$

$$T=-V\sin(\theta-\theta'),$$

$$S=\quad Us(1+s^2)^{-\frac{1}{2}};$$

and also

$$S-Ps=\quad Vs\cos(\theta-\theta').$$

From these we find

$$P=\frac{\mu u^2}{(1+s^2)^{\frac{3}{2}}}-\frac{m'}{r'^3u}\left[\frac{1}{2}+\frac{3}{2}\cos 2(\theta-\theta')+\frac{E-M}{E+M}\frac{1}{r'u}\left\{\left(\frac{9}{8}-\frac{3}{2}s^2\right)\cos(\theta-\theta')\right.\right.$$
$$\left.\left.+\frac{15}{8}\cos 3(\theta-\theta')\right\}+\ldots\right],$$

$$T=\quad -\frac{m'}{r'^3u}\left[\quad\frac{3}{2}\sin 2(\theta-\theta')+\frac{E-M}{E+M}\frac{1}{r'u}\left\{\left(\frac{3}{8}-\frac{3}{2}s^2\right)\sin(\theta-\theta')\right.\right.$$
$$\left.\left.+\frac{15}{8}\sin 3(\theta-\theta')\right\}+\ldots\right],$$

$$S-Ps=\quad\frac{m's}{r'^3u}\left[\frac{3}{2}+\frac{3}{2}\cos 2(\theta-\theta')+\frac{E-M}{E+M}\frac{1}{r'u}\left\{\left(\frac{33}{8}-\frac{3}{2}s^2\right)\cos(\theta-\theta')\right.\right.$$
$$\left.\left.+\frac{15}{8}\cos 3(\theta-\theta')\right\}+\ldots\right].$$

Hence with the time as independent variable we have the equations of motion

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = T,$$

$$\frac{d^2}{dt^2}(rs) = -S,$$

Or we may write these with θ as independent variable; let

$$r^2\frac{d\theta}{dt} = H,$$

so that

$$\frac{d\theta}{dt} = Hu^2.$$

Then

$$H\frac{dH}{d\theta} = \frac{T}{u^3},$$

$$\frac{d^2r}{dt^2} = -H^2u^2\frac{d^2u}{d\theta^2} - u^2\frac{du}{d\theta}H\frac{dH}{d\theta},$$

$$r\left(\frac{d\theta}{dt}\right)^2 = H^2u^3,$$

whence

$$H^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) + H\frac{dH}{d\theta}u^2\frac{du}{d\theta} = P;$$

again,

$$\frac{d^2}{dt^2}(rs) = H^2u^2\left(u\frac{d^2s}{d\theta^2} - s\frac{d^2u}{d\theta^2}\right) + H\frac{dH}{d\theta}u^2\left(u\frac{ds}{d\theta} - s\frac{du}{d\theta}\right),$$

whence

$$H^2u^3\left(\frac{d^2s}{d\theta^2} + s\right) + H\frac{dH}{d\theta}u^3\frac{ds}{d\theta} = Ps - S;$$

or the equations of motion may be written

$$H^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = P - T\frac{du}{ud\theta},$$

$$H\frac{dH}{d\theta} = \frac{T}{u^3},$$

$$H^2u^3\left(\frac{d^2s}{d\theta^2} + s\right) = Ps - S - T\frac{ds}{d\theta}.$$

Our problem is to discuss these equations and to obtain from them expressions for the Moon's position at any time. The integration is best effected by observing what kinds of terms will disappear on substitution in the equations, and then assuming for the desired expressions for the coordinates a series of such terms multiplied by undetermined coefficients. Our procedure will be to discuss one by one the irregularities which can be isolated from one another. This will permit a survey of the entire field without involving needless complexity; but if the Lunar Theory is to be accurate, the combinations of such terms with one another must also be included, and the number of terms employed and the labour of manipulating them becomes very great.

LECTURE III.

THE SUN'S COORDINATES IN TERMS OF THE TIME.

To obtain the Moon's coordinates in terms of the time from the equations found in Lecture II., we must substitute in the expressions for the forces the developments of the Sun's coordinates which we now proceed to give.

Employing as coordinates r', θ', of the last lecture, we have seen that the Sun's motion may be regarded as purely elliptical, so that

$$\frac{a'}{r'} = \frac{1 + e' \cos(\theta' - \varpi')}{1 - e'^2},$$

$$\theta' - \varpi' = n't - \varpi' + 2e' \sin(n't - \varpi') + \frac{5}{4} e'^2 \sin 2(n't - \varpi') + \ldots$$

in which we have written for convenience $n't$ in place of $n't + \epsilon'$.

The quantities that enter the equations are

$$\left(\frac{a'}{r'}\right)^3,$$

$$\left(\frac{a'}{r'}\right)^3 \begin{matrix}\cos\\ \sin\end{matrix} 2(\theta - \theta'),$$

$$\left(\frac{a'}{r'}\right)^4 \begin{matrix}\cos\\ \sin\end{matrix} (\theta - \theta'),$$

$$\left(\frac{a'}{r'}\right)^4 \begin{matrix}\cos\\ \sin\end{matrix} 3(\theta - \theta').$$

Making the substitutions we find without difficulty

$$\left(\frac{a'}{r'}\right)^3 = 1 + \frac{3}{2} e'^2 + 3e' \cos(n't - \varpi') + \frac{9}{2} e'^2 \cos 2(n't - \varpi') + \ldots.$$

$$\left(\frac{a'}{r'}\right)^3 \frac{\cos}{\sin} 2(\theta-\theta') = \left(1-\frac{5}{2}e'^2\right)\frac{\cos}{\sin} 2(\theta-n't)$$

$$+\frac{7}{2}e'\frac{\cos}{\sin}\{2(\theta-n't)-(n't-\varpi')\}$$

$$-\frac{1}{2}e'\frac{\cos}{\sin}\{2(\theta-n't)+(n't-\varpi')\}$$

$$+\frac{17}{2}e'^2\frac{\cos}{\sin}\{2(\theta-n't)-2(n't-\varpi')\}$$

$$+\ldots\ldots.$$

$$\left(\frac{a'}{r'}\right)^4 \frac{\cos}{\sin} (\theta-\theta') = (1+2e'^2)\frac{\cos}{\sin}(\theta-n't)$$

$$+3e'\frac{\cos}{\sin}\{(\theta-n't)-(n't-\varpi')\}$$

$$+e'\frac{\cos}{\sin}\{(\theta-n't)+(n't-\varpi')\}$$

$$+\frac{53}{8}e'^2\frac{\cos}{\sin}\{(\theta-n't)-2(n't-\varpi')\}$$

$$+\frac{11}{8}e'^2\frac{\cos}{\sin}\{(\theta-n't)+2(n't-\varpi')\}$$

$$+\ldots\ldots.$$

$$\left(\frac{a'}{r'}\right)^4 \frac{\cos}{\sin} 3(\theta-\theta') = (1-6e'^2)\frac{\cos}{\sin} 3(\theta-n't)$$

$$+5e'\frac{\cos}{\sin}\{3(\theta-n't)-(n't-\varpi')\}$$

$$-e'\frac{\cos}{\sin}\{3(\theta-n't)+(n't-\varpi')\}$$

$$+\frac{127}{8}e'^2\frac{\cos}{\sin}\{3(\theta-n't)-2(n't-\varpi')\}$$

$$+\frac{1}{8}e'^2\frac{\cos}{\sin}\{3(\theta-n't)+2(n't-\varpi')\}$$

$$+\ldots\ldots.$$

These quantities are to be substituted where they occur in the expressions for the forces found in Lecture II.

Let us now make a few general remarks upon the result of the substitution.

It will be observed that the disturbing forces all involve the coefficient $m'a'^{-3}$. It is very important to notice that the Sun's parallax is not required for the evaluation of this quantity. By Kepler's Third Law it is derivable from observations of the Sun's mean motion alone. Other terms however, namely those with the coefficient m'/a'^4u, involve the Sun's parallax directly; and that constant may be obtained by comparing the observed with the theoretical values of the coefficients of those inequalities, with an accuracy probably greater than that of any other method.

The mean disturbing force is radial, and is equal to

$$-\frac{1}{2}\frac{m'a}{a'^3}\left(1+\frac{3}{2}e'^2\right);$$

or the mean effect of the Sun's disturbance is to diminish the Moon's gravity towards the Earth; and to diminish it more, the greater is the eccentricity of the Sun's orbit. Now e' has been diminishing for ages; hence the Moon's gravity towards the Earth has been increasing, and its average time for accomplishing a revolution about the Earth has been diminishing.

This is one cause of the Secular Acceleration of the Moon's mean motion which Halley derived from the records of ancient eclipses.

It may also be noticed that the coefficient of the chief periodic part of the disturbing force, which involves $1-\frac{5}{2}e'^2$, increases as e' diminishes.

Finally let it be observed that the term with argument

$$2(\theta-n't)+2(n't-\varpi'),$$

which does not involve the Sun's Mean Longitude, is absent from the development of $\left(\frac{a'}{r'}\right)^3 \frac{\cos}{\sin} 2(\theta-\theta')$.

LECTURE IV.

THE VARIATION.

THE Variation is the first inequality we shall consider; this is the inequality which is independent of eccentricities and mutual inclination in the orbits of the Sun and Moon.

Let us first take the equations in the first form in which they are given in Lecture II., namely with t as independent variable:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \quad T;$$

we omit the equation of motion in latitude, and in the expressions for P, T we suppose $s=0$; moreover it is possible and convenient to discuss separately the terms that involve the Sun's parallax; let these be omitted and we have

$$\frac{1}{r}\frac{d^2r}{dt^2} - \left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^3} = \frac{1}{2}\frac{m'}{r'^3} + \frac{3}{2}\frac{m'}{r'^3}\cos 2(\theta-\theta'),$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{3}{2}\frac{m'}{r'^3}\sin 2(\theta-\theta');$$

and if $$e'=0,\ r'=a',\ m'/a'^3=n'^2,\ \theta'=n't+\epsilon',$$

$$\frac{1}{r}\frac{d^2r}{dt^2} - \left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^3} = \frac{1}{2}n'^2 + \frac{3}{2}n'^2\cos 2(\theta-n't-\epsilon'),$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{3}{2}n'^2\sin 2(\theta-n't-\epsilon');$$

these are the equations to discuss.

Assume as a first approximation

$$\theta = nt + \epsilon + b_2 \sin \{2(nt+\epsilon) - 2(n't+\epsilon')\}$$

$$\equiv nt + \epsilon + b_2 \sin 2\psi, \text{ say};$$

$$\frac{1}{r} = \frac{1}{a}[1 + a_2 \cos 2\psi],$$

and we shall suppose a_2, b_2 so small that in the first instance we may neglect their squares and products.

Substitute in the equations; then

$$4(n-n')^2 a_2 \cos 2\psi - \{n^2 + 4n(n-n') b_2 \cos 2\psi\} + \frac{\mu}{a^3}\{1 + 3a_2 \cos 2\psi\}$$

$$= \frac{1}{2}n'^2 + \frac{3}{2}n'^2 \cos 2\psi$$

$$-4(n-n')^2 b_2 \sin 2\psi + 4(n-n') n a_2 \sin 2\psi = \quad -\frac{3}{2}n'^2 \sin 2\psi.$$

Hence, equating the coefficients of similar terms, we have

$$\frac{\mu}{a^3} = n^2 + \frac{1}{2}n'^2,$$

which gives the relation between n the Moon's mean motion, and $\frac{1}{a}$, the mean of the reciprocal of the distance; also

$$\left[4(n-n')^2 + \frac{3\mu}{a^3}\right] a_2 - 4n(n-n') b_2 = \quad \frac{3}{2}n'^2 \ldots\ldots\ldots\ldots\ldots (1),$$

$$-4(n-n')^2 b_2 + 4n(n-n') a_2 = -\frac{3}{2}n'^2 \ldots\ldots\ldots\ldots\ldots (2).$$

From (2) $\qquad 4n(n-n') b_2 - 4n^2 a_2 \quad = \quad \frac{3}{2}\frac{nn'^2}{n-n'}.$

Add to (1), and substitute for μ/a^3;

$$\left[4(n-n')^2 - n^2 + \frac{3}{2}n'^2\right] a_2 = \frac{3}{2}n'^2 \frac{2n-n'}{n-n'},$$

$$a_2 = \frac{3}{2}n'^2 \cdot \frac{2n-n'}{n-n'} \cdot \frac{1}{3n^2 - 8nn' + \frac{11}{2}n'^2},$$

$$b_2 = \frac{n}{n-n'} a_2 + \frac{3}{8} \frac{n'^2}{(n-n')^2}$$

$$= \frac{3}{2} n'^2 . \frac{n(2n-n')}{(n-n')^2} . \frac{1}{3n^2 - 8nn' + \frac{11}{2} n'^2} + \frac{3}{8} \frac{n'^2}{(n-n')^2}.$$

Calling $\frac{n'}{n} = m$, we have

$$a_2 = \frac{3}{2} m^2 . \frac{2-m}{1-m} . \frac{1}{3 - 8m + \frac{11}{2} m^2},$$

$$b_2 = \frac{3}{2} m^2 . \frac{2-m}{(1-m)^2} . \frac{1}{3 - 8m + \frac{11}{2} m^2} + \frac{3}{8} \frac{m^2}{(1-m)^2},$$

or, calling $\frac{n'}{n-n'}$, or $\frac{m}{1-m} = m_1$, we have

$$a_2 = \frac{3}{2} m_1^2 . \frac{2+m_1}{3 - 2m_1 + \frac{1}{2} m_1^2},$$

$$b_2 = \frac{3}{2} m_1^2 . \frac{(1+m_1)(2+m_1)}{3 - 2m_1 + \frac{1}{2} m_1^2} + \frac{3}{8} m_1^2.$$

These are convenient expressions, and, as it happens, very approximate. If we wish to develope in ascending powers of m or m_1, it appears that the latter development will be the more convergent.

We find by observation $\frac{n'}{n} = \cdot 07480$, very nearly.

Hence $a_2 = \cdot 00717{,}95$,

$b_2 = \cdot 01021{,}2 = 2106''\cdot 4$.

Hence the ratio of the greatest and least distances will be

$1\cdot 00717{,}95 : 0\cdot 99282{,}05$,

and the greatest angular deviation from the mean longitude will be

$35' \cdot 6'' \cdot 4$,

a very close approximation to the truth.

Also we have found

$$\frac{\mu}{a^3} = n^2 + \frac{1}{2} n'^2 = n^2 \left(1 + \frac{1}{2} m^2\right)$$
$$= n^2 \times 1{\cdot}00280,$$

which is the relation between the actual mean motion and the actual mean distance (or rather mean reciprocal distance) of the Moon.

Without the Sun's disturbing action, the relation between the mean distance and the mean motion, or rather between the radius of the orbit supposed circular and the uniform rate of angular motion along it would be

$$\frac{\mu}{a^3} = n^2.$$

Hence in the actual orbit, the mean motion for a given mean distance is smaller than it would be without disturbance;

Or, for a given mean motion, the mean distance is smaller than it would be without disturbance.

In fact, the relation between the mean distance and the mean motion is the same as it would be if the sum of the masses of the Earth and Moon were diminished in the ratio of 1·00280 to 1.

LECTURE V.

THE VARIATION, (*continued*).

We will now proceed to substitute in the differential equations the values of $1/r$ and θ which we have obtained, retaining terms of the order of the squares and products of a_2, b_2 and m^2 or m_1^2.

The values to be thus substituted are

$$\frac{1}{r}=\frac{1}{a}(1+a_2\cos 2\psi),$$

$$\theta = nt+\epsilon+b_2\sin 2\psi,$$

where
$$\psi = nt+\epsilon-(n't+\epsilon'),$$

$$a_2=\frac{3}{2}m_1^2\frac{2+m_1}{3-2m_1+\frac{1}{2}m_1^2},$$

$$b_2=(1+m_1)a_2+\frac{3}{8}m_1^2.$$

Hence
$$r=a\left[1-a_2\cos 2\psi+\frac{1}{2}a_2^2(1+\cos 4\psi)\right],$$

$$\frac{d^2r}{dt^2}=4a(n-n')^2\left[a_2\cos 2\psi-2a_2^2\cos 4\psi\right],$$

$$\frac{1}{r}\frac{d^2r}{dt^2}=4(n-n')^2\left[\frac{1}{2}a_2^2+a_2\cos 2\psi-\frac{3}{2}a_2^2\cos 4\psi\right];$$

again, $$\frac{d\theta}{dt} = n + 2(n - n')\, b_2 \cos 2\psi,$$

$$\left(\frac{d\theta}{dt}\right)^2 = n^2 + 4n(n - n')\, b_2 \cos 2\psi + 2(n - n')^2 b_2^2 [1 + \cos 4\psi],$$

$$\frac{1}{r^3} = \frac{1}{a^3}\left[1 + 3a_2 \cos 2\psi + \frac{3}{2} a_2^2 (1 + \cos 4\psi)\right].$$

Also, $$\frac{1}{r}\frac{dr}{dt} = 2(n - n')\left[a_2 \sin 2\psi - \frac{1}{2} a_2^2 \sin 4\psi\right],$$

$$\frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt} = 2(n - n')\left[n a_2 \sin 2\psi + \left\{(n - n')\, a_2 b_2 - \frac{1}{2} n a_2^2\right\} \sin 4\psi\right],$$

$$\frac{d^2\theta}{dt^2} = -4(n - n')^2 b_2 \sin 2\psi.$$

And $$\cos 2(\theta - n't - \epsilon') = \cos 2\psi - b_2 (1 - \cos 4\psi),$$

$$\sin 2(\theta - n't - \epsilon') = \sin 2\psi + b_2 \sin 4\psi.$$

Substitute these in the differential equations, and we get, on transposing all the terms to the left-hand sides from the first equation

$$\begin{aligned} &4(n - n')^2 \left[\frac{1}{2} a_2^2 + a_2 \cos 2\psi - \frac{3}{2} a_2^2 \cos 4\psi\right] \\ &- [n^2 + 2(n - n')^2 b_2^2 + 4n(n - n')\, b_2 \cos 2\psi + 2(n - n')^2 b_2^2 \cos 4\psi] \\ &+ \frac{\mu}{a^3}\left[1 + \frac{3}{2} a_2^2 + 3a_2 \cos 2\psi + \frac{3}{2} a_2^2 \cos 4\psi\right] \\ &- \frac{1}{2} n'^2 - \frac{3}{2} n'^2 [-b_2 + \cos 2\psi + b_2 \cos 4\psi], \end{aligned}$$

and from the second equation

$$\begin{aligned} &-4(n - n')^2 b_2 \sin 2\psi \\ &+ 4(n - n')\left[n a_2 \sin 2\psi + \left\{(n - n')\, a_2 b_2 - \frac{1}{2} n a_2^2\right\} \sin 4\psi\right] \\ &+ \frac{3}{2} n'^2 [\sin 2\psi + b_2 \sin 4\psi]. \end{aligned}$$

The coefficient of $\cos 2\psi$ in the first of these expressions, and that of $\sin 2\psi$ in the second, are respectively

$$4(n-n')^2 a_2 - 4n(n-n')b_2 + 3\frac{\mu}{a^3}a_2 - \frac{3}{2}n'^2,$$

and
$$-4(n-n')^2 b_2 + 4n(n-n')a_2 \qquad + \frac{3}{2}n'^2,$$

and these are evidently reduced to zero by giving a_2, b_2 the values previously found, if we substitute for μ/a^3 the approximate value $n^2+\frac{1}{2}n'^2$. To find the more correct value of μ/a^3, equate to zero the constant term in the first expression;

$$2(n-n')^2 a_2^2 - n^2 - 2(n-n')^2 b_2^2 + \frac{\mu}{a^3}\left(1+\frac{3}{2}a_2^2\right) - \frac{1}{2}n'^2 + \frac{3}{2}n'^2 b_2 = 0,$$

that is

$$\frac{\mu}{a^3}\left(1+\frac{3}{2}a_2^2\right) = n^2 + \frac{1}{2}n'^2 - 2(n-n')^2 a_2^2 + 2(n-n')^2 b_2^2 - \frac{3}{2}n'^2 b_2$$

$$= n^2 + \frac{1}{2}n'^2 + 2(n-n')^2\left[(2m_1+m_1^2)a_2^2 - \frac{9}{64}m_1^4\right].$$

Hence we see that μ/a^3 differs from $n^2+\frac{1}{2}n'^2$ only in terms of the fourth order, if we consider m_1 a quantity of the first order and consequently a_2, b_2 quantities of the second order. Hence also by taking

$$\frac{\mu}{a^3} = n^2 + \frac{1}{2}n'^2,$$

in the multiplier of a_2, when we equate to zero the coefficient of $\cos 2\psi$, we only neglected a quantity of the sixth order in m_1, and the error in the resulting values of a_2, b_2 is of that order.

We see that the substitution just made in our equations leaves outstanding terms of the fourth order in $\cos 4\psi$ and $\sin 4\psi$. In order to get rid of these we must add terms of this form to the assumed values of $1/r$ and θ, respectively. Suppose that

$$\frac{1}{r} = \frac{1}{a}[1 + a_2\cos 2\psi + a_4\cos 4\psi],$$

$$\theta = nt + \epsilon + b_2\sin 2\psi + b_4\sin 4\psi,$$

where, as we shall find, a_4 and b_4 are small quantities of the fourth order.

It may be readily seen that the additional terms introduced are the following:—

$$\text{in}\quad \frac{1}{r}\frac{d^2r}{dt^2}\qquad 16(n-n')^2 a_4 \cos 4\psi,$$

$$-\left(\frac{d\theta}{dt}\right)^2\qquad -8n(n-n')\,b_4\cos 4\psi,$$

$$\frac{\mu}{r^3}\qquad \frac{\mu}{a^3}3a_4\cos 4\psi,$$

$$\frac{d^2\theta}{dt^2}\qquad -16(n-n')^2 b_4 \sin 4\psi,$$

$$\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\qquad 8n(n-n')\,a_4\sin 4\psi,$$

and also that we may neglect the terms added in the expressions for

$$\frac{3}{2}n'^2\cos 2(\theta-n't-\epsilon'),\qquad \frac{3}{2}n'^2\sin 2(\theta-n't-\epsilon').$$

If we write the terms thus produced along with the several terms left outstanding in our equations and then equate the whole to zero, we have

$$16(n-n')^2 a_4 - 8n(n-n')\,b_4 + \frac{\mu}{a^3}3a_4 - 6(n-n')^2 a_2^2 - 2(n-n')^2 b_2^2$$

$$+\frac{\mu}{a^3}\frac{3}{2}a_2^2 - \frac{3}{2}n'^2 b_2 = 0,$$

$$-16(n-n')^2 b_4 + 8n(n-n')\,a_4 + 4(n-n')^2 a_2 b_2 - 2n(n-n')\,a_2^2 + \frac{3}{2}n'^2 b_2 = 0,$$

from which we must determine a_4 and b_4.

Put
$$\frac{\mu}{a^3} = n^2 + \frac{1}{2}n'^2,$$

$$b_2 = (1+m_1)\,a^2 + \frac{3}{8}m_1^2,$$

and divide both equations by $(n-n')^2$; we get

$$\left[16+3\left(1+2m_1+\frac{3}{2}m_1^2\right)\right]a_4-8(1+m_1)b_4-\left[6+2(1+m_1)^2-\frac{3}{2}\left(1+2m_1+\frac{3}{2}m_1^2\right)\right]a_2^2$$
$$-3(1+m_1)m_1^2a_2-\frac{27}{32}m_1^4=0,$$

$$8(1+m_1)a_4-16b_4+[4(1+m_1)-2(1+m_1)]a_2^2+3m_1^2\left(1+\frac{1}{2}m_1\right)a_2+\frac{9}{16}m_1^4=0.$$

Simplify and multiply the last equation by $\frac{1}{2}(1+m_1)$,

$$\left(19+6m_1+\frac{9}{2}m_1^2\right)a_4-8(1+m_1)b_4-\left(\frac{13}{2}+m_1-\frac{1}{4}m_1^2\right)a_2^2$$
$$-3(1+m_1)m_1^2a_2-\frac{27}{32}m_1^4=0,$$

$$(4+8m_1+4m_1^2)a_4-8(1+m_1)b_4+(1+2m_1+m_1^2)a_2^2$$
$$+\frac{3}{2}(1+m_1)\left(1+\frac{1}{2}m_1\right)m_1^2a_2+\frac{9}{32}(1+m_1)m_1^4=0.$$

Subtract the latter from the former and b_4 will be eliminated; we get

$$\left(15-2m_1+\frac{1}{2}m_1^2\right)a_4-\left(\frac{15}{2}+3m_1+\frac{3}{4}m_1^2\right)a_2^2$$
$$-\frac{3}{2}(1+m_1)\left(3+\frac{1}{2}m_1\right)m_1^2a_2-\frac{9}{32}(4+m_1)m_1^4=0,$$

which gives a_4; and this being known b_4 is found from

$$b_4=\frac{1}{2}(1+m_1)a_4+\frac{1}{8}(1+m_1)a_2^2+\frac{3}{16}\left(1+\frac{1}{2}m_1\right)m_1^2a_2+\frac{9}{256}m_1^4.$$

Taking $m=\cdot07480$ as in Lecture IV, we find

$$a_4=\cdot00004{,}580,$$

$$b_4=\cdot00004{,}237=8''\cdot740.$$

LECTURE VI.

THE VARIATION, (*continued*).

LET us consider the problem of the Variation over again, taking now θ as independent variable.

The equations of motion are given in Lecture II :—

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{H^2u^2}-\frac{T}{H^2u^3}\frac{du}{d\theta},$$

$$H\frac{dH}{d\theta}=\frac{T}{u^3},$$

where

$$\frac{P}{u^2}=\mu-\frac{1}{2}\frac{n'^2}{u^3}-\frac{3}{2}\frac{n'^2}{u^3}\cos 2(\theta-\theta'),$$

$$\frac{T}{u^3}=\qquad\qquad -\frac{3}{2}\frac{n'^2}{u^4}\sin 2(\theta-\theta'),$$

so that the second equation may be written

$$\frac{1}{H^2}\frac{d(H^2)}{d\theta}=-3n'^2\left(\frac{dt}{d\theta}\right)^2\sin 2(\theta-\theta').$$

Our aim is to express t and u in terms of θ and constant quantities. Now since the orbit of the Moon does not differ widely from a circle we may write the difference of $nt+\epsilon$ from θ, and the difference of au from unity as series of small periodic terms depending upon θ. Inspecting the form of the equations, it is evident that these periodic terms are of argument $2(\theta-\theta')$ and its multiples; that is

$$nt+\epsilon=\theta+\text{periodic terms of argument } 2(\theta-\theta'), \text{ \&c.};$$

but

$$n't+\epsilon'=\theta';$$

therefore

$$\theta-\theta'=(1-m)\theta-\beta+\text{periodic terms of argument } 2(1-m)\theta-2\beta, \text{ \&c.},$$

where we have written

$$\beta = \epsilon' - m\epsilon;$$

this constant β is associated with $(1-m)\,\theta$ wherever the latter occurs; for brevity in writing, we shall omit it.

We may then assume as a first approximation

$$au = 1 + a_2 \cos(2-2m)\,\theta,$$

$$nt + \epsilon = \theta + b_2 \sin(2-2m)\,\theta;$$

whence

$$2(\theta - \theta') = (2-2m)\,\theta - 2mb_2 \sin(2-2m)\,\theta,$$

$$\cos 2(\theta-\theta') = mb_2 + \cos(2-2m)\,\theta - mb_2\cos(4-4m)\,\theta,$$

$$\sin 2(\theta-\theta') = \qquad \sin(2-2m)\,\theta - mb_2 \sin(4-4m)\,\theta,$$

$$n\frac{dt}{d\theta} = 1 + (2-2m)\,b_2\cos(2-2m)\,\theta.$$

Substitute in the right-hand member of the second equation:—

$$\frac{1}{H^2}\frac{d(H^2)}{d\theta} = -3m^2\left[\sin(2-2m)\,\theta + (2-3m)\,b_2\sin(4-4m)\,\theta\right].$$

Therefore

$$\log_e\left(\frac{H^2}{h^2}\right) = \frac{3}{2}\,\frac{m^2}{1-m}\cos(2-2m)\,\theta + \frac{3}{4}\,\frac{2-3m}{1-m}\,m^2b_2\cos(4-4m)\,\theta,$$

which we may write

$$\log_e\left(\frac{H^2}{h^2}\right) = 2h_2\cos(2-2m)\,\theta + 2h_4\cos(4-4m)\,\theta,$$

where h is an arbitrary constant of integration, h_2 is a known quantity. and h_4 involves b_2. If we take as a second approximation

$$au = 1 + a_2\cos(2-2m)\,\theta + a_4\cos(4-4m)\,\theta,$$

$$nt + \epsilon = \theta + b_2\sin(2-2m)\,\theta + b_4\sin(4-4m)\,\theta,$$

the above value of $\log_e(H^2/h^2)$ will not require modification and will supply equations of condition for determining the coefficients a_2, b_2, a_4, b_4.

Thus

$$n\frac{dt}{d\theta} = \frac{n}{Hu^2} = \frac{na^2}{h}\,\frac{h}{H}\,\frac{1}{(au)^2},$$

so that

$$\log_e\left(n\frac{dt}{d\theta}\right)=\log_e\left(\frac{na^2}{h}\right)-\frac{1}{2}\log_e\left(\frac{H^2}{h^2}\right)-2\log_e au\,;$$

but

$$\log_e\left(n\frac{dt}{d\theta}\right)=-(1-m)^2b_2^2+(2-2m)\,b_2\cos(2-2m)\,\theta$$
$$+\left[(4-4m)\,b_4-(1-m)^2b_2^2\right]\cos(4-4m)\,\theta,$$

$$\log_e au=-\frac{a_2^2}{4}+a_2\cos(2-2m)\,\theta+\left(a_4-\frac{a_2^2}{4}\right)\cos(4-4m)\,\theta.$$

Hence we find

$$-(1-m)^2b_2^2=\log_e\left(\frac{na^2}{h}\right)+\frac{1}{2}a_2^2,$$

$$(2-2m)\,b_2=-h_2-2a_2,$$

$$(4-4m)\,b_4-(1-m)^2b_2^2=-h_4-2a_4+\frac{1}{2}a_2^2.$$

The remaining equations of condition that we require are obtained from the first equation of motion; this may be written

$$\frac{d^2(au)}{d\theta^2}+au\left[1+\frac{1}{2}\left(n'\frac{dt}{d\theta}\right)^2+\frac{3}{2}\left(n'\frac{dt}{d\theta}\right)^2\cos 2(\theta-\theta')\right]$$
$$-\frac{d(au)}{d\theta}\,\frac{3}{2}\left(n'\frac{dt}{d\theta}\right)^2\sin 2(\theta-\theta')=\frac{\mu a}{H^2}.$$

Now

$$au=1+a_2\cos(2-2m)\,\theta+a_4\cos(4-4m)\,\theta,$$

whence

$$\frac{d(au)}{d\theta}=-(2-2m)\,a_2\sin(2-2m)\,\theta-(4-4m)\,a_4\sin(4-4m)\,\theta,$$

$$\frac{d^2(au)}{d\theta^2}=-(2-2m)^2a_2\cos(2-2m)\,\theta-(4-4m)^2a_4\cos(4-4m)\,\theta,$$

and

$$\left(n'\frac{dt}{d\theta}\right)^2=m^2\left[1+(4-4m)\,b_2\cos(2-2m)\,\theta\right],$$

$$\left(n'\frac{dt}{d\theta}\right)^2\sin 2(\theta-\theta')=m^2\left[\sin(2-2m)\,\theta+(2-3m)\,b_2\sin(4-4m)\,\theta\right],$$

$$\left(n'\frac{dt}{d\theta}\right)^2\cos 2(\theta-\theta') = m^2\left[(2-m)\,b_2+\cos(2-2m)\,\theta+(2-3m)\,b_2\cos(4-4m)\,\theta\right],$$

$$\frac{1}{H^2}=\frac{1}{h^2}\left[1+h_2^2-2h_2\cos(2-2m)\,\theta+(h_2^2-2h_4)\cos(4-4m)\,\theta\right].$$

Substitute these in the equation above, and equate the coefficients of corresponding terms,

$$1+\frac{1}{2}m^2+\frac{3}{2}m^2(2-m)\,b_2+\frac{3}{4}m^2a_2+\frac{3}{4}m^2(2-2m)\,a_2=\frac{\mu a}{h^2}(1+h_2^2)$$

$$-(2-2m)^2a_2+\left(1+\frac{1}{2}m^2\right)a_2+\frac{3}{2}m^2+(2-2m)\,m^2b_2=\frac{\mu a}{h^2}(-2h_2)$$

$$-(4-4m)^2a_4+\left(1+\frac{1}{2}m^2\right)a_4+\frac{3}{2}m^2(2-3m)\,b_2+\frac{3}{4}m^2a_2-\frac{3}{4}m^2(2-2m)\,a_2$$
$$=\frac{\mu a}{h^2}(h_2^2-2h_4).$$

If we neglect at first terms of the fourth order, we find from the first of these equations

$$\frac{\mu a}{h^2}=1+\frac{1}{2}m^2.$$

From the earlier set of equations we have

$$(2-2m)\,b_2=-h_2-2a_2\,;$$

substitute this in the second equation above. We get

$$\left[-(2-2m)^2+1+\frac{1}{2}m^2-2m^2\right]a_2-m^2h_2+\frac{3}{2}m^2=\left(1+\frac{1}{2}m^2\right)(-2h_2),$$

or

$$\left(3-8m+\frac{11}{2}m^2\right)a_2=\frac{3}{2}m^2+2h_2=\frac{3}{2}m^2+\frac{3}{2}\frac{m^2}{1-m}=\frac{3}{2}m^2\frac{2-m}{1-m},$$

so that

$$a_2=\frac{3}{2}m^2\frac{2-m}{1-m}\,\frac{1}{3-8m+\frac{11}{2}m^2},$$

and

$$b_2=-\frac{1}{1-m}a_2-\frac{1}{2-2m}h_2$$
$$=-\frac{1}{1-m}a_2-\frac{3}{8}\frac{m^2}{(1-m)^2}.$$

These are numerically equal to the quantities denoted by the same symbols in Lecture IV, but b_2 bears the contrary sign. We further find

$$\left(15 - 32m + \frac{31}{2} m^2\right) a_4 = \frac{3}{4} \frac{m^2}{1-m} \{(1 + 3m - 2m^2)\, a_2 + (8 - 15m + 6m^2)\, b_2\},$$

$$b_4 = -\frac{1}{2-2m} a_4 - \frac{3}{8} a_2 b_2 - \frac{3}{64} \frac{m^2}{(1-m)^2} \{a_2 + (6 - 8m)\, b_2\};$$

or reduced to numbers

$$a_4 = -\cdot 00002{,}210,$$

$$b_4 = \cdot 00005{,}414 = 11''\cdot 17.$$

Finally let us exhibit the relation between the constants employed in this investigation and those of Lectures IV, V; to distinguish them, attach accents to the latter, so that

$$\theta = nt + \epsilon + b_2' \sin 2\psi + b_4' \sin 4\psi,$$

$$\frac{a'}{r} = 1 + a_2' \cos 2\psi + a_4' \cos 4\psi,$$

and, omitting the constant β as before,

$$(1-m)\,\theta = \psi + (1-m)\, b_2' \sin 2\psi + (1-m)\, b_4' \sin 4\psi.$$

Then

$$2\psi = (2-2m)\,\theta - (2-2m)\, b_2' \sin (2-2m)\,\theta,$$

$$\sin 2\psi = \sin (2-2m)\,\theta - (1-m)\, b_2' \sin (4-4m)\,\theta,$$

$$\cos 2\psi = (1-m)\, b_2' + \cos (2-2m)\,\theta - (1-m)\, b_2' \cos (4-4m)\,\theta.$$

Substitute in the equation for θ; we find

$$nt + \epsilon = \theta - b_2' \sin (2-2m)\,\theta - [b_4' - (1-m)\, b_2'^2] \sin (4-4m)\,\theta,$$

and similarly

$$a'u = 1 + (1-m)\, a_2' b_2' + a_2' \cos (2-2m)\,\theta + [a_4' - (1-m)\, a_2' b_2'] \cos (4-4m)\,\theta.$$

We observe that a' differs from a by quantities of the fourth order.

LECTURE VII.

CORRECTION OF APPROXIMATE SOLUTIONS.

We may simplify the equations we have been dealing with, by a proper choice of units. Let the unit of distance be the radius of the circular orbit which the Moon, if undisturbed, would describe about the Earth in its actual periodic time; then

$$\mu = n^2.$$

Also choose the unit of time so that

$$n - n' = 1,$$

so that, if we take as the result of observation of the mean motions of the Sun and Moon,

$$n' : n = \cdot 07480{,}13,$$

we get

$$n' = \cdot 08084{,}9 = m_1,$$

where m_1 is the quantity so called in Lecture IV; and

$$\mu = 1\cdot 16823{,}4.$$

We shall frequently adopt these simplifications in what follows.

Now let

$$l = \log_e (r/a),$$

so that

$$\frac{1}{r}\frac{dr}{dt} = \frac{dl}{dt}, \quad \frac{1}{r}\frac{d^2r}{dt^2} = \frac{d^2l}{dt^2} + \left(\frac{dl}{dt}\right)^2, \quad \frac{\mu}{r^3} = \frac{\mu}{a^3} e^{-3l};$$

and the equations discussed in Lecture IV become

$$\frac{d^2l}{dt^2} + \left(\frac{dl}{dt}\right)^2 - \left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{a^3} e^{-3l} - n'^2 \left[\frac{1}{2} + \frac{3}{2}\cos 2\omega\right] = 0,$$

$$\frac{d^2\theta}{dt^2} + 2\frac{dl}{dt}\frac{d\theta}{dt} + n'^2 \left[\frac{3}{2}\sin 2\omega\right] = 0,$$

where

$$\omega = \theta - \theta'.$$

Now these equations are defective, for they have been formed by omitting certain terms from the complete equations as given in Lecture II. Hence, calling l_0, θ_0 the values of l, θ, which we have proved in Lecture IV to be solutions of the above equations, if we substitute l_0, θ_0 in the complete equations of Lecture II, residuals are left, say X and Y respectively. And if l, θ be solutions of the complete equations, and if we write

$$l = l_0 + \delta l,$$

$$\theta = \theta_0 + \delta\theta,$$

where δl, $\delta\theta$ are small quantities whose squares and products may be neglected in the first instance, we obtain the following equations for determining δl, $\delta\theta$, the corrections to approximate solutions l_0, θ_0 already found:—

$$X + \frac{d^2\delta l}{dt^2} + 2\frac{dl_0}{dt}\frac{d\delta l}{dt} - 2\frac{d\theta_0}{dt}\frac{d\delta\theta}{dt} - 3\frac{\mu}{a^3}e^{-3l_0}\delta l + 3n'^2 \sin 2\omega\delta\theta = 0,$$

$$Y + \frac{d^2\delta\theta}{dt^2} + 2\frac{dl_0}{dt}\frac{d\delta\theta}{dt} + 2\frac{d\theta_0}{dt}\frac{d\delta l}{dt} \qquad + 3n'^2 \cos 2\omega\delta\theta = 0.$$

Now let us write

$$\frac{d\theta_0}{dt} = 1 + n' + v, \qquad \frac{\mu}{r_0^3} = \frac{\mu}{a^3}e^{-3l_0} = c + w,$$

where

$$c = (1 + n')^2 + \frac{1}{2}n'^2.$$

The quantity v consists wholly of periodic terms of the form $\cos 2i\psi$ multiplied by small coefficients; w contains, besides periodic terms, a small constant term, which however might be removed if we were to choose c as the constant part of μ/r_0^3 in place of according to the definition above.

Let $\delta' l$, $\delta'\theta$ be quantities defined by the equations

$$X + \frac{d^2\delta' l}{dt^2} - 2(1 + n')\frac{d\delta'\theta}{dt} - 3c\delta' l = 0,$$

$$Y + \frac{d^2\delta'\theta}{dt^2} + 2(1 + n')\frac{d\delta' l}{dt} \qquad = 0;$$

then $\delta' l$, $\delta'\theta$ are approximations to the complete corrections δl, $\delta\theta$, which if substituted in the equations that give those corrections will leave residuals, say X' and Y', where

$$X' = 2\frac{dl_0}{dt}\frac{d\delta' l}{dt} - 2v\frac{d\delta'\theta}{dt} - 3w\delta' l + 3n'^2 \sin 2\omega\delta'\theta,$$

$$Y' = 2\frac{dl_0}{dt}\frac{d\delta'\theta}{dt} + 2v\frac{d\delta' l}{dt} \qquad\qquad + 3n'^2 \cos 2\omega\delta'\theta.$$

We see that their value is known when $\delta' l$, $\delta'\theta$ are determined.

Now $\delta' l$, $\delta'\theta$ may be determined as follows.

Let $$X = p_0 + \Sigma p_i \cos i\psi, \quad Y = \Sigma q_i \sin i\psi,$$
where i takes all positive integral values; and assume
$$\delta' l = a_0 + \Sigma a_i \cos i\psi, \quad \delta'\theta = \Sigma b_i \sin i\psi.$$

Then substituting and equating coefficients, the constant term gives
$$p_0 - 3ca_0 = 0,$$
and the terms in $i\psi$ give
$$\begin{aligned} p_i - i^2 a_i - 2(1+n')\, ib_i - 3ca_i &= 0, \\ q_i - i^2 b_i - 2(1+n')\, ia_i &= 0; \end{aligned}$$
the second of these may be written
$$2(1+n')\frac{q_i}{i} - 4(1+n')^2 a_i - 2(1+n')\, ib_i = 0;$$
subtract from the first and we have
$$p_i - 2(1+n')\frac{q_i}{i} - a_i[i^2 - 4(1+n')^2 + 3c] = 0,$$
or
$$a_i = \frac{p_i - 2(1+n')\frac{q_i}{i}}{i^2 - (1+n')^2 + \frac{3}{2}n'^2};$$
and
$$b_i = -2(1+n')\frac{a_i}{i} + \frac{q_i}{i^2}.$$

We see that a_i, b_i will be of the same order of small quantities as p_i, q_i, in general. And therefore the coefficients of the terms of X', Y' will be of order higher than those of X, Y. Proceed then to determine further corrections $\delta'' l$, $\delta''\theta$ satisfying the equations
$$X' + \frac{d^2\delta'' l}{dt^2} - 2(1+n')\frac{d\delta''\theta}{dt} - 3c\delta'' l = 0,$$
$$Y' + \frac{d^2\delta''\theta}{dt^2} + 2(1+n')\frac{d\delta'' l}{dt} \qquad = 0;$$

then if $\delta' l + \delta'' l$, $\delta'\theta + \delta''\theta$ are substituted in the complete equations for δl, $\delta\theta$ the residuals become

$$X'' = 2\frac{dl_0}{dt}\frac{d\delta'' l}{dt} - 2v\frac{d\delta''\theta}{dt} - 3w\delta'' l + 3n'^2 \sin 2\omega\delta''\theta,$$

$$Y'' = 2\frac{dl_0}{dt}\frac{d\delta''\theta}{dt} + 2v\frac{d\delta'' l}{dt} \qquad\qquad + 3n'^2 \cos 2\omega\delta''\theta,$$

expressions which, if developed in series of cosines and sines of multiples of ψ, will have coefficients of higher order than the corresponding coefficients in X', Y'. The like process may be repeated until the residuals become insensible; we then have sensibly correct values of δl, $\delta\theta$, giving

$$l = l_0 + \delta l = l_0 + \delta' l + \delta'' l + \ldots\ldots,$$

$$\theta = \theta_0 + \delta\theta = \theta_0 + \delta'\theta + \delta''\theta + \ldots\ldots.$$

We may now take into account squares and products of the small quantities δl, $\delta\theta$ by treating $l_0 + \delta l$, $\theta_0 + \delta\theta$ as given approximate solutions just as we have here treated l_0, θ_0; substitute them in the complete equations of motion, and determine the residuals X, Y which they leave. These residuals will form the basis of a second approximation, and the operation may be repeated until no further correction is necessary. It is to be observed that if δl, $\delta\theta$ depend upon some such constant as the eccentricity of the Earth's orbit around the Sun, or the parallax of the Sun, then successive approximations yield correctly and separately the terms which depend upon the first, second, ... powers of that constant.

LECTURE VIII.

THE PARALLACTIC INEQUALITY.

We shall now apply the method of the last lecture to find the terms in the Moon's coordinates which depend upon the parallax of the Sun.

The values of l, θ found in Lecture IV are

$$l_0 = \log_e (r/a) = -a_2 \cos 2\psi,$$

$$\theta_0 = nt + \epsilon + b_2 \sin 2\psi,$$

and these satisfy the equations of motion in which the terms involving the Sun's parallax are omitted. Hence the residuals they leave from the complete equations are

$$X = -\lambda n'^2 \frac{r}{a} \left\{\frac{9}{8} \cos (\theta_0 - \theta') + \frac{15}{8} \cos 3 (\theta_0 - \theta')\right\},$$

$$Y = \lambda n'^2 \frac{r}{a} \left\{\frac{3}{8} \sin (\theta_0 - \theta') + \frac{15}{8} \sin 3 (\theta_0 - \theta')\right\},$$

where

$$\lambda = \frac{E - M}{E + M} \frac{a}{a'}.$$

Now from above

$$\theta_0 - \theta' = \psi + b_2 \sin 2\psi;$$

hence we have

$$\sin (\theta_0 - \theta') = \sin \psi + \frac{1}{2} b_2 (\sin \psi + \sin 3\psi),$$

$$\cos (\theta_0 - \theta') = \cos \psi - \frac{1}{2} b_2 (\cos \psi - \cos 3\psi),$$

$$\sin 3 (\theta_0 - \theta') = \sin 3\psi + \frac{3}{2} b_2 (\sin \psi + \sin 5\psi),$$

$$\cos 3 (\theta_0 - \theta') = \cos 3\psi - \frac{3}{2} b_2 (\cos \psi - \cos 5\psi);$$

and

$$\frac{r}{a}\left\{\frac{9}{8}\cos(\theta_0-\theta')+\frac{15}{8}\cos 3(\theta_0-\theta')\right\}=\left(\frac{9}{8}-\frac{27}{8}b_2-\frac{3}{2}a_2\right)\cos\psi$$
$$+\left(\frac{15}{8}+\frac{9}{16}b_2-\frac{9}{16}a_2\right)\cos 3\psi+\left(\frac{45}{16}b_2-\frac{15}{16}a_2\right)\cos 5\psi,$$

$$\frac{r}{a}\left\{\frac{3}{8}\sin(\theta_0-\theta')+\frac{15}{8}\sin 3(\theta_0-\theta')\right\}=\left(\frac{3}{8}-\frac{21}{8}b_2-\frac{3}{4}a_2\right)\sin\psi$$
$$+\left(\frac{15}{8}+\frac{3}{16}b_2-\frac{3}{16}a_2\right)\sin 3\psi+\left(\frac{45}{16}b_2+\frac{15}{16}a_2\right)\sin 5\psi.$$

Assume

$$-\delta l=\lambda a_1\cos\psi+\lambda a_3\cos 3\psi,$$
$$\delta\theta=\lambda b_1\sin\psi+\lambda b_3\sin 3\psi,$$

neglecting for the present the terms in 5ψ. In the present case it happens that it is more advantageous to substitute these expressions directly in the complete equations for δl, $\delta\theta$ given in the last lecture than to follow exactly the process for finding them by successive approximation. Omitting the factor λ, we get

$$a_1\cos\psi+9a_3\cos 3\psi+4a_2\sin 2\psi\,[a_1\sin\psi+3a_3\sin 3\psi]$$
$$+3\frac{\mu}{a^3}[a_1\cos\psi+a_3\cos 3\psi]+3\frac{\mu}{a^3}[3a_2\cos 2\psi][a_1\cos\psi+a_3\cos 3\psi]$$
$$-[2(1+n')+4b_2\cos 2\psi][b_1\cos\psi+3b_3\cos 3\psi]$$
$$+3n'^2[\sin 2\psi+b_2\sin 4\psi][b_1\sin\psi+b_3\sin 3\psi]$$
$$-n'^2\left(\frac{9}{8}-\frac{27}{8}b_2-\frac{3}{2}a_2\right)\cos\psi-n'^2\left(\frac{15}{8}+\frac{9}{16}b_2-\frac{9}{16}a_2\right)\cos 3\psi$$
$$-n'^2\left(\frac{45}{16}b_2-\frac{15}{16}a_2\right)\cos 5\psi=0,$$

$$-b_1\sin\psi-9b_3\sin 3\psi+4a_2\sin 2\psi\,[b_1\cos\psi+3b_3\cos 3\psi]$$
$$+3n'^2[-b_2+\cos 2\psi+b_2\cos 4\psi][b_1\sin\psi+b_3\sin 3\psi]$$
$$+[2(1+n')+4b_2\cos 2\psi][a_1\sin\psi+3a_3\sin 3\psi]$$
$$+n'^2\left(\frac{3}{8}-\frac{21}{8}b_2-\frac{3}{4}a_2\right)\sin\psi+n'^2\left(\frac{15}{8}+\frac{3}{16}b_2-\frac{3}{16}a_2\right)\sin 3\psi$$
$$+n'^2\left(\frac{45}{16}b_2-\frac{15}{16}a_2\right)\sin 5\psi=0.$$

If we equate to zero the coefficients of $\cos\psi$ and $\cos 3\psi$ in the first, and those of $\sin\psi$ and $\sin 3\psi$ in the second, we obtain the following equations for a_1, b_1, a_3, b_3; the terms in 5ψ remain outstanding, and the effect of a_5, b_5 in modifying the other coefficient is neglected.

$$a_1\left[1+2a_2+3\frac{\mu}{a^3}\left(1+\frac{3}{2}a_2\right)\right]-b_1\left[2(1+n')+2b_2-\frac{3}{2}n'^2\right]$$
$$+a_3\left[6a_2+\frac{9}{2}\frac{\mu}{a^3}a_2\right]-b_3\left[6b_2-\frac{3}{2}n'^2(1+b_2)\right]=\frac{3}{8}n'^2(3-9b_2-4a_2),$$

$$a_1[2(1+n')-2b_2]-b_1\left[1+\frac{3}{2}n'^2-2a_2+3n'^2b_2\right]$$
$$+a_3[6b_2]\qquad -b_3\left[6a_2-\frac{3}{2}n'^2+\frac{3}{2}n'^2b_2\right]=-\frac{3}{8}n'^2(1-7b_2-2a_2),$$

$$a_1\left[-2a_2+\frac{9}{2}\frac{\mu}{a^3}a_2\right]+b_1\left[-2b_2-\frac{3}{2}n'^2+\frac{3}{2}n'^2b_2\right]$$
$$+a_3\left[9+3\frac{\mu}{a^3}\right]\qquad -b_3[6(1+n')]\qquad =\frac{3}{8}n'^2\left(5+\frac{3}{2}b_2-\frac{3}{2}a_2\right),$$

$$a_1[2b_2]+b_1\left[2a_2+\frac{3}{2}n'^2-\frac{3}{2}n'^2b_2\right]$$
$$+a_3[6(1+n')]\qquad -b_3[9+3n'^2b_2]\qquad =-\frac{3}{8}n'^2\left(5+\frac{1}{2}b_2-\frac{1}{2}a_2\right).$$

If we require the formal values of a_1, b_1, a_3, b_3, we must substitute for a_2, b_2, μ/a^3 the expressions we have found for them, and it will then be best to develope the coefficients in ascending powers of n'. But it is difficult to obtain by this process such good numerical results as we can get by substituting the numerical values of a_2, b_2, μ/a^3 immediately in the equations above. If we do so we get the equations

$$4{\cdot}56672a_1-2{\cdot}17232b_1+\ {\cdot}08093a_3-\ {\cdot}05137b_3=\ \frac{3}{8}n'^2\times 2{\cdot}87937,$$

$$2{\cdot}14128a_1-0{\cdot}99564b_1+\ {\cdot}06127a_3-\ {\cdot}03338b_3=-\frac{3}{8}n'^2\times 0{\cdot}91416,$$

$${\cdot}02349a_1-\ {\cdot}03012b_1+12{\cdot}51451a_3-6{\cdot}48508b_3=\ \frac{3}{8}n'^2\times 5{\cdot}00455,$$

$${\cdot}02042a_1+\ {\cdot}02406b_1+\ 6{\cdot}48508a_3-9{\cdot}00020b_3=-\frac{3}{8}n'^2\times 5{\cdot}00152.$$

We notice that the first equation is not very different from the second doubled; it is this fact that makes successive approximation a disadvantageous method and renders it advisable to include small quantities from the beginning.

Eliminate a_3, b_3 in succession from the third and fourth equations, thus:—

Multiply the third equation by

$$9{\cdot}00020 \div [12{\cdot}51451 \times 9{\cdot}00020 - 6{\cdot}48508 \times 6{\cdot}48508] = \quad 0{\cdot}127523,$$

and the fourth by

$$-6{\cdot}48508 \div [12{\cdot}51451 \times 9{\cdot}00020 - 6{\cdot}48508 \times 6{\cdot}48508] = -0{\cdot}091887,$$

and add; b_3 will be eliminated.

Again multiply the third equation by ·091887 and the fourth by −·177317 and add; a_3 will be eliminated. Hence we find

$$\cdot 001119 a_1 - \cdot 006052 b_1 + a_3 = \frac{3}{8} n'^2 \times 1{\cdot}09776,$$

$$-\cdot 001463 a_1 - \cdot 007034 b_1 + b_3 = \frac{3}{8} n'^2 \times 1{\cdot}34669.$$

Multiply these by −·08093 and ·05137 respectively and add to the first equation:—

$$\begin{aligned} 4{\cdot}56672 a_1 - 2{\cdot}17232 b_1 + \cdot 08093 a_3 - \cdot 05137 b_3 &= \frac{3}{8} n'^2 \times \quad 2{\cdot}87937, \\ -0{\cdot}00009 a_1 + 0{\cdot}00049 b_1 - \cdot 08093 a_3 \qquad\qquad &= \frac{3}{8} n'^2 \times -0{\cdot}08884, \\ -0{\cdot}00008 a_1 - 0{\cdot}00036 b_1 \qquad\qquad + \cdot 05137 b_3 &= \frac{3}{8} n'^2 \times \quad 0{\cdot}06918; \end{aligned}$$

hence

$$4{\cdot}56665 a_1 - 2{\cdot}17219 b_1 = \frac{3}{8} n'^2 \times 2{\cdot}85971.$$

Eliminate a_3, b_3 in a similar manner from the second equation;

$$\begin{aligned} 2{\cdot}14128 a_1 - 0{\cdot}99546 b_1 + \cdot 06127 a_3 - \cdot 03338 b_3 &= -\frac{3}{8} n'^2 \times \quad 0{\cdot}91416, \\ -0{\cdot}00007 a_1 + 0{\cdot}00037 b_1 - \cdot 06127 a_3 \qquad\qquad &= -\frac{3}{8} n'^2 \times \quad 0{\cdot}06726, \\ -0{\cdot}00005 a_1 - 0{\cdot}00023 b_1 \qquad\qquad + \cdot 03338 b_3 &= -\frac{3}{8} n'^2 \times -0{\cdot}04495; \end{aligned}$$

hence

$$2{\cdot}14116a_1 - 0{\cdot}99550b_1 = -\frac{3}{8}n'^2 \times 0{\cdot}93647.$$

From these equations we find

$$a_1 = -\frac{3}{8}n'^2 \times 46{\cdot}4814 = -{\cdot}11392{,}8,$$

$$b_1 = -\frac{3}{8}n'^2 \times 99{\cdot}0336 = -{\cdot}24273{,}4,$$

$$a_3 = \frac{3}{8}n'^2 \times {\cdot}55042 = {\cdot}00134{,}9,$$

$$b_3 = \frac{3}{8}n'^2 \times {\cdot}58209 = {\cdot}00142{,}7.$$

LECTURE IX.

THE PARALLACTIC INEQUALITY, (*continued*).

LET us now consider the terms in 5ψ which have been left outstanding.

Include additional terms $\lambda a_5 \cos 5\psi$, $\lambda b_5 \sin 5\psi$ in $-\delta l$, $\delta\theta$, and equate to zero the coefficients of $\cos 5\psi$, $\sin 5\psi$ in the differential equations that give δl, $\delta\theta$.

We have

$$25a_5 + \frac{3\mu}{a^3}a_5 - 10(1+n')b_5 - 6a_2a_3 - 6b_2b_3 + \frac{9}{2}\frac{\mu}{a^3}a_2a_3 - \frac{3}{2}n'^2b_3 - n'^2\left(\frac{45}{16}b_2 - \frac{15}{16}a_2\right) = 0,$$

$$-25b_5 \qquad + 10(1+n')a_5 + 6a_2b_3 + 6a_3b_2 \qquad + \frac{3}{2}n'^2b_3 + n'^2\left(\frac{45}{16}b_2 - \frac{15}{16}a_2\right) = 0.$$

In these equations substitute

$$\frac{\mu}{a^3} = 1 + 2n' + \frac{3}{2}n'^2.$$

Then

$$\left(28 + 6n' + \frac{9}{2}n'^2\right)a_5 - 10(1+n')b_5 = \left(\frac{3}{2} - 9n' - \frac{27}{4}n'^2\right)a_2a_3 + \left(6b_2 + \frac{3}{2}n'^2\right)b_3$$
$$+ n'^2\left(\frac{45}{16}b_2 - \frac{15}{16}a_2\right),$$

$$-25b_5 + 10(1+n')a_5 = -6a_2b_3 - 6b_2a_3 - \frac{3}{2}n'^2b_3 - n'^2\left(\frac{45}{16}b_2 - \frac{15}{16}a_2\right).$$

Eliminate b_5:

$$\left(24 - 2n' + \frac{1}{2}n'^2\right)a_5 = \left[\left(\frac{3}{2} - 9n' - \frac{27}{4}n'^2\right)a_2 + \frac{12}{5}(1+n')b_2\right]a_3$$
$$+ \left[6b_2 + \frac{3}{2}n'^2 + \frac{12}{5}(1+n')a_2 + \frac{3}{5}(1+n')n'^2\right]b_3$$
$$+ (7+2n')n'^2\left(\frac{9}{16}b_2 - \frac{3}{16}a_2\right),$$

and then b_5 is given by

$$b_5 = \frac{2}{5}(1+n')\,a_5 + \frac{6}{25}b_2a_3 + \left(\frac{6}{25}a_2 + \frac{3}{50}n'^2\right)b_3 + \frac{1}{5}n'^2\left(\frac{9}{16}b_2 - \frac{3}{16}a_2\right).$$

From these we find

$$a_5 = \frac{3}{8}n'^2 \times {\cdot}00595{,}3 = {\cdot}00001{,}4591,$$

$$b_5 = \frac{3}{8}n'^2 \times {\cdot}00710{,}3 = {\cdot}00001{,}7410.$$

These numbers being so small, we see that we may safely ignore, as we have done, their effect in modifying the earlier coefficients.

To find the effect of these coefficients upon the Moon's coordinates we must multiply by the factor $\lambda = \frac{E-M}{E+M}\cdot\frac{a}{a'}$.

We shall take in accordance with the results given in *Monthly Notices*, Vol. 13, p. 177, and Appendix to the Nautical Almanac, 1856,

$$\frac{E}{M} = 81{\cdot}5.$$

Constant of Moon's Parallax $= 3422''{\cdot}325$.

Also we shall take in the first place, the Sun's Mean Parallax to be $8''{\cdot}8$, and in the next place $8''{\cdot}9$, and we will find the corresponding values of the coefficients of the Parallactic Inequalities.

We find

$8''{\cdot}8$			$8''{\cdot}9$		
$\lambda =$	$\cdot00250{,}9$		$\lambda =$	$\cdot00253{,}76$	
$\lambda a_1 =$	$-\cdot00028{,}585$		$\lambda a_1 =$	$-\cdot00028{,}910$	
$\lambda b_1 =$	$-\cdot00060{,}903$	$= -125''{\cdot}62$	$\lambda b_1 =$	$-\cdot00061{,}596$	$= -127''{\cdot}05$
$\lambda a_3 =$	$\cdot00000{,}3385$		$\lambda a_3 =$	$\cdot00000{,}3423$	
$\lambda b_3 =$	$\cdot00000{,}3580$	$= 0''{\cdot}7384$	$\lambda b_3 =$	$\cdot00000{,}3620$	$= 0''{\cdot}7468$
$\lambda a_5 =$	$\cdot00000{,}00366$		$\lambda a_5 =$	$\cdot00000{,}00370$	
$\lambda b_5 =$	$\cdot00000{,}00437$	$= 0''{\cdot}00901$	$\lambda b_5 =$	$\cdot00000{,}00442$	$= 0''{\cdot}00911.$

These results are very fairly accurate; but in order to get good values for a_1, b_1, we were obliged to discuss a_1, b_1, a_3, b_3 simultaneously. Let us consider the peculiarity of the equations from which this difficulty arose.

Following the method of approximation of Lecture VII, if we neglect at first the products of δl, $\delta\theta$, $d\delta l/dt$, $d\delta\theta/dt$ with the small quantities a_2, b_2, n'^2, the equations become

$$\frac{d^2\delta l}{dt^2} - 2n\frac{d\delta\theta}{dt} - 3\frac{\mu}{a^3}\delta l + X = 0,$$

$$\frac{d^2\delta\theta}{dt^2} + 2n\frac{d\delta l}{dt} \qquad\qquad + Y = 0.$$

Now suppose the following is a set of terms that appear

in X $p_i \cos(it+\gamma)$, in Y $q_i \sin(it+\gamma)$,

δl $a_i \cos(it+\gamma)$, $\delta\theta$ $b_i \sin(it+\gamma)$;

then as in Lecture VII, we find

$$a_i = \frac{p_i - 2\frac{n}{i}q_i}{i^2 - n^2 + \frac{3}{2}n'^2},$$

$$b_i = -2\frac{n}{i}a_i + \frac{1}{i^2}q.$$

Therefore if i differs little from n, the divisor in a_i will be small, and a small error or omission in the numerator of a_i will appear magnified in the values of both a_i and b_i. In the case of the first term of the Parallactic Inequality,

$$i = n - n',$$

$$i^2 - n^2 + \frac{3}{2}n'^2 = -2nn' + \frac{5}{2}n'^2;$$

and if we take

$$p_i = -\frac{9}{8}n'^2, \quad q_i = \frac{3}{8}n'^2,$$

which differ from the correct values by quantities of the fourth order, then

$$p_i - 2\frac{n}{i}q_i = -\frac{3}{8}n'^2\frac{5n-3n'}{n-n'},$$

and the formulae give

$$a_i = \frac{3}{4}\frac{n'(5n-3n')}{(n-n')(4n-5n')},$$

$$b_i = -\frac{2n}{n-n'}a_i + \frac{3}{8}\frac{n'^2}{(n-n')^2}.$$

Now if we develope these expressions in ascending powers of m, i.e. n'/n, the first terms are

$$a_i = \frac{15}{16}m, \quad b_i = -\frac{15}{8}m,$$

and these are the only terms which the formulae derived from our method of approximation will give correctly.

LECTURE X.

THE ANNUAL EQUATION.

Let us next take into account the effect of the first power of the eccentricity of the Earth's orbit. We shall find that it produces an inequality in the Moon's coordinates, the chief part of which has a period of one year, and is therefore called the Annual Equation.

In the formulae of Lecture VII, let the known approximate solutions l_0, θ_0, include the Variation only; then the equations for the corrections δl, $\delta\theta$ are

$$X+\frac{d^2\delta l}{dt^2}+4a_2\sin 2\psi\frac{d\delta l}{dt}-3\frac{\mu}{a^3}(1+3a_2\cos 2\psi)\,\delta l$$
$$-2\left[(1+n')+2b_2\cos 2\psi\right]\frac{d\delta\theta}{dt}+3n'^2(\sin 2\psi+b_2\sin 4\psi)\,\delta\theta=0,$$

$$Y+\frac{d^2\delta\theta}{dt^2}+4a_2\sin 2\psi\frac{d\delta\theta}{dt}+3n'^2(-b_2+\cos 2\psi+b_2\cos 4\psi)\,\delta\theta$$
$$+2\left[(1+n')+2b_2\cos 2\psi\right]\frac{d\delta l}{dt}\qquad=0,$$

where a_2, b_2, μ/a^3 are known quantities whose values are given in Lectures IV, V.

Refer now to Lecture III, and we find that the terms that are left outstanding when the terms of the Variation are substituted, and the parallactic terms omitted are the following:—

$$X=-\frac{3}{2}n'^2e'\cos(n't-\varpi')-\frac{21}{4}n'^2e'\cos\{2(\theta-n't)-(n't-\varpi')\}$$
$$+\frac{3}{4}n'^2e'\cos\{2(\theta-n't)+(n't-\varpi')\}$$

$$Y=\qquad+\frac{21}{4}n'^2e'\sin\{2(\theta-n't)-(n't-\varpi')\}$$
$$-\frac{3}{4}n'^2e'\sin\{2(\theta-n't)+(n't-\varpi')\}.$$

Write α for $n't-\varpi'$; then

$$\begin{aligned}
\cos\{2(\theta-n't)-\alpha\} &= \quad \cos(2\psi-\alpha)-(2b_2\sin 2\psi)\sin(2\psi-\alpha)\\
&= -b_2\cos\alpha+\cos(2\psi-\alpha)+b_2\cos(4\psi-\alpha),\\
\sin\{2(\theta-n't)-\alpha\} &= +b_2\sin\alpha+\sin(2\psi-\alpha)+b_2\sin(4\psi-\alpha),\\
\cos\{2(\theta-n't)+\alpha\} &= -b_2\cos\alpha+\cos(2\psi+\alpha)+b_2\cos(4\psi+\alpha),\\
\sin\{2(\theta-n't)+\alpha\} &= -b_2\sin\alpha+\sin(2\psi+\alpha)+b_2\sin(4\psi+\alpha).
\end{aligned}$$

Hence

$$\begin{aligned}
X = -\frac{3}{2}n'^2(1-3b_2)e'\cos\alpha &- \frac{21}{4}n'^2e'\cos(2\psi-\alpha)+\frac{3}{4}n'^2e'\cos(2\psi+\alpha)\\
&-\frac{21}{4}n'^2b_2e'\cos(4\psi-\alpha)+\frac{3}{4}n'^2b_2e'\cos(4\psi+\alpha),
\end{aligned}$$

$$\begin{aligned}
Y = 6n'^2b_2e'\sin\alpha &+ \frac{21}{4}n'^2e'\sin(2\psi-\alpha)-\frac{3}{4}n'^2e'\sin(2\psi+\alpha)\\
&+\frac{21}{4}n'^2b_2e'\sin(4\psi-\alpha)-\frac{3}{4}n'^2b_2e'\sin(4\psi+\alpha).
\end{aligned}$$

For our present purpose we shall ignore the small terms in $4\psi-\alpha$ and $4\psi+\alpha$ which are of the sixth order.

Assume

$$\begin{aligned}
-\delta l &= a_5e'\cos\alpha+a_6e'\cos(2\psi-\alpha)+a_7e'\cos(2\psi+\alpha),\\
\delta\theta &= b_5e'\sin\alpha+b_6e'\sin(2\psi-\alpha)+b_7e'\sin(2\psi+\alpha).
\end{aligned}$$

Now the terms which arise in the left-hand members of the equations owing to terms $-a_p\cos pt$ in δl, and $b_p\sin pt$ in $\delta\theta$, will be

$$\begin{aligned}
&p^2a_p\cos pt+2a_2pa_p[\cos(pt-2\psi)-\cos(pt+2\psi)]\\
&\quad+\frac{3\mu}{a^3}a_p\left[\cos pt+\frac{3}{2}a_2\cos(pt-2\psi)+\frac{3}{2}a_2\cos(pt+2\psi)\right]\\
&\quad-2(1+n')pb_p\cos pt-2b_2pb_p[\cos(pt-2\psi)+\cos(pt+2\psi)]\\
&\quad+\frac{3}{2}n'^2b_p[\cos(pt-2\psi)-\cos(pt+2\psi)],
\end{aligned}$$

and

$$\begin{aligned}
&-p^2b_p\sin pt+2a_2pb_p[-\sin(pt-2\psi)+\sin(pt+2\psi)]\\
&\quad+\frac{3}{2}n'^2b_p[\sin(pt-2\psi)-2b_2\sin pt+\sin(pt+2\psi)]\\
&\quad+2(1+n')pa_p\sin pt+2b_2pa_p[\sin(pt-2\psi)+\sin(pt+2\psi)],
\end{aligned}$$

respectively, neglecting the very small quantities in 4ψ.

Hence we get the equations following:—

Equate to zero the coefficients of $\cos\alpha$, $\sin\alpha$:

$$\left(n'^2+\frac{3\mu}{a^3}\right)a_5-2n'(1+n')\,b_5+2\,(2-n')\,a_2a_6+\frac{3\mu}{a^3}\cdot\frac{3}{2}\,a_2a_6$$

$$-2\,(2-n')\,b_2b_6+\frac{3}{2}n'^2b_6+2\,(2+n')\,a_2a_7+\frac{3\mu}{a^3}\cdot\frac{3}{2}\,a_2a_7$$

$$-2\,(2+n')\,b_2b_7+\frac{3}{2}n'^2b_7=\frac{3}{2}n'^2(1-3b_2)$$

$$-n'^2b_5-3n'^2b_2b_5+2\,(1+n')\,n'a_5+2\,(2-n')\,a_2b_6$$

$$-2\,(2-n')\,b_2a_6-\frac{3}{2}n'^2b_6-2\,(2+n')\,a_2b_7$$

$$+2\,(2+n')\,b_2a_7+\frac{3}{2}n'^2b_7=-6n'^2b_2.$$

Equate the coefficients of $\cos(2\psi-\alpha)$, $\sin(2\psi-\alpha)$:—

$$(2-n')^2a_6+3\,\frac{\mu}{a^3}\,a_6-2\,(1+n')\,(2-n')\,b_6+2n'a_2a_5+3\,\frac{\mu}{a^3}\cdot\frac{3}{2}\,a_2a_5$$

$$-2n'b_2b_5+\frac{3}{2}\,n'^2b_5=\ \frac{21}{4}\,n'^2,$$

$$-(2-n')^2\,b_6-3n'^2b_2b_6+2\,(1+n')\,(2-n')\,a_6+2n'a_2b_5-2n'b_2a_5$$

$$-\frac{3}{2}\,n'^2b_5=-\frac{21}{4}\,n'^2.$$

Equate the coefficients of $\cos(2\psi+\alpha)$, $\sin(2\psi+\alpha)$:—

$$(2+n')^2a_7+3\,\frac{\mu}{a^3}\,a_7-2\,(1+n')\,(2+n')\,b_7-2n'a_2a_5+3\,\frac{\mu}{a^3}\cdot\frac{3}{2}\,a_2a_5$$

$$-2n'b_2b_5-\frac{3}{2}\,n'^2b_5=-\frac{3}{4}\,n'^2,$$

$$-(2+n')^2\,b_7-3n'^2b_2b_7+2\,(1+n')\,(2+n')\,a_7+2n'a_2b_5+2n'b_2a_5$$

$$+\frac{3}{2}\,n'^2b_5=\ \frac{3}{4}\,n'^2.$$

In equations of this class, as a general rule we would determine a_5, b_5 approximately from the first pair, substitute them in the second pair and determine a_6, b_6 approximately, and similarly a_7, b_7 from the third pair, and repeat this approximation as often as might be necessary.

But if we refer to the second equation, we see that b_5 must be determined by means of a small divisor, and this puts any method of approximation at a disadvantage. In order to obtain readily satisfactory values for the new coefficients, we shall treat the six equations simultaneously, substituting first the numerical values of the known quantities.

We have found

$$a_2 = {\cdot}00717{,}95, \quad b_2 = {\cdot}01021{,}20, \quad \mu/a^3 = 1{\cdot}17150{,}3.$$

Hence

$$3{\cdot}52105a_5 - 0{\cdot}174763b_5 + {\cdot}065405a_6 - {\cdot}029393b_6 + {\cdot}067727a_7 - {\cdot}032695b_7 = \frac{3}{2}n'^2 \times 0{\cdot}969364,$$

$$0{\cdot}174763a_5 - 0{\cdot}006736b_5 - {\cdot}039197a_6 + {\cdot}017753b_6 + {\cdot}042499a_7 - {\cdot}020075b_7 = \frac{3}{2}n'^2 \times -0{\cdot}040848,$$

$${\cdot}039009a_5 + {\cdot}008153b_5 + 7{\cdot}19766a_6 - 4{\cdot}14858b_6 = \frac{3}{2}n'^2 \times 3{\cdot}50,$$

$$-{\cdot}001651a_5 - {\cdot}008643b_5 + 4{\cdot}14858a_6 - 3{\cdot}68335b_6 = \frac{3}{2}n'^2 \times -3{\cdot}50,$$

$${\cdot}036687a_5 - {\cdot}011455b_5 + 7{\cdot}84443a_7 - 4{\cdot}49811b_7 = \frac{3}{2}n'^2 \times -0{\cdot}50,$$

$${\cdot}001651a_5 + {\cdot}010965b_5 + 4{\cdot}49811a_7 - 4{\cdot}33012b_7 = \frac{3}{2}n'^2 \times 0{\cdot}50.$$

From the second and third pairs we find

$${\cdot}016186a_5 + {\cdot}007084b_5 + a_6 = \frac{3}{2}n'^2 \times 2{\cdot}94739,$$

$${\cdot}018678a_5 + {\cdot}010326b_5 + b_6 = \frac{3}{2}n'^2 \times 4{\cdot}26994,$$

$${\cdot}011026a_5 - {\cdot}007203b_5 + a_7 = \frac{3}{2}n'^2 \times -0{\cdot}321398,$$

$${\cdot}011072a_5 - {\cdot}010015b_5 + b_7 = \frac{3}{2}n'^2 \times -0{\cdot}449338.$$

Eliminate a_6, b_6, a_7, b_7 from the first pair.

Hence

$$3{\cdot}52015a_5 - {\cdot}174762b_5 = \frac{3}{2}n'^2 \times 0{\cdot}909170,$$

$$0{\cdot}174819a_5 - {\cdot}006536b_5 = \frac{3}{2}n'^2 \times 0{\cdot}003516.$$

Hence

$$a_5 = \frac{3}{2}n'^2 \times -\ 0{\cdot}70619{,}8 = -{\cdot}00692{,}37,$$

$$b_5 = \frac{3}{2}n'^2 \times -19{\cdot}4268 \quad = -{\cdot}19046{,}3.$$

Now e' is a constant found by observation; taking $e' = 3459''{\cdot}28$, its value in 1850, we get

$$a_5e' = -{\cdot}00011{,}61,$$

$$b_5e' = -{\cdot}00319{,}4 \quad = -658''{\cdot}9,$$

and further

$$\begin{array}{llll} a_6 = \ \ {\cdot}03035{,}8, & a_6e' = \ \ {\cdot}00050{,}9, & & \\ b_6 = \ \ {\cdot}04396{,}7, & b_6e' = \ \ {\cdot}00073{,}73 = & 152''{\cdot}09, & \\ a_7 = -{\cdot}00444{,}7, & a_7e' = -{\cdot}00007{,}457, & & \\ b_7 = -{\cdot}00623{,}6, & b_7e' = -{\cdot}00010{,}46 = - & 21''{\cdot}57. & \end{array}$$

LECTURE XI.

THE EQUATION OF THE CENTRE AND THE EVECTION.

We have seen that the equations of motion

$$\frac{d^2l}{dt^2} + \left(\frac{dl}{dt}\right)^2 - \left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^3} - n'^2\left[\frac{1}{2} + \frac{3}{2}\cos 2(\theta - n't)\right] = 0,$$

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dl}{dt} \qquad\qquad + n'^2\left[\qquad \frac{3}{2}\sin 2(\theta - n't)\right] = 0,$$

are satisfied very approximately by the values

$$l = \log\frac{r}{a} = -a_2\cos 2\psi,$$

$$\theta \qquad = \qquad nt + \epsilon + b_2\sin 2\psi,$$

where
$$\psi = nt + \epsilon - (n't + \epsilon'),$$

and a_2, b_2 are small quantities depending upon the ratio n'/n, and a is a quantity depending upon n in such a way that

$$\frac{\mu}{a^3} = n^2 + \frac{1}{2}n'^2 - \frac{9}{32}\frac{n'^4}{(n-n')^2} + 2n'(2n-n')\,a_2^2,$$

while n, ϵ are arbitrary, though subject to the assumption that the ratio n'/n is small.

This solution, then, expresses a possible case of motion; nevertheless it is no more than a particular case because it involves only two arbitrary constants, whereas the complete and general solution must contain four, in order that it may be able to satisfy any given initial conditions, that is, in order that the initial coordinates and their initial velocities may have any given values.

When there is no disturbance the four arbitrary constants are n and ϵ,—which denote quantities similar to those expressed by the same symbols above,—and the two elliptic elements e and ϖ, of which e denotes the eccentricity of the orbit and ϖ the longitude of the apse.

We will now shew how to complete the solution by introducing into $\log(a/r)$ and θ additional terms depending on quantities similar to e, ϖ, of which the former is constant and the latter varies slowly and uniformly with t; and for the sake of simplicity we will suppose at first that e is so small that its square and higher powers may be neglected though it is otherwise arbitrary in magnitude.

Let us assume then

$$\log\frac{a}{r} = \qquad a_2 \cos 2\psi + e\cos(nt-\varpi),$$

$$\theta = nt + \epsilon + b_2 \sin 2\psi + 2e(1+b_0)\sin(nt-\varpi),$$

in which the elliptic terms are of the same form as in the undisturbed orbit, and ϖ is supposed to be slowly variable, so that

$$\frac{d\varpi}{dt} = p,$$

where p is supposed to be a small quantity of the order of the disturbing force.

We will now substitute these assumed values in the differential equations. We have

$$\frac{dl}{dt} = \qquad 2(n-n')\,a_2 \sin 2\psi + (n-p)\,e\sin(nt-\varpi),$$

$$\frac{d^2l}{dt^2} = \qquad 4(n-n')^2 a_2 \cos 2\psi + (n-p)^2 e\cos(nt-\varpi),$$

$$\frac{d\theta}{dt} = n + 2(n-n')\,b_2 \cos 2\psi + 2(n-p)(1+b_0)\,e\cos(nt-\varpi),$$

$$\frac{d^2\theta}{dt^2} = \quad -4(n-n')\,b_2 \sin 2\psi - 2(n-p)^2(1+b_0)\,e\sin(nt-\varpi).$$

Hence

$$\begin{aligned}
&4(n-n')^2 a_2 \cos 2\psi + (n-p)^2 e\cos(nt-\varpi)\\
&\quad + 2(n-n')(n-p)\,a_2 e\left[\cos(2\psi-nt+\varpi) - \cos(2\psi+nt-\varpi)\right]\\
&\quad - \{n^2 + 4n(n-n')\,b_2\cos 2\psi + 4n(n-p)(1+b_0)\,e\cos(nt-\varpi)\\
&\qquad\quad + 4(n-n')(n-p)(1+b_0)\,b_2 e\left[\cos(2\psi-nt+\varpi) + \cos(2\psi+nt-\varpi)\right]\}
\end{aligned}$$

$$+\frac{\mu}{a^3}\{1+3a_2\cos 2\psi\}\{1+3e\cos(nt-\varpi)\}$$

$$-n'^2\left\{\frac{1}{2}+\frac{3}{2}\cos 2\psi-3\sin 2\psi\left[2(1+b_0)e\sin(nt-\varpi)\right]\right\}=0,$$

and

$$\begin{aligned}-4\ &(n-n')^2 b_2\sin 2\psi-2(n-p)^2(1+b_0)e\sin(nt-\varpi)\\&+4n(n-n')a_2\sin 2\psi+2n(n-p)e\sin(nt-\varpi)\\&+\ 4(n-n')(n-p)(1+b_0)ea_2\left[\sin(2\psi-nt+\varpi)+\sin(2\psi+nt-\varpi)\right]\\&+\ 2(n-n')(n-p)eb_2\left[-\sin(2\psi-nt+\varpi)+\sin(2\psi+nt-\varpi)\right]\\&+n'^2\left\{\frac{3}{2}\sin 2\psi+3\cos 2\psi\left[2(1+b_0)e\sin(nt-\varpi)\right]\right\}=0.\end{aligned}$$

It will of course be found that with the values of a_2, b_2 of Lecture IV, the terms independent of e vanish identically.

Equating to zero the coefficients of $\cos(nt-\varpi)$ in the first equation and $\sin(nt-\varpi)$ in the second, we get

$$(n-p)^2-4n(n-p)(1+b_0)+\frac{3\mu}{a^3}=0,$$

$$-2(n-p)^2(1+b_0)+2n(n-p)\qquad=0.$$

Therefore $$(n-p)(1+b_0)=n,$$

and $$(n-p)^2=4n^2-\frac{3\mu}{a^3}$$

$$=n^2-\frac{3}{2}n'^2, \text{ approximately},$$

or $$\frac{p}{n}=\frac{3}{4}m^2=b_0, \text{ approximately}.$$

Now terms have been left outstanding with the arguments $2\psi-nt+\varpi$, $2\psi+nt-\varpi$. These may be removed by assuming

$$\log\frac{a}{r}=a_2\cos 2\psi+e\cos(nt-\varpi)+a_{21}e\cos(2\psi-nt+\varpi)+a_{22}e\cos(2\psi+nt-\varpi),$$

$$\begin{aligned}\theta=nt+\epsilon+b_2\sin 2\psi+2e(1+b_0)\sin(nt-\varpi)&+b_{21}e\sin(2\psi-nt+\varpi)\\&+b_{22}e\sin(2\psi+nt-\varpi).\end{aligned}$$

Hence in place of the former equations, we get the following

$$(n-p)^2 - 4n(n-p)(1+b_0) + \frac{3\mu}{a^3}$$
$$+\left[2(n-n')(n-2n'+p) + \frac{9}{2}\frac{\mu}{a^3}\right]a_2a_{21} + \left[2(n-n')(3n-2n'-p) + \frac{9}{2}\frac{\mu}{a^3}\right]a_2a_{22}$$
$$+\left[-2(n-n')(n-2n'+p)b_2 + \frac{3}{2}n'^2\right]b_{21}$$
$$+\left[-2(n-n')(3n-2n'-p)b_2 + \frac{3}{2}n'^2\right]b_{22} = 0$$

$$-2(n-p)^2(1+b_0) + 2n(n-p) - 6n'^2b_2(1+b_0)$$
$$-2(n-n')(n-2n'+p)b_2a_{21} + 2(n-n')(3n-2n'-p)b_2a_{22}$$
$$+\left[2(n-n')(n-2n'+p)a_2 - \frac{3}{2}n'^2\right]b_{21}$$
$$+\left[-2(n-n')(3n-2n'-p)a_2 + \frac{3}{2}n'^2\right]b_{22} = 0.$$

Multiply the second by $-\dfrac{2n}{n-p}$, and add to the first; this will eliminate $1+b_0$.

$$(n-p)^2 - 4n^2 + \frac{3\mu}{a^3} + \frac{12nn'^2}{n-p}b_2(1+b_0) + 2(n-n')(n-2n'+p)[a_2a_{21} - b_2b_{21}]$$
$$+\frac{9}{2}\frac{\mu}{a^3}[a_2a_{21} + a_2a_{22}] + \frac{3}{2}n'^2[b_{21} + b_{22}] + 2(n-n')(3n-2n'-p)[a_2a_{22} - b_2b_{22}]$$
$$+4\frac{n}{n-p}(n-n')(n-2n'+p)[b_2a_{21} - a_2b_{21}]$$
$$-4\frac{n}{n-p}(n-n')(3n-2n'-p)[b_2a_{22} - a_2b_{22}] + 3n'^2\frac{n}{n-p}[b_{21} - b_{22}] = 0.$$

Also the equations obtained by equating the coefficients of $e\cos(2\psi - nt + \varpi)$ and $e\sin(2\psi - nt + \varpi)$ to zero are

$$\left[2(n-n')(n-p) + \frac{9}{2}\frac{\mu}{a^3}\right]a_2 - 4(n-n')(n-p)(1+b_0)b_2 + 3n'^2(1+b_0)$$
$$+\left[(n-2n'+p)^2 + \frac{3\mu}{a^3}\right]a_{21} - 2n(n-2n'+p)b_{21} = 0,$$

$$4(n-n')(n-p)(1+b_0)a_2 - 2(n-n')(n-p)b_2 - 3n'^2(1+b_0) - (n-2n'+p)^2b_{21}$$
$$+2n(n-2n'+p)a_{21} = 0.$$

Multiply the second by $-\dfrac{2n}{n-2n'+p}$, and add to the first; this will eliminate b_{21}, and gives

$$\left[2(n-n')(n-p)+\frac{9}{2}\frac{\mu}{a^3}-\frac{8n}{n-2n'+p}(n-n')(n-p)(1+b_0)\right]a_2$$
$$+\left[-4(n-n')(n-p)(1+b_0)+4n\frac{n-n'}{n-2n'+p}(n-p)\right]b_2$$
$$+\left[3n'^2+\frac{6nn'^2}{n-2n'+p}\right](1+b_0)+\left[(n-2n'+p)^2-4n^2+\frac{3\mu}{a^3}\right]a_{21}=0.$$

Lastly the equations obtained by equating the coefficients of

$$e\cos(2\psi+nt-\varpi) \text{ and } e\sin(2\psi+nt-\varpi)$$

to zero are

$$\left[-2(n-n')(n-p)+\frac{9}{2}\frac{\mu}{a^3}\right]a_2-4(n-n')(n-p)(1+b_0)b_2-3n'^2(1+b_0)$$
$$+\left[(3n-2n'-p)^2+\frac{3\mu}{a^3}\right]a_{22}-2n(3n-2n'-p)b_{22}=0,$$

and

$$4(n-n')(n-p)(1+b_0)a_2+2(n-n')(n-p)b_2+3n'^2(1+b_0)$$
$$-(3n-2n'-p)^2b_{22}+2n(3n-2n'-p)a_{22}=0.$$

Multiply the second by $-\dfrac{2n}{3n-2n'-p}$ and add to the first; this will eliminate b_{22}, and gives

$$\left[-2(n-n')(n-p)+\frac{9}{2}\frac{\mu}{a^3}-\frac{8n}{3n-2n'-p}(n-n')(n-p)(1+b_0)\right]a_2$$
$$+\left[-4(n-n')(n-p)(1+b_0)-\frac{4n}{3n-2n'-p}(n-n')(n-p)\right]b_2$$
$$+\left[-3n'^2-\frac{6nn'^2}{3n-2n'-p}\right](1+b_0)+\left[(3n-2n'-p)^2-4n^2+\frac{3\mu}{a^3}\right]a_{22}=0.$$

These six equations are to be solved by successive approximation; taking the first rough values of p/n and b_0, we find from the last two pairs values for a_{21}, b_{21}, a_{22}, b_{22}; these are substituted in the first pair and yield more approximate values of p/n and b_0, and so on.

It will be noticed that this complexity is made necessary by the fact that a_{21}, b_{21} are found by means of a small divisor $(n-2n'+p)^2-4n^2+\dfrac{3\mu}{a^3}$.

LECTURE XII.

THE EVECTION AND THE MOTION OF THE APSE.

We proceed to the conversion into numbers of the formulae of Lecture XI.

Take
$$n - n' = 1,$$
$$n = 1{\cdot}08084{,}9,$$
$$n' = {\cdot}08084{,}9,$$
$$\frac{\mu}{a^3} = 1{\cdot}17150{,}3,$$
$$\log a_2 = 7{\cdot}85609,$$
$$\log b_2 = 8{\cdot}00911.$$

First approximation.

$$(n-p)^2 - 4n^2 + 3\frac{\mu}{a^3} = 0,$$

$$\begin{aligned}
4n^2 &\quad 4{\cdot}67293{,}7,\\
-3\frac{\mu}{a^3} &\quad -3{\cdot}51450{,}9,\\
(n-p)^2 &= 1{\cdot}15842{,}8,\\
n-p &= 1{\cdot}07630{,}3,\\
p &= {\cdot}00454{,}6,\\
n-2n'+p &= {\cdot}92369{,}7,\\
3n-2n'-p &= 3{\cdot}07630{,}3,\\
1+b_0 &= 1{\cdot}00422{,}4 = n/(n-p).
\end{aligned}$$

Substitute in the equation for a_{21},

$$\left[(n-2n'+p)^2-4n^2+\frac{3\mu}{a^3}\right]a_{21}+2(n-n')(n-p)\,a_2+\frac{9}{2}\frac{\mu}{a^3}a_2$$

$$-8\frac{n}{n-2n'+p}(n-n')(n-p)(1+b_0)\,a_2-4(n-n')(n-p)(1+b_0)\,b_2$$

$$+4\frac{n}{n-2n'+p}(n-n')(n-p)\,b_2+3n'^2(1+b_0)+6\frac{nn'^2}{n-2n'+p}(1+b_0)=0.$$

The various terms give

$2(n-n')(n-p)\,a_2$	·01545,45
$\frac{9}{2}\frac{\mu}{a^3}a_2$	·03784,83
$-8\frac{n}{n-2n'+p}(n-n')(n-p)(1+b_0)\,a_2$	− ·07264,10
$-4(n-n')(n-p)(1+b_0)\,b_2$	− ·04415,05
$4\frac{n}{n-2n'+p}(n-n')(n-p)\,b_2$	·05144,47
$3n'^2(1+b_0)$	·01969,25
$6\frac{nn'^2}{n-2n'+p}(1+b_0)$	·04608,58
	·05373,43
$(n-2n'+p)^2$	·85321,6
$-4n^2+3\frac{\mu}{a^3}$	− 1·15842,8
	− ·30521,2

$$a_{21}=\cdot 17605{,}6.$$

The equation for b_{21} is

$$b_{21}=\frac{2n}{n-2n'+p}a_{21}+4\frac{n-n'}{(n-2n'+p)^2}(n-p)(1+b_0)\,a_2-2\frac{n-n'}{(n-2n'+p)^2}(n-p)\,b_2$$

$$-3\frac{n'^2}{(n-2n'+p)^2}(1+b_0),$$

$\frac{2n}{n-2n'+p}a_{21}$	$\cdot 41201{,}8$
$4\frac{n-n'}{(n-2n'+p)^2}(n-p)(1+b_0)a_2$	$\cdot 03637{,}95$
$-2\frac{n-n'}{(n-2n'+p)^2}(n-p)b_2$	$-\cdot 02576{,}42$
$-3\frac{n'^2}{(n-2n'+p)^2}(1+b_0)$	$-\cdot 02308{,}03$
$b_{21}=$	$\cdot 39955{,}3$

The equation for a_{22} is

$$\left[(3n-2n'-p)^2-4n^2+3\frac{\mu}{a^3}\right]a_{22}-2(n-n')(n-p)a_2+\frac{9}{2}\frac{\mu}{a^3}a_2$$

$$-8\frac{n}{3n-2n'-p}(n-n')(n-p)(1+b_0)a_2-4(n-n')(n-p)(1+b_0)b_2$$

$$-4\frac{n}{3n-2n'-p}(n-n')(n-p)b_2-3n'^2(1+b_0)-6\frac{nn'^2}{3n-2n'-p}(1+b_0)=0.$$

Here

$-2(n-n')(n-p)a_2$	$-\cdot 01545{,}45$
$\frac{9}{2}\frac{\mu}{a^3}a_2$	$\cdot 03784{,}83$
$-8\frac{n}{3n-2n'-p}(n-n')(n-p)(1+b_0)a_2$	$-\cdot 02181{,}13$
$-4(n-n')(n-p)(1+b_0)b_2$	$-\cdot 04415{,}05$
$-4\frac{n}{3n-2n'-p}(n-n')(n-p)b_2$	$-\cdot 01544{,}69$
$-3n'^2(1+b_0)$	$-\cdot 01969{,}25$
$-6\frac{nn'^2}{3n-2n'-p}(1+b_0)$	$-\cdot 01383{,}78$
	$-\cdot 09254{,}52$

$(3n-2n'-p)^2$	$9\cdot 46363{,}4$
$-4n^2+3\frac{\mu}{a^3}$	$-1\cdot 15842{,}8$
	$8\cdot 30520{,}6$

$$a_{22}=\cdot 01114{,}30.$$

And

$$b_{22} = 2\frac{n}{3n-2n'-p}a_{22} + 4\frac{n-n'}{(3n-2n'-p)^2}(n-p)(1+b_0)a_2$$
$$+ 2\frac{n-n'}{(3n-2n'-p)^2}(n-p)b_2 + 3\frac{n'^2}{(3n-2n'-p)^2}(1+b_0),$$

$2\frac{n}{3n-2n'-p}a_{22}$	·00783,011
$4\frac{n-n'}{(3n-2n'-p)^2}(n-p)(1+b_0)a_2$	·00327,988
$2\frac{n-n'}{(3n-2n'-p)^2}(n-p)b_2$	·00232,283
$3\frac{n'^2}{(3n-2n'-p)^2}(1+b_0)$	·00208,086
$b_{22} =$	·01551,37

Second Approximation. The complete equation for $n-p$ is,

$$(n-p)^2 - 4n^2 + 3\frac{\mu}{a^3} + 12\frac{nn'^2}{n-p}b_2(1+b_0) + 2(n-n')(n-2n'+p)[a_2a_{21} - b_2b_{21}]$$
$$+ \frac{9}{2}\frac{\mu}{a^3}[a_3a_{21} + a_2a_{22}] + \frac{3}{2}n'^2[b_{21} + b_{22}] + 2(n-n')(3n-2n'-p)[a_2a_{22} - b_2b_{22}]$$
$$+ 4\frac{n}{n-p}(n-n')(n-2n'-p)[b_2a_{21} - a_2b_{21}]$$
$$- 4\frac{n}{n-p}(n-n')(3n-2n'-p)[b_2a_{22} - a_2b_{22}]$$
$$+ 3\frac{nn'^2}{n-p}[b_{21} - b_{22}] = 0.$$

$-4n^2 + 3\frac{\mu}{a^3}$	$-$ 1·15842,8
$12\frac{nn'^2}{n-p}b_2(1+b_0)$	·00080,8
$2(n-n')(n-2n'+p)[a_2a_{21} - b_2b_{21}]$	$-$ ·00520,1
$\frac{9}{2}\frac{\mu}{a^3}[a_2a_{21} + a_2a_{22}]$	·00708,4
$\frac{3}{2}n'^2[b_{21} + b_{22}]$	·00406,9
$2(n-n')(3n-2n'-p)[a_2a_{22} - b_2b_{22}]$	$-$ ·00048,2
$4\frac{n}{n-p}(n-n')(n-2n'+p)[b_2a_{21} - a_2b_{21}]$	$-$ ·00397,5

$$-4\frac{n}{n-p}(n-n')(3n-2n'-p)[b_2a_{22}-a_2b_{22}] \qquad -\ \cdot 00003{,}0$$

$$3\frac{nn'^2}{n-p}[b_{21}-b_{22}] \qquad \cdot 00756{,}1$$

$$(n-p)^2 = \quad 1\cdot 14859{,}4$$

$$n-p = 1\cdot 07172{,}5$$

$$p = \ \cdot 00912{,}4$$

$$p : n = \ \cdot 00844{,}2.$$

Apply these numbers in the equation for $1+b_0$:—

$$1+b_0 = \frac{n}{n-p} - 3\frac{n'^2}{(n-p)^2}b_2(1+b_0) - \frac{n-n'}{(n-p)^2}(n-2n'+p)[b_2a_{21}-a_2b_{21}]$$

$$+\frac{n-n'}{(n-p)^2}(3n-2n'-p)[b_2a_{22}-a_2b_{22}] - \frac{3}{4}\frac{n'^2}{(n-p)^2}[b_{21}-b_{22}],$$

$$\frac{n}{n-p} \qquad 1\cdot 00851{,}33$$

$$-3\frac{n'^2}{(n-p)^2}b_2(1+b_0) \qquad -\ \cdot 00017{,}51$$

$$-\frac{n-n'}{(n-p)^2}(n-2n'+p)[b_2a_{21}-a_2b_{21}] \qquad \cdot 00086{,}55$$

$$\frac{n-n'}{(n-p)^2}(3n-2n'-p)[b_2a_{22}-a_2b_{22}] \qquad \cdot 00000{,}64$$

$$-\frac{3}{4}\frac{n'^2}{(n-p)^2}[b_{21}-b_{22}] \qquad -\ \cdot 00163{,}92$$

$$1+b_0 = \quad 1\cdot 00757{,}1$$

Continuing the approximation for a_{21}, b_{21}, a_{22}, b_{22} the various terms found are the following:—

·01538,88	divisor	– ·01538,88	divisor
·03784,83	·86169,5	·03784,83	9·43549,4
– ·07221,53	– 1·15842,8	– ·02182,34	– 1·15842,8
– ·04410,93	– ·29673,3	– ·04410,93	8·27706,6
·05097,33		– ·01540,41	
·01975,81		– ·01975,82	
·04601,14	a_{21} = ·18082,0	– ·01390,46	a_{22} = ·01118,03
·05365,53		– ·09254,01	

	·42108,0		·00786,805
	·03598,79		·00328,659
	− ·02540,22		·00231,985
	− ·02292,94		·00209,403
$b_{21}=$	·40873,6	$b_{22}=$	·01556,85

Third Approximation. We find

	1·15842,8			1·00860,75
	− ·00081,40			− ·00017,57
	·00535,02	$n-p=1·07162,5$		·00087,94
	− ·00723,54	$p=$ ·00922,4		·00000,64
	− ·00416,02	$p:n=$ ·00853,5		− ·00167,84
	·00048,35		$1+b_0=$	1·00763,9
	·00407,40			
	·00002,97			
	− ·00777,57			
$(n-p)^2=$	1·14838,0			

·01538,73	divisor	− ·01538,73	divisor
·03784,83	·86188,04	·03784,83	9·43488,0
− ·07220,54	− 1·15842,8	− ·02182,35	− 1·15842,8
− ·04410,81	− ·29654,8	− ·04410,81	8·27645,2
·05096,30		− ·01540,32	
·01975,95		− ·01975,95	
·04600,96		− ·01390,60	
·05365,42	$a_{21}=$ ·18092,9.	− ·09253,93	$a_{22}=$ ·01118,10.

	·42128,90		·00786,881
	·03597,92		·00328,671
	− ·02539,43		·00231,978
	− ·02292,60		·00209,430
$b_{21}=$	·40894,8	$b_{22}=$	·01556,96

Fourth Approximation.

	1·15842,8			1·00862,79
	− ·00081,41			− ·00017,57
	·00534,22	$n-p=1·07160,3$		·00088,04
	− ·00727,13	$p=$ ·00924,6		·00000,64
	− ·00416,24	$p:n=$ ·00855,4		− ·00167,94
	·00048,35		$1+b_0=$	1·00766,0
	·00407,66			
	·00002,97			
	− ·00778,05			
$(n-p)^2=$	1·14833,2			

The values already found for the remaining quantities are sufficiently exact.

These numbers give, taking after Hansen,

$$e(1+b_0) = \cdot 05491$$
$$e = \cdot 05449$$
$$2(1+b_0)e = \cdot 10982{,}0 = 22651''\cdot 9$$
$$a_{21}e = \cdot 00986{,}03$$
$$b_{21}e = \cdot 02228{,}44 = 4596''\cdot 6$$
$$a_{22}e = \cdot 00060{,}93$$
$$b_{22}e = \cdot 00084{,}85 = \ \ 175''\cdot 1,$$

and taking the Moon's mean annual motion 17325593″, the annual motion of the apse is

$$148202'' = 41^\circ\, 10'\, 2''.$$

LECTURE XIII.

THE MOTION OF THE APSE, AND THE CHANGE OF THE ECCENTRICITY.

We have seen that when the eccentricity of the Moon's orbit is not considered we may write

$$\frac{1}{r}=\frac{1}{a}[1+a_2\cos 2(\theta-\theta')],$$

$$H=na^2[1+h_2\cos 2(\theta-\theta')],$$

where $$a_2=\frac{3}{2}m^2\cdot\frac{2-m}{1-m}\cdot\frac{1}{3-8m+\frac{11}{2}m^2};\quad h_2=\frac{3}{4}\frac{m^2}{1-m}.$$

Let us introduce the two new arbitraries e, ϖ by writing

$$H=hna^2[1+h_2\cos(2-2m)\theta],$$

$$\frac{1}{r}=\frac{1}{h^2a}[1+a_2\cos(2-2m)\theta+e\cos(\theta-\varpi)],$$

where h is a third arbitrary, which may be chosen to suit our convenience; it must be unity when $e=0$.

Then $$\frac{dH}{d\theta}=\frac{dH}{dt}\cdot\frac{dt}{d\theta}=-\frac{3}{2}m^2n^2r^2\sin 2(\theta-\theta')\frac{dt}{d\theta}$$

$$=-\frac{3}{2}m^2n^2\frac{r^4}{H}[\sin(2-2m)\theta+4me\cos(2-2m)\theta\sin(\theta-\varpi)]$$

$$=-\frac{3}{2}m^2n^2\frac{h^8a^4}{hna^2}[1-(4a_2+h_2)\cos(2-2m)\theta-4e\cos(\theta-\varpi)]$$

$$\times[\sin(2-2m)\theta+4me\cos(2-2m)\theta\sin(\theta-\varpi)],$$

and also

$$\frac{dH}{d\theta}=\frac{dh}{d\theta}na^2[1+h_2\cos(2-2m)\theta]-na^2h(2-2m)h_2\sin(2-2m)\theta.$$

Now we may put $h=1+\eta$, where η vanishes with e. Neglecting powers of e above the first

$$\frac{d\eta}{d\theta}=\frac{dh}{d\theta}=3m^2(1+m)\,e\sin(\overline{1-2m}\,\theta+\varpi)+3m^2(1-m)\,e\sin(\overline{3-2m}\,\theta-\varpi)$$
$$-9m^2\eta\sin(2-2m)\,\theta.$$

Neglect at first the last term:

$$\eta=-3m^2\frac{1+m}{1-2m}\,e\cos(\overline{1-2m}\,\theta+\varpi)-3m^2\frac{1-m}{3-2m}\,e\cos(\overline{3-2m}\,\theta-\varpi).$$

Substitute this in the last term, and we get

$$\frac{d\eta}{d\theta}=3m^2(1+m)\,e\sin(\overline{1-2m}\,\theta+\varpi)+3m^2(1-m)\,e\sin(\overline{3-2m}\,\theta-\varpi)$$
$$+\frac{27}{2}m^4\frac{2+4m-4m^2}{(1-2m)(3-2m)}\,e\sin(\theta-\varpi).$$

Now consider the other equation

$$\frac{d^2r}{dt^2}=\frac{H^2}{r^3}-\frac{\mu}{r^2}+\frac{1}{2}m^2n^2r+\frac{3}{2}m^2n^2r\cos2(\theta-\theta')$$
$$=\frac{H^2}{r^3}-\frac{\mu}{r^2}+\frac{1}{2}m^2n^2r+\frac{3}{2}m^2n^2r\left[\cos(2-2m)\,\theta-2me\cos(\overline{1-2m}\,\theta+\varpi)\right.$$
$$\left.+2me\cos(\overline{3-2m}\,\theta-\varpi)\right].$$

Differentiate the assumed expression for $\frac{1}{r}$, and let h be chosen so that the first differential coefficient shall have the same form as if h, e, ϖ were constant.

Thus

$$\frac{1}{r^2}\frac{dr}{dt}=\frac{1}{h^2a}(2-2m)\,a_2\sin(2-2m)\,\theta\frac{d\theta}{dt}+\frac{1}{h^2a}e\sin(\theta-\varpi)\frac{d\theta}{dt},$$

where

$$-\frac{2}{h}\frac{dh}{d\theta}\left[1+a_2\cos(2-2m)\,\theta\right]+\frac{de}{d\theta}\cos(\theta-\varpi)+e\frac{d\varpi}{d\theta}\sin(\theta-\varpi)=0,$$

or

$$\frac{de}{d\theta}\cos(\theta-\varpi)+e\frac{d\varpi}{d\theta}\sin(\theta-\varpi)=\ 6m^2(1+m)\,e\sin(\overline{1-2m}\,\theta+\varpi)$$
$$+\ 6m^2(1-m)\,e\sin(\overline{3-2m}\,\theta-\varpi)$$
$$+\left\{27m^4\frac{2+4m-4m^2}{(1-2m)(3-2m)}-6m^3a_2\right\}e\sin(\theta-\varpi),$$

and

$$\frac{dr}{dt}=\frac{H}{h^2a}(2-2m)\,a_2\sin(2-2m)\,\theta+\frac{H}{h^2a}e\sin(\theta-\varpi)$$

$$=\frac{na}{h}(2-2m)\,a_2\sin(2-2m)\,\theta+\frac{na}{h}[1+h_2\cos(2-2m)\,\theta]\,e\sin(\theta-\varpi),$$

$$\frac{d^2r}{dt^2}=\frac{na}{h}(2-2m)^2a_2\cos(2-2m)\,\theta\frac{d\theta}{dt}-\frac{na}{h}(2-2m)\,h_2\sin(2-2m)\,\theta\, e\sin(\theta-\varpi)\frac{d\theta}{dt}$$

$$+\frac{na}{h}[1+h_2\cos(2-2m)\,\theta]\,e\cos(\theta-\varpi)\frac{d\theta}{dt}$$

$$-\frac{na}{h^2}\frac{dh}{d\theta}(2-2m)\,a_2\sin(2-2m)\,\theta\frac{d\theta}{dt}$$

$$+\frac{na}{h}[1+h_2\cos(2-2m)\,\theta]\left[\frac{de}{d\theta}\sin(\theta-\varpi)-e\frac{d\varpi}{d\theta}\cos(\theta-\varpi)\right]\frac{d\theta}{dt}.$$

Multiply by r^2/n^2a^3; then since

$$r^2\frac{d\theta}{dt}=H=hna^2[1+h_2\cos(2-2m)\,\theta],$$

we have

$$\frac{r^2}{n^2a^3}\frac{d^2r}{dt^2}=(2-2m)^2a_2\cos(2-2m)\,\theta-(2-2m)\,h_2\sin(2-2m)\,\theta\, e\sin(\theta-\varpi)$$

$$+e\cos(\theta-\varpi)+2h_2\cos(2-2m)\,\theta\, e\cos(\theta-\varpi)$$

$$-\frac{1}{h}\frac{dh}{d\theta}(2-2m)\,a_2\sin(2-2m)\,\theta$$

$$+[1+h_2\cos(2-2m)\,\theta]^2\left[\frac{de}{d\theta}\sin(\theta-\varpi)-e\frac{d\varpi}{d\theta}\cos(\theta-\varpi)\right],$$

and this is equal to

$$\frac{H^2}{n^2a^3r}-\frac{\mu}{n^2a^3}+\frac{1}{2}m^2\frac{r^3}{a^3}+\frac{3}{2}m^2\frac{r^3}{a^3}[\cos(2-2m)\,\theta-2me\cos(\overline{1-2m}\,\theta+\varpi)$$

$$+2me\cos(\overline{3-2m}\,\theta-\varpi)]$$

$$=\left[1+\frac{h_2^2}{2}+2h_2\cos(2-2m)\,\theta\right][1+a_2\cos(2-2m)\,\theta+e\cos(\theta-\varpi)]$$

$$-\frac{\mu}{n^2a^3}+\frac{1}{2}m^2h^6[1-3a_2\cos(2-2m)\,\theta-3e\cos(\theta-\varpi)]$$

$$+\frac{3}{2}m^2h^6\Big[\cos(2-2m)\,\theta+6a_2e\cos(\theta-\varpi)$$

$$-\left(\frac{3}{2}+2m\right)e\cos(\overline{1-2m}\,\theta+\varpi)-\left(\frac{3}{2}-2m\right)e\cos(\overline{3-2m}\,\theta-\varpi)\Big].$$

The terms in these two expressions which are independent of e give no new information; equating the others:—

$$-(2-2m)\,h_2 e \sin(2-2m)\,\theta \sin(\theta-\varpi) - \frac{1}{h}\frac{dh}{d\theta}(2-2m)\,a_2 \sin(2-2m)\,\theta$$

$$+\left[1+h_2\cos(2-2m)\,\theta\right]^2\left[\frac{de}{d\theta}\sin(\theta-\varpi) - e\frac{d\varpi}{d\theta}\cos(\theta-\varpi)\right]$$

$$=3m^2\eta - \frac{3}{2}m^2 e\cos(\theta-\varpi) + \frac{3}{2}m^2\left[6a_2 e\cos(\theta-\varpi)\right.$$

$$\left. -\left(\frac{3}{2}+2m\right)e\cos(\overline{1-2m}\,\theta+\varpi) - \left(\frac{3}{2}-2m\right)e\cos(\overline{3-2m}\,\theta-\varpi)\right]$$

$$+9m^2\eta\cos(2-2m)\,\theta.$$

Reducing this expression

$$-\frac{de}{d\theta}\sin(\theta-\varpi) + e\frac{d\varpi}{d\theta}\cos(\theta-\varpi) = \left(\frac{3}{2}m^2 - \frac{3}{8}m^4\right)e\cos(\theta-\varpi)$$

$$+\left(\frac{3}{2}m^2 + 3m^3 + \frac{63}{8}m^4\right)e\cos(\overline{1-2m}\,\theta+\varpi)$$

$$+\left(3\,m^2 - 3m^3 + \frac{15}{8}m^4\right)e\cos(\overline{3-2m}\,\theta-\varpi)$$

and from before

$$\frac{de}{d\theta}\cos(\theta-\varpi) + e\frac{d\varpi}{d\theta}\sin(\theta-\varpi) = 18m^4 e\sin(\theta-\varpi)$$

$$+(6m^2+6m^3)\,e\sin(\overline{1-2m}\,\theta+\varpi)$$

$$+(6m^2-6m^3)\,e\sin(\overline{3-2m}\,\theta-\varpi).$$

Hence,

$$\frac{de}{d\theta} = \left(-\frac{3}{4}m^2 \qquad + \frac{147}{16}m^4\right)e\sin 2(\theta-\varpi)$$

$$+\left(\frac{27}{4}m^2 - 3m^3 - 3\,m^4\right)e\sin(2-2m)\,\theta$$

$$+\left(-\frac{15}{4}m^2 - \frac{9}{2}m^3 - \frac{63}{16}m^4\right)e\sin(2m\theta - 2\varpi)$$

$$+\left(\frac{3}{2}m^2 - \frac{3}{2}m^3 - \frac{15}{16}m^4\right)e\sin(\overline{4-2m}\,\theta - 2\varpi),$$

$$\frac{d\varpi}{d\theta} = \frac{3}{4}m^2 + \frac{141}{16}m^4 + \left(\frac{3}{4}m^2 - \frac{147}{16}m^4\right)\cos 2(\theta - \varpi)$$
$$+\left(\frac{9}{4}m^2 - 6m^3 + \frac{39}{8}m^4\right)\cos(2-2m)\theta$$
$$+\left(\frac{15}{4}m^2 + \frac{9}{2}m^3 + \frac{63}{16}m^4\right)\cos(2m\theta - 2\varpi)$$
$$+\left(-\frac{3}{2}m^2 + \frac{3}{2}m^3 + \frac{15}{16}m^4\right)\cos(\overline{4-2m}\,\theta - 2\varpi).$$

We notice that among these terms, one is of long period, approximately semiannual, and will become of greater relative importance than the others on integration.

To effect this integration, assume

$$\Pi = \varpi + \alpha \sin 2(\theta - \varpi) + \beta \sin(2-2m)\theta + \gamma \sin\{(4-2m)\theta - 2\varpi\},$$

so that the mean motion of π is the same as that of ϖ, and substitute in the equation.

Then

$$\frac{d\Pi}{d\theta} = \frac{3}{4}m^2 + \frac{141}{16}m^4 - \frac{3}{4}m^2\alpha + \frac{3}{2}m^2\gamma$$
$$+\left[\frac{3}{4}m^2 - \frac{147}{16}m^4 + 2\alpha - \frac{3}{2}m^2\alpha + \frac{15}{4}m^4\beta - \frac{9}{4}m^2\gamma\right]\cos 2(\theta - \varpi)$$
$$+\left[\frac{9}{4}m^2 - 6m^3 + \frac{39}{8}m^4 + (2-2m)\beta - 6m^2\alpha - \frac{3}{4}m^2\gamma\right]\cos(2-2m)\theta$$
$$+\left[\frac{15}{4}m^2 + \frac{9}{2}m^3 + \frac{63}{16}m^4 - \frac{9}{4}m^2\alpha\right]\cos(2m\theta - 2\Pi)$$
$$+\left[-\frac{3}{2}m^2 + \frac{3}{2}m^3 + \frac{15}{16}m^4 + (4-2m)\gamma - \frac{9}{4}m^2\alpha - \frac{3}{2}m^2\gamma\right]\cos\{(4-2m)\theta - 2\varpi\},$$

so that if we take

$$\alpha = -\frac{3}{8}m^2 + \frac{219}{32}m^4,$$

$$\beta = -\frac{9}{8}m^2 + \frac{15}{8}m^3 - \frac{99}{64}m^4,$$

$$\gamma = \frac{3}{8}m^2 - \frac{3}{16}m^3 - \frac{51}{128}m^4,$$

we have

$$\frac{d\Pi}{d\theta} = \frac{3}{4}m^2 + \frac{309}{32}m^4 + \left[\frac{15}{4}m^2 + \frac{9}{2}m^3 + \frac{45}{8}m^4\right]\cos(2m\theta - 2\Pi).$$

If we write

$$m\theta - \Pi = \psi,$$

this becomes

$$\frac{d\psi}{d\theta} = a - b\cos 2\psi,$$

where

$$a = m - \frac{3}{4}m^2 - \frac{309}{32}m^4, \qquad b = \frac{15}{4}m^2 + \frac{9}{2}m^3 + \frac{45}{8}m^4,$$

and the solution is

$$\tan^{-1}\left(\sqrt{\frac{a+b}{a-b}}\tan\psi\right) = \theta\sqrt{a^2 - b^2} + \text{constant}.$$

Hence if we denote by $\dfrac{d\varpi_0}{d\theta}$ the mean rate of change of ϖ, we have

$$m - \frac{d\varpi_0}{d\theta} = \sqrt{a^2 - b^2}$$

$$= m - \frac{3}{4}m^2 - \frac{225}{32}m^3 - \frac{4071}{128}m^4,$$

or

$$\frac{d\varpi_0}{d\theta} = \frac{3}{4}m^2 + \frac{225}{32}m^3 + \frac{4071}{128}m^4.$$

We observe that $a+b$ and $a-b$ are the rates of separation of the Sun from the apse when the Sun and the apse are at quadratures and syzygies with one another, respectively,—that is if we take Π for the longitude of the apse, or, what is the same thing, if we ignore small terms of short period. Hence the mean rate of separation of the Sun from the apse is a mean proportional between its rates when at quadratures and syzygies respectively with the apse*.

[* This is the analogue for the case of the apse of Machin and Pemberton's theorem on the motion of the node, inserted in the third edition of the *Principia* as a scholium to prop. XXXIII., lib. III. See some notes by Adams in Brewster's *Life of Newton*, Appendix XXX.]

LECTURE XIV.

THE LATITUDE AND THE MOTION OF THE NODE.

LET us first treat this problem on the supposition that the latitude is so small that its square may be neglected. The equation of motion, taken from Lecture II., may be written

$$\frac{d^2z}{dt^2} = -\frac{z}{r}\left[\frac{\mu}{r^2} + \frac{m'r}{r'^3}\left(1 + \frac{E-M}{E+M}\frac{r}{r'}3\cos\omega\right)\right],$$

where $z = r\sin(\text{latitude})$ and the cube of s is omitted; or neglecting the parallactic terms

$$\frac{d^2z}{dt^2} = -z\left[\frac{\mu}{r^3} + \frac{m'}{r'^3}\right].$$

The value of μ/r^3 may be considered known by the operations which have determined the motion in an orbit coinciding with the ecliptic; that is to say,

$$\frac{\mu}{r^3} = \frac{\mu}{a^3}\left[1 + \frac{3}{2}\alpha_2^2 + 3\alpha_2\cos 2\psi + \left(\frac{3}{2}\alpha_2^2 + 3\alpha_4\right)\cos 4\psi\right],$$

where α has the definition of Lectures IV, V; or numerically, taking

$$n - n' = 1,$$

$$\frac{\mu}{r^3} = 1{\cdot}17150{,}3 + {\cdot}02523{,}0\cos 2t + {\cdot}00025{,}15\cos 4t.$$

And

$$\frac{m'}{r'^3} = n'^2 = {\cdot}00653{,}6.$$

Hence

$$\frac{d^2z}{dt^2} = -z[1{\cdot}17803{,}9 + {\cdot}02523{,}0\cos 2t + {\cdot}00025{,}15\cos 4t].$$

Let us now consider the equation

$$\frac{d^2z}{dt^2}+Pz=0,$$

where
$$P=q_0+2q_1\cos 2t+2q_2\cos 4t,$$

in which q_1, q_2 are supposed small.

Suppose a term in z to be $c\cos(kt+\beta)$; when this is substituted in Pz there will arise terms

$$c\cos(\overline{k-2}\,t+\beta) \qquad c\cos(\overline{k+2}\,t+\beta)$$

$$c\cos(\overline{k-4}\,t+\beta) \qquad c\cos(\overline{k+4}\,t+\beta).$$

Let us therefore assume

$$\begin{aligned} z=c[\cos(kt+\beta)&+c_1\cos(\overline{k+2}\,t+\beta)+c_2\cos(\overline{k+4}\,t+\beta)+\dots \\ &+c_{-1}\cos(\overline{k-2}\,t+\beta)+c_{-2}\cos(\overline{k-4}\,t+\beta)+\dots] \end{aligned}$$

c is arbitrary; we have to determine k, c_1, c_{-1}, &c.

Substitute and equate coefficients:

$$\begin{aligned}
&\dots+[-(k-4)^2+q_0]\,c_{-2}+q_1c_{-1}+q_2+\dots &&=0\\
&\dots\dots\quad +q_1c_{-2}+[-(k-2)^2+q_0]\,c_{-1}+q_1+q_2c_1+\dots &&=0\\
&\dots\dots\quad +q_2c_{-2}\quad +q_1c_{-1}\quad +[-k^2+q_0]+q_1c_1+q_2c_2+\dots &&=0\\
&\dots\dots\quad +q_2c_{-1}\quad +\quad q_1\quad +[-(k+2)^2+q_0]\,c_1+q_1c_2+\dots &&=0\\
&\dots\dots\quad +\quad q_2\quad +\quad q_1c_1\quad +[-(k+4)^2+q_0]\,c_2+\dots &&=0.
\end{aligned}$$

If q_1, q_2, ... are neglected, we have simply

$$-k^2+q_0=0,$$

this is a first approximation to the value of k.

Taking q_1 into account and neglecting q_2

$$c_{-1}=-\frac{q_1}{q_0-(k-2)^2},$$

$$c_1=-\frac{q_1}{q_0-(k+2)^2}.$$

In the actual case considered we notice that q_0 does not differ widely from unity. Hence k is nearly equal to unity also, and the denominator in c_{-1} is small, and makes c_{-1} much more important than c_1.

If we substitute these values in the third equation above, we have

$$k^2 - q_0 + q_1^2 \left\{ \frac{1}{q_0 - (k+2)^2} + \frac{1}{q_0 - (k-2)^2} \right\} = 0,$$

whence

$$(k^2 - q_0)^3 - 8(k^2 - q_0)^2 - \{16(q_0 - 1) + 2q_1^2\}(k^2 - q_0) - 8q_1^2 = 0,$$

which may be put under the form

$$(k^2 - q_0)^2 + 2(q_0 - 1)(k^2 - q_0) = -q_1^2 + \frac{1}{4} q_1^2 (k^2 - q_0) + \frac{1}{8}(k^2 - q_0)^3,$$

whence

$$(k^2 - 1)^2 = (q_0 - 1)^2 - q_1^2 - \frac{1}{4} q_1^2 (k^2 - q_0) + \frac{1}{8}(k^2 - q_0)^3.$$

With this equation we can approximate very rapidly to the value of k. Taking as a first approximation

$$k^2 - q_0 = 0,$$

substitute this value of k in the small terms and we get as a second approximation

$$k = 1{\cdot}08516{,}9.$$

Whence the ratio of the retrograde motion of the node to the Moon's mean motion is

$$k/n - 1 = g - 1 = {\cdot}00399{,}7,$$

where g is written for k/n. This value is very correct. Taking the Moon's mean annual motion as 17325593″, the resulting annual retrograde motion of the node is

$$69252'' = 19^\circ\ 14'\ 12''.$$

Next find the values of the coefficients c_{-1}, c_1, c_{-2}, c_2. We have

$$\begin{aligned} q_1 &= {\cdot}01261{,}5, \\ q_0 - (k-2)^2 &= {\cdot}34112{,}3, \\ q_0 - (k+2)^2 &= -8{\cdot}34022{,}8; \end{aligned}$$

whence as a first approximation

$$c_{-1} = -{\cdot}03698{,}19, \qquad c_1 = {\cdot}00151{,}26.$$

Hence

$$q_1 c_{-1} + q_2 = -\cdot 00034{,}02, \qquad q_1 c_1 + q_2 = \cdot 00014{,}49,$$

and

$$q_0 - (k-4)^2 = -7\cdot 31821, \qquad q_0 - (k+4)^2 = -24\cdot 6809,$$

whence

$$c_{-2} = -\cdot 00004{,}650,$$
$$c_2 = \cdot 00000{,}587.$$

A second approximation to c_{-1}, c_1 gives

$$[-(k-2)^2 + q_0]\, c_{-1} = -(q_1 + q_2 c_1 + q_1 c_{-2}),$$
$$[-(k+2)^2 + q_0]\, c_1 = -(q_1 + q_2 c_{-1} + q_1 c_2);$$

with the above values

$$q_1 + q_2 c_1 + q_1 c_{-2} = \cdot 01261{,}47, \qquad q_1 + q_2 c_{-1} + q_1 c_2 = \cdot 01261{,}03,$$

so that

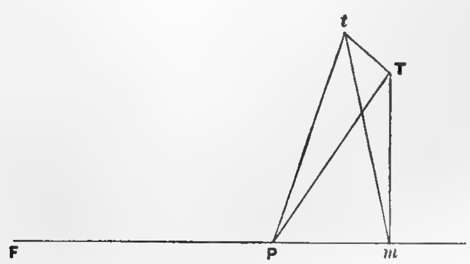
$$c_{-1} = -\cdot 03698{,}00,$$
$$c_1 = \cdot 00151{,}20.$$

LECTURE XV.

MOTION IN AN ORBIT OF ANY INCLINATION.

Let us consider the change in the plane of the orbit produced in an indefinitely small time dt by the action of a given disturbing force. Let Z be the resolved part of the disturbing force at any time in a direction perpendicular to the plane in which the body is moving at the instant. Imagine the force Z to act by impulses at the small intervals of time dt, then Zdt will be the indefinitely small velocity generated by the force Z in the time dt, in the direction perpendicular to the plane of orbit at the instant.

Let FP be the radius vector and P the position of the body at the instant. Also let PT represent the velocity at the instant in magnitude



and direction; then if Tt be taken perpendicular to the plane FPT and equal to Zdt, the velocity and its direction after the impulse will be represented by Pt, and the new plane of the orbit by FPt. Draw Tm perpendicular to FP and join tm; then tmT is the angle through which the plane of the orbit has been turned about the radius vector FP in the indefinitely short time dt.

Now $$tmT = \frac{tT}{Tm} = \frac{Zdt}{v},$$

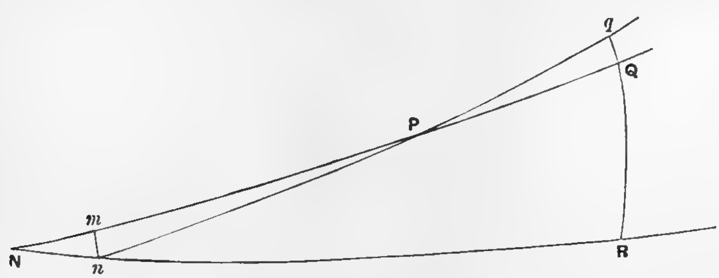
where v is the resolved part of the velocity at P perpendicular to the radius vector. But

$$H = vr;$$

hence the angle through which the orbit is turned in an indefinitely short time dt is

$$\frac{rZ}{H} dt.$$

To find the corresponding changes in the elements that determine the plane of the orbit, namely, the inclination of the orbit to a fixed plane, and the longitude of the node on that plane. Let NPQ be the great circle which



represents the plane of the orbit at the time t, NR the plane of reference, usually the plane of the ecliptic, P the position of the body at the same time, and let $i = PNR$, the inclination, and let N be the longitude of the node.

Let nPq be the position of the orbit at time $t + dt$.

Take $NQ = 90°$; draw nm perpendicular to NPQ and qQR perpendicular to NR; then $QR = i$; and by what we have just proved

$$NPn = \frac{Zr}{H} dt.$$

Therefore $$nm = \frac{Zr}{H} dt \,.\, \sin\theta, \quad qQ = \frac{Zr}{H} dt \,.\, \cos\theta,$$

where $$\theta = nP.$$

But $$nm = Nn \sin i = \sin i \, dN; \; qQ = di.$$

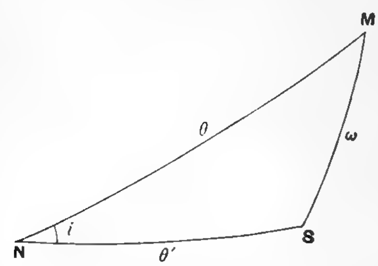
Therefore

$$\frac{di}{dt} = \frac{Zr\cos\theta}{H},$$

$$\frac{dN}{dt} = \frac{Zr\sin\theta}{H\sin i},$$

which give the changes of the elements required.

Now let NMS be a spherical triangle, the centre of the sphere being G, the centre of gravity of the Earth and Moon; and let GS, GM, GN



point respectively to the Sun, the Moon, and the node of the Moon's orbit upon the ecliptic, so that NM is the plane of the Moon's orbit and NS the ecliptic. Let $MS = \omega$, $NS = \theta'$, $NM = \theta$, of which the first is identical with the quantity denoted by the same symbol in Lecture II, but the second and third are not so.

Then, following Lecture II, the forces on the Moon are

$$\frac{\mu}{r^2} + \frac{m'r}{r'^3} \qquad \text{in } MG,$$

$$-\frac{m'r}{r'^3}\, 3\cos\omega, \quad \text{in } SG,$$

if we ignore the parallactic terms.

This latter may be resolved into

$$-\frac{m'r}{r'^3}\, 3\cos\omega \times (\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos i) \qquad \text{in } MG,$$

$$\frac{m'r}{r'^3}\, 3\cos\omega \times (\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos i) \qquad \text{perpendicular to } MG \text{ in the plane of the orbit},$$

$$\frac{m'r}{r'^3}\, 3\cos\omega \times \sin\theta'\sin i \qquad \text{perpendicular to the plane of the orbit.}$$

Now

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos i = \cos(\theta-\theta')\cos^2\frac{i}{2} + \cos(\theta+\theta')\sin^2\frac{i}{2},$$

$$\sin\theta\cos\theta' - \cos\theta\sin\theta'\cos i = \sin(\theta-\theta')\cos^2\frac{i}{2} + \sin(\theta+\theta')\sin^2\frac{i}{2}.$$

Hence we have the following expressions for the three forces:

$$P = \frac{\mu}{r^2} + \frac{m'r}{r'^3}$$
$$-\frac{3}{2}\frac{m'r}{r'^3}\left[\{1+\cos 2(\theta-\theta')\}\cos^4\frac{i}{2} + \{\cos 2\theta + \cos 2\theta'\}\,2\cos^2\frac{i}{2}\sin^2\frac{i}{2}\right.$$
$$\left. + \{1+\cos 2(\theta+\theta')\}\sin^4\frac{i}{2}\right],$$

$$T = \frac{3}{2}\frac{m'r}{r'^3}\left[\sin 2(\theta-\theta')\cos^4\frac{i}{2} + \sin 2\theta \,.\, 2\cos^2\frac{i}{2}\sin^2\frac{i}{2}\right.$$
$$\left. + \sin 2(\theta+\theta')\sin^4\frac{i}{2}\right],$$

$$Z = \frac{3}{2}\frac{m'r}{r'^3}\sin i\left[-\sin(\theta-2\theta')\cos^2\frac{i}{2} + \sin\theta\cos i + \sin(\theta+2\theta')\sin^2\frac{i}{2}\right].$$

Now we have seen

$$\frac{di}{dt} = -\frac{Zr\cos\theta}{H}, \qquad \frac{dN}{dt} = -\frac{Zr\sin\theta}{H\sin i};$$

also, the rate of advance of the node along the orbit is

$$-\frac{Zr\sin\theta}{H\tan i}.$$

Thus the equations of motion become

$$\frac{H}{r^2} = \frac{d\theta}{dt} - \frac{Zr\sin\theta}{H\tan i},$$

together with

$$\frac{d^2r}{dt^2} - \frac{H^2}{r^3} = -P,$$

$$\frac{dH}{dt} = -rT.$$

LECTURE XVI.

MOTION IN AN ORBIT OF ANY INCLINATION, (*continued*).

To satisfy the equation at the end of Lecture XV, assume

$$r = a\left[1 + A_1 \cos 2(\theta - \theta') + A_2 \cos 2\theta + A_3 \cos 2\theta' + A_4 \cos 2(\theta + \theta')\right],$$

neglecting the square of the disturbing force and the eccentricity; thus in the small terms we write

$$r = a, \quad \frac{d\theta}{dt} = n, \quad r' = a', \quad \frac{d\theta'}{dt} = n'.$$

Hence

$$-\frac{d^2r}{dt^2} = n^2a\left[(2-2m)^2 A_1 \cos 2(\theta - \theta') + 4A_2 \cos 2\theta + 4m^2 A_3 \cos 2\theta' \right.$$
$$\left. + (2+2m)^2 A_4 \cos 2(\theta + \theta')\right];$$

substitute in the equation

$$H^2 = r^3 P + r^3 \frac{d^2r}{dt^2};$$

therefore

$$H^2 = \mu a\left[1 + A_1 \cos 2(\theta - \theta') + A_2 \cos 2\theta + A_3 \cos 2\theta' + A_4 \cos 2(\theta + \theta')\right]$$
$$+ m^2n^2a^4$$
$$-\frac{3}{2} m^2n^2a^4 \left[\{1 + \cos 2(\theta - \theta')\} \cos^4 \frac{i}{2} + \{\cos 2\theta + \cos 2\theta'\} 2 \cos^2 \frac{i}{2} \sin^2 \frac{i}{2} \right.$$
$$\left. + \{1 + \cos 2(\theta + \theta')\} \sin^4 \frac{i}{2}\right]$$
$$- n^2a^4\left[(2-2m)^2 A_1 \cos 2(\theta - \theta') + 4A_2 \cos 2\theta + 4m^2 A_3 \cos 2\theta' \right.$$
$$\left. + (2+2m)^2 A_4 \cos 2(\theta + \theta')\right].$$

Again, we have the equation

$$\frac{dH}{dt} = -rT,$$

which may be written

$$H\frac{dH}{dt} = -\frac{3}{2}n^3m^2a^4\left[\sin 2(\theta-\theta')\cos^4\frac{i}{2} + \sin 2\theta\,.\,2\sin^2\frac{i}{2}\cos^2\frac{i}{2} + \sin 2(\theta+\theta')\sin^4\frac{i}{2}\right].$$

Substitute for μa its approximate value n^2a^4 in the small terms; and we find from these two equations

$$-(1-m)A_1 + 4(1-m)^3A_1 + \frac{3}{2}m^2(1-m)\cos^4\frac{i}{2} = -\frac{3}{2}m^2\cos^4\frac{i}{2},$$

$$-A_2 + 4A_2 + \frac{3}{2}m^2\,.\,2\cos^2\frac{i}{2}\sin\frac{i}{2} = -\frac{3}{2}m^2\,.\,2\cos^2\frac{i}{2}\sin^2\frac{i}{2},$$

$$-mA_3 + 4m^3A_3 + \frac{3}{2}m^3\,.\,2\cos^2\frac{i}{2}\sin^2\frac{i}{2} = 0,$$

$$-(1+m)A_4 + 4(1+m)^3A_4 + \frac{3}{2}m^2(1+m)\sin^4\frac{i}{2} = -\frac{3}{2}m^2\sin^4\frac{i}{2}.$$

Therefore

$$A_1 = -\frac{3}{2}m^2\cos^4\frac{i}{2}\frac{2-m}{(1-m)(1-2m)(3-2m)},$$

$$A_2 = -2m^2\cos^2\frac{i}{2}\sin^2\frac{i}{2},$$

$$A_3 = 3m^2\cos^2\frac{i}{2}\sin^2\frac{i}{2}\frac{1}{(1-2m)(1+2m)},$$

$$A_4 = -\frac{3}{2}m^2\sin^4\frac{i}{2}\frac{2+m}{(1+m)(1+2m)(3+2m)},$$

and

$$H^2 = n^2a^4\left[1 + m^2 - \frac{3}{2}m^2\left(\cos^4\frac{i}{2} + \sin^4\frac{i}{2}\right)\right.$$

$$\left.+\frac{3}{2}\frac{m^2}{1-m}\cos^4\frac{i}{2}\cos 2(\theta-\theta') + 3m\cos^2\frac{i}{2}\sin^2\frac{i}{2}\cos 2\theta + \frac{3}{2}\frac{m^2}{1+m}\sin^4\frac{i}{2}\cos 2(\theta+\theta')\right];$$

where a is defined by

$$\mu = n^2a^3.$$

If we preferred to define a so that the constant term in H^2 were equal to n^2a^4, we should have

$$n^2a^4 = \mu a + m^2n^2a^4 - \frac{3}{2}m^2n^2a^4\left(\cos^4\frac{i}{2} + \sin^4\frac{i}{2}\right),$$

or

$$\mu = n^2a^3\left[1 + \frac{1}{2}m^2 - 3m^2\sin^2\frac{i}{2}\cos^2\frac{i}{2}\right].$$

Let us next find the latitude and the motion of the node.

Suppose that

$$i = i_0 + \Delta i,$$
$$N = N_0 + \Delta N,$$

in which Δi, ΔN are small, i_0 is a constant, and N_0 varies slowly in proportion to the time, so that we may assume

$$\frac{dN_0}{dt} = -qn,$$

$$\Delta N = N_1 \sin 2(\theta - \theta') + N_2 \sin 2\theta + N_3 \sin 2\theta' + N_4 \sin 2(\theta + \theta'),$$

$$\Delta i = I_1 \cos 2(\theta - \theta') + I_2 \cos 2\theta + I_3 \cos 2\theta' + I_4 \cos 2(\theta + \theta').$$

Then remembering that

$$\frac{d\theta'}{dt} = mn - \frac{dN}{dt},$$

an expression that must be used in the terms of chief importance, we have

$$\begin{aligned}\frac{di}{dt} = \frac{d\Delta i}{dt} = &-2(1-m)\,nI_1 \sin 2(\theta - \theta') - 2nI_2 \sin 2\theta \\ &-2\left(m - \frac{1}{n}\frac{dN}{dt}\right)nI_3 \sin 2\theta' - 2(1+m)\,nI_4 \sin 2(\theta + \theta'),\end{aligned}$$

$$\begin{aligned}\frac{dN}{dt} = \frac{dN_0}{dt} + \frac{d\Delta N}{dt} = &-qn + 2(1-m)\,nN_1 \cos 2(\theta - \theta') + 2nN_2 \cos 2\theta \\ &+2\left(m - \frac{1}{n}\frac{dN}{dt}\right)nN_3 \cos 2\theta' + 2(1+m)\,nN_4 \cos 2(\theta + \theta') \\ &+\frac{dN_3}{di}\frac{di}{dt}\sin 2\theta',\end{aligned}$$

in which the last term will be found to be required to get the constant q correctly to the order m^3.

These must be equated to $-\dfrac{Zr\cos\theta}{H}$, $-\dfrac{Zr\sin\theta}{H\sin i}$ respectively.

Hence

$$-q-2mN_3^2-mI_3\frac{dN_3}{di}=-\frac{3}{4}m^2\cos i;$$

therefore as a first approximation

$$q=\frac{3}{4}m^2\cos i;$$

hence

$$-2(1-m)I_1=\frac{3}{4}m^2\sin i\cos^2\frac{i}{2},\qquad I_1=-\frac{3}{8}\frac{m^2}{1-m}\sin i\cos^2\frac{i}{2},$$

$$-\quad 2I_2=-\frac{3}{4}m^2\sin i\cos i,\qquad I_2=\frac{3}{8}m^2\sin i\cos i,$$

$$-2\left(m+\frac{3}{4}m^2\cos i\right)I_3=-\frac{3}{4}m^2\sin i,\qquad I_3=\frac{3}{8}\frac{m\sin i}{1+\frac{3}{4}m\cos i},$$

$$-2(1+m)I_4=-\frac{3}{4}m^2\sin i\sin^2\frac{i}{2},\qquad I_4=\frac{3}{8}\frac{m^2}{1+m}\sin i\sin^2\frac{i}{2},$$

and

$$2(1-m)N_1=-\frac{3}{4}m^2\cos^2\frac{i}{2},\qquad N_1=-\frac{3}{8}\frac{m^2}{1-m}\cos^2\frac{i}{2},$$

$$2N_2=\frac{3}{4}m^2\cos i,\qquad N_2=\frac{3}{8}m^2\cos i,$$

$$2\left(m+\frac{3}{4}m^2\cos i\right)N_3=\frac{3}{4}m^2\cos i,\qquad N_3=\frac{3}{8}\frac{m\cos i}{1+\frac{3}{4}m\cos i},$$

$$2(1+m)N_4=\frac{3}{4}m^2\sin^2\frac{i}{2},\qquad N_4=\frac{3}{8}\frac{m^2}{1+m}\sin^2\frac{i}{2}.$$

Substitute above for the quantities I_3, N_3 and we get the second approximation to q,

$$q=\frac{3}{4}m^2\cos i-\frac{9}{32}m^3\cos^2 i+\frac{9}{64}m^3\sin^2 i.$$

It will be observed that I_3, N_3 are of lower order than the other coefficients, so that in order to obtain them correctly to the same order as the others we were obliged to retain small terms in $\frac{d\theta'}{dt}$ arising from the variability of N.

If we take the variable plane defined by the longitude of the node N_0 and the inclination i_0 as the plane to which the position of the Moon is referred, we have the latitude of the Moon above this plane

$$= \Delta i \sin\theta - \Delta N \sin i \cos\theta$$

$$= \frac{3}{8} m \sin i \cos^2\frac{i}{2}\left[\frac{1}{1+\frac{3}{4}m\cos i} + \frac{m}{1-m}\right]\sin(\theta - 2\theta')$$

$$-\frac{3}{8} m^2 \sin i \cos i \sin\theta$$

$$+\frac{3}{8} m \sin i \sin^2\frac{i}{2}\left[\frac{1}{1+\frac{3}{4}m\cos i} - \frac{m}{1+m}\right]\sin(\theta + 2\theta').$$

LECTURE XVII.

ON HILL'S METHOD OF TREATING THE LUNAR THEORY.

LET us suppose the Moon to move in the plane of the ecliptic, and let us refer its motion to rectangular axes in rotation, the rotation being such that the axis of x passes always through the mean position of the Sun; that is, the axes rotate with angular velocity n', and if we suppose the Sun describes a circular orbit about the origin, its coordinates are

$$x' = a', \quad y' = 0.$$

Let x, y be the coordinates of the Moon.

Then the disturbing forces of the Sun upon the Moon, relative to the Earth are

$$-\frac{m'}{\rho^2}\frac{x-a'}{\rho} - \frac{m'}{a'^2}, \quad -\frac{m'}{\rho^2}\frac{y}{\rho}$$

parallel to the axes of x and y respectively, where

$$\rho^2 = (x-a')^2 + y^2,$$

and the forces of the Earth on the Moon relative to the Earth are

$$-\frac{\mu}{r^2}\frac{x}{r}, \quad -\frac{\mu}{r^2}\frac{y}{r},$$

where $$r^2 = x^2 + y^2.$$

Now these forces may be written

$$\frac{d\Omega}{dx}, \quad \frac{d\Omega}{dy},$$

where $$\Omega = \frac{\mu}{r} + \frac{m'}{\rho} - \frac{m'x}{a'^2}.$$

But $$\frac{1}{\rho} = \frac{1}{a'} + \frac{x}{a'^2} + \frac{1}{a'^3}\left(x^2 - \frac{1}{2}y^2\right) + \frac{1}{a'^4}\left(x^3 - \frac{3}{2}xy^2\right) + \ldots.$$

Hence $$\Omega = \frac{\mu}{r} + \frac{m'}{a'^3}\left(x^2 - \frac{1}{2}y^2\right) + \frac{m'}{a'^4}\left(x^3 - \frac{3}{2}xy^2\right) + \ldots.$$

We have tacitly assumed the origin to be at the centre of the Earth; if we prefer to place it at the centre of gravity of the Earth and Moon, the necessary change is effected by multiplying the last terms, which correspond to the Parallactic Inequalities, by $(E-M)/(E+M)$.

Equating these forces to the accelerations of the Moon parallel to the coordinate axes, we have the equations of motion in the form

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} - n'^2x = \frac{d\Omega}{dx},$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} - n'^2y = \frac{d\Omega}{dy},$$

or, as they may be written,

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} = \frac{dR}{dx},$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} = \frac{dR}{dy},$$

where $$R = \Omega + \frac{1}{2}n'^2(x^2 + y^2)$$

$$= \frac{\mu}{r} + \frac{3}{2}n'^2x^2 + \frac{n'^2}{a'}\left(x^3 - \frac{3}{2}xy^2\right) + \ldots.$$

Now suppose we have found values of x and y which satisfy this pair of equations and which involve two arbitrary constants. This may be accomplished by taking assumed developments

$$x = \Sigma a_i \cos i(t + \gamma),$$

$$y = \Sigma b_i \sin i(t + \gamma),$$

substituting in the equations, and equating coefficients of the various terms. The solution found will include the Variation and the Parallactic Inequalities. Let it be required to amend this solution by the introduction of the remaining two arbitrary constants that are required for a complete solution.

Let the additional terms that we seek be δx, δy, which we shall suppose so small that their squares and products may be neglected, let us consider first the terms which are multiplied by the first power of one of the new arbitraries, the original particular solution corresponding to the case in which this arbitrary is zero.

Then δx, δy are determined by the equations

$$\frac{d^2\delta x}{dt^2} - 2n'\frac{d\delta y}{dt} = \frac{d^2R}{dx^2}\delta x + \frac{d^2R}{dxdy}\delta y + X,$$

$$\frac{d^2\delta y}{dt^2} + 2n'\frac{d\delta x}{dt} = \frac{d^2R}{dxdy}\delta x + \frac{d^2R}{dy^2}\delta y + Y,$$

where X, Y are supposed known functions of x, y or of t, and have been added here to include disturbing causes not allowed for in the above form of R.

Multiply the original equations by $\frac{dx}{dt}$, $\frac{dy}{dt}$ and add:

$$\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} = \frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt}$$

$$= \frac{dR}{dt},$$

since x, y are the only functions of t that R involves; whence

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2R + C,$$

where C is an arbitrary constant; this is the integral known as Jacobi's Integral.

Let us write

$$\frac{dx}{dt} = V\cos\phi, \quad \frac{dy}{dt} = V\sin\phi;$$

then we have $$V^2 = 2R + C;$$

and from the original equations themselves

$$\frac{dV}{dt} = \frac{d^2x}{dt^2}\cos\phi + \frac{d^2y}{dt^2}\sin\phi$$

$$= \left(\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt}\right)\cos\phi + \left(\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt}\right)\sin\phi$$

$$= \frac{dR}{dx}\cos\phi + \frac{dR}{dy}\sin\phi;$$

and

$$V\frac{d\phi}{dt} = -\frac{d^2x}{dt^2}\sin\phi + \frac{d^2y}{dt^2}\cos\phi$$

$$= -\left(\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt}\right)\sin\phi + \left(\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt}\right)\cos\phi$$

$$+ 2n'\left(\frac{dx}{dt}\cos\phi + \frac{dy}{dt}\sin\phi\right),$$

or

$$V\left(\frac{d\phi}{dt} + 2n'\right) = -\frac{dR}{dx}\sin\phi + \frac{dR}{dy}\cos\phi.$$

And from these, differentiating and substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, we get

$$\frac{d^2V}{dt^2} - V\frac{d\phi}{dt}\left(\frac{d\phi}{dt} + 2n'\right) = V\left[\frac{d^2R}{dx^2}\cos^2\phi + 2\frac{d^2R}{dxdy}\cos\phi\sin\phi + \frac{d^2R}{dy^2}\sin^2\phi\right],$$

$$V\frac{d^2\phi}{dt^2} + 2\frac{dV}{dt}\left(\frac{d\phi}{dt} + n'\right) = V\left[-\frac{d^2R}{dx^2}\sin\phi\cos\phi + \frac{d^2R}{dxdy}(\cos^2\phi - \sin^2\phi)\right.$$

$$\left. + \frac{d^2R}{dy^2}\sin\phi\cos\phi\right].$$

LECTURE XVIII.

ON HILL'S METHOD OF TREATING THE LUNAR THEORY, (*continued*).

THE equations for δx, δy are

$$\frac{d^2\delta x}{dt^2} - 2n'\frac{d\delta y}{dt} = \frac{d^2R}{dx^2}\delta x + \frac{d^2R}{dx\,dy}\delta y + X,$$

$$\frac{d^2\delta y}{dt^2} + 2n'\frac{d\delta x}{dt} = \frac{d^2R}{dx\,dy}\delta x + \frac{d^2R}{dy^2}\delta y + Y;$$

the equations for x, y are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} = \frac{dR}{dx},$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} = \frac{dR}{dy}.$$

Multiply the former pair by $\frac{dx}{dt}$, $\frac{dy}{dt}$ respectively, and the latter pair by $\frac{d\delta x}{dt}$, $\frac{d\delta y}{dt}$, and add all together; we get

$$\frac{dx}{dt}\frac{d^2\delta x}{dt^2} + \frac{d^2x}{dt^2}\frac{d\delta x}{dt} + \frac{dy}{dt}\frac{d^2\delta y}{dt^2} + \frac{d^2y}{dt^2}\frac{d\delta y}{dt}$$
$$= \left(\frac{d^2R}{dx^2}\frac{dx}{dt} + \frac{d^2R}{dx\,dy}\frac{dy}{dt}\right)\delta x + \left(\frac{d^2R}{dx\,dy}\frac{dx}{dt} + \frac{d^2R}{dy^2}\frac{dy}{dt}\right)\delta y$$
$$+ \frac{dR}{dx}\frac{d\delta x}{dt} + \frac{dR}{dy}\frac{d\delta y}{dt} + X\frac{dx}{dt} + Y\frac{dy}{dt}.$$

Now

$$\frac{d^2R}{dx^2}\frac{dx}{dt} + \frac{d^2R}{dx\,dy}\frac{dy}{dt} = \frac{d}{dt}\frac{dR}{dx},$$

$$\frac{d^2R}{dx\,dy}\frac{dx}{dt} + \frac{d^2R}{dy^2}\frac{dy}{dt} = \frac{d}{dt}\frac{dR}{dy}.$$

Then our equation may be integrated

$$\frac{dx}{dt}\frac{d\delta x}{dt}+\frac{dy}{dt}\frac{d\delta y}{dt}=\frac{dR}{dx}\delta x+\frac{dR}{dy}\delta y+T,$$

where

$$T=\int\left(X\frac{dx}{dt}+Y\frac{dy}{dt}\right)dt,$$

so that T is a known function of t, which involves an arbitrary constant.

Now let us assume

$$\delta x=v\cos\phi-w\sin\phi,$$

$$\delta y=v\sin\phi+w\cos\phi.$$

Substitute above for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d\delta x}{dt}$, $\frac{d\delta y}{dt}$; we find

$$V\left(\frac{dv}{dt}-w\frac{d\phi}{dt}\right)=\left(\frac{dR}{dx}\cos\phi+\frac{dR}{dy}\sin\phi\right)v+\left(-\frac{dR}{dx}\sin\phi+\frac{dR}{dy}\cos\phi\right)w+T.$$

But

$$\frac{dR}{dx}\cos\phi+\frac{dR}{dy}\sin\phi=\frac{dV}{dt},$$

$$-\frac{dR}{dx}\sin\phi+\frac{dR}{dy}\cos\phi=V\left(\frac{d\phi}{dt}+2n'\right).$$

Therefore

$$V\left(\frac{dv}{dt}-w\frac{d\phi}{dt}\right)=\frac{dV}{dt}v+V\left(\frac{d\phi}{dt}+2n'\right)w+T,$$

or

$$V\frac{dv}{dt}-\frac{dV}{dt}v=2wV\left(\frac{d\phi}{dt}+n'\right)+T,$$

whence

$$\frac{v}{V}=\int\frac{2}{V}\left(\frac{d\phi}{dt}+n'\right)wdt+\int\frac{T}{V^2}dt.$$

An arbitrary constant is included on the right. This equation shews that when w is known, v can be found; it remains to determine w.

Now by actual differentiation

$$\cos\phi\frac{d\delta x}{dt}+\sin\phi\frac{d\delta y}{dt}=\frac{dv}{dt}-w\frac{d\phi}{dt}$$

$$-\sin\phi\frac{d^2\delta x}{dt^2}+\cos\phi\frac{d^2\delta y}{dt^2}=\frac{d^2w}{dt^2}+2\frac{dv}{dt}\frac{d\phi}{dt}-w\left(\frac{d\phi}{dt}\right)^2+v\frac{d^2\phi}{dt^2}.$$

Also multiplying the differential equations for δx, δy by $-\sin\phi$, $\cos\phi$, respectively and adding

$$-\sin\phi\frac{d^2\delta x}{dt^2}+\cos\phi\frac{d^2\delta y}{dt^2}+2n'\left(\cos\phi\frac{d\delta x}{dt}+\sin\phi\frac{d\delta y}{dt}\right)$$

$$=\left(-\frac{d^2R}{dx^2}\sin\phi+\frac{d^2R}{dxdy}\cos\phi\right)\delta x+\left(-\frac{d^2R}{dxdy}\sin\phi+\frac{d^2R}{dy^2}\cos\phi\right)\delta y$$

$$-X\sin\phi+Y\cos\phi.$$

Substitute and we find

$$\frac{d^2w}{dt^2}+2\frac{dv}{dt}\frac{d\phi}{dt}-w\left(\frac{d\phi}{dt}\right)^2+v\frac{d^2\phi}{dt^2}+2n'\left(\frac{dv}{dt}-w\frac{d\phi}{dt}\right)$$

$$=\quad v\left[-\frac{d^2R}{dx^2}\sin\phi\cos\phi+\frac{d^2R}{dxdy}(\cos^2\phi-\sin^2\phi)+\frac{d^2R}{dy^2}\sin\phi\cos\phi\right]$$

$$+w\left[\frac{d^2R}{dx^2}\sin^2\phi\qquad-2\frac{d^2R}{dxdy}\sin\phi\cos\phi\qquad+\frac{d^2R}{dy^2}\cos^2\phi\right]$$

$$-X\sin\phi+Y\cos\phi.$$

Now we have seen

$$\frac{dv}{dt}=\frac{v}{V}\frac{dV}{dt}+2\left(\frac{d\phi}{dt}+n'\right)w+\frac{T}{V}.$$

Substitute for $2\left(\frac{d\phi}{dt}+n'\right)\frac{dv}{dt}$ on the left.

We get

$$\frac{d^2w}{dt^2}+v\left[\frac{d^2\phi}{dt^2}+\frac{2}{V}\frac{dV}{dt}\left(\frac{d\phi}{dt}+n'\right)\right]+w\left[4\left(\frac{d\phi}{dt}+n'\right)^2-\left(\frac{d\phi}{dt}\right)^2-2n'\frac{d\phi}{dt}\right]$$

$$+2\left(\frac{d\phi}{dt}+n'\right)\frac{T}{V}$$

$$=\quad v\left[-\frac{d^2R}{dx^2}\sin\phi\cos\phi+\frac{d^2R}{dxdy}(\cos^2\phi-\sin^2\phi)+\frac{d^2R}{dy^2}\sin\phi\cos\phi\right]$$

$$+w\left[\frac{d^2R}{dx^2}\sin^2\phi\qquad-2\frac{d^2R}{dxdy}\sin\phi\cos\phi\qquad+\frac{d^2R}{dy^2}\cos^2\phi\right]$$

$$-X\sin\phi+Y\cos\phi.$$

But by the equations proved at the end of Lecture XVII., the terms in v cancel one another, and we are left with the equation for w:

$$\frac{d^2w}{dt^2} + w\left[3\left(\frac{d\phi}{dt}\right)^2 + 6n'\frac{d\phi}{dt} + 4n'^2 - \frac{d^2R}{dx^2}\sin^2\phi + 2\frac{d^2R}{dxdy}\sin\phi\cos\phi - \frac{d^2R}{dy^2}\cos^2\phi\right]$$

$$= -2\left(\frac{d\phi}{dt} + n'\right)\frac{T}{V} - X\sin\phi + Y\cos\phi.$$

Or since

$$\cos\phi = \frac{1}{V}\frac{dx}{dt}, \qquad \sin\phi = \frac{1}{V}\frac{dy}{dt}$$

$$\frac{d\phi}{dt} + 2n' = \frac{1}{V}\left(-\frac{dR}{dx}\sin\phi + \frac{dR}{dy}\cos\phi\right)$$

the coefficient of w is

$$\frac{3}{V^4}\left(-\frac{dR}{dx}\frac{dy}{dt} + \frac{dR}{dy}\frac{dx}{dt}\right)^2 - 6\frac{n'}{V^2}\left(-\frac{dR}{dx}\frac{dy}{dt} + \frac{dR}{dy}\frac{dx}{dt}\right) + 4n'^2$$

$$-\frac{1}{V^2}\left\{\frac{d^2R}{dx^2}\left(\frac{dy}{dt}\right)^2 - 2\frac{d^2R}{dxdy}\frac{dx}{dt}\frac{dy}{dt} + \frac{d^2R}{dy^2}\left(\frac{dx}{dt}\right)^2\right\}$$

$= P$, say.

This function P is a known function of t; it may be seen that if

$$x = \Sigma a_i \cos i(t+\gamma),$$

$$y = \Sigma b_i \sin i(t+\gamma),$$

then P may be developed in the form

$$P = \Sigma A_i \cos i(t+\gamma).$$

Hence if we omit the terms X, Y, due to other disturbances not yet allowed for, the equation for w assumes the form

$$\frac{d^2w}{dt^2} + w[A_0 + A_1\cos(t+\gamma) + A_2\cos 2(t+\gamma) + \ldots] = 0.$$

This is identical in form with the equation treated in Lecture XIV., to find the motion of the node. The value of w may be found by the method there employed, and the value of v deduced from it.

2.

DEVELOPMENT OF A CERTAIN INFINITE DETERMINANT ARISING IN RELATION TO THE MOTION OF THE NODE OF THE MOON'S ORBIT.

[THE aim of the following pages will be made clear by an extract from a paper of Adams "On the Motion of the Moon's Node" (*Mon. Not.* XXXVIII. Nov. 1877; *Works,* Vol. I., p. 181). This paper was evoked by Dr G. W. Hill's now famous work on the Lunar Perigee. It appears that one part of the process invented by Dr Hill for evaluating the motion of the perigee had already been found by Adams to yield the motion of the node to a high order with rapid approximation. After describing, in the paper in question, his views of the most advantageous method of treating the Lunar Theory, and mentioning his early determination of the Variation terms, he continues:—"In the next place I proceeded to consider the inequalities of latitude, or rather the disturbed value of the Moon's co-ordinate perpendicular to the Ecliptic, omitting the eccentricities as before, and taking account only of the first power of γ.

"In this case the differential equation for finding z presents itself naturally in the form to which Mr Hill reduces, with so much skill, the equations depending on the first power of the eccentricity of the Moon's orbit.

"In solving this equation I fell upon the same infinite determinant as that considered by Mr Hill, and I developed it in a similar manner in a series of powers and products of small quantities, the coefficient of each such term being given in a finite form.

"The terms of the fourth order in the determinant were thus obtained by me on the 26th December, 1868. I then laid aside the further investigation of this subject for a considerable time, but resumed it in 1874

and 1875, and on the 2nd of December in the latter year I carried the approximation to the value of the determinant as far as terms of the twelfth order, or to the same extent as that which has been attained by Mr Hill......

"The equation which I had obtained by equating the above-mentioned determinant to zero differed in form from Mr Hill's, and on making the reductions required to make the two results immediately comparable, I found that there was an agreement between them except in one of the twelfth order. On examining my work I found that this arose from a simple error of transcription in a portion of my work, and that when this had been rectified my result was in entire accordance with Mr Hill's.

"The calculations by which I have found the value of the determinant are very different in detail from those required by Mr Hill's method, and appear to be considerably more laborious. I have not yet had time to copy out and arrange the details of the calculations from my old papers, but I hope soon to do so, thinking that they may not be without interest for the Society."

This intention was not fulfilled; the details referred to appear for the first time in the following pages.]

[With the date 26 Dec. 1868 we find the following.]

If we call $\dfrac{1}{r^3}+\dfrac{m'}{r'^3}=1+a_0+a_1\cos kt$, where $k=2-2m$, the equation for finding z is*

$$\frac{d^2z}{dt^2}+z\,(1+a_0+a_1\cos kt)=0.$$

If now

$$\begin{aligned} z=c_0\sin gt+c_1\ \sin(g+k)\,t+c_2\ \sin(g+2k)\,t+\dots \\ +c_{-1}\sin(g-k)\,t+c_{-2}\sin(g-2k)\,t+\dots \end{aligned}$$

we obtain by equating to zero the coefficient of each sine in the result of substituting in the above differential equation

....

$$0=\dots[(g-2k)^2-(1+a_0)]\,c_{-2}-\frac{1}{2}a_1c_{-1}$$

$$0=\qquad -\frac{1}{2}a_1c_{-2}\qquad +[(g-k)^2-(1+a_0)]\,c_{-1}\qquad -\frac{1}{2}a_1c_0$$

* [See *Lecture XIV. on Lunar Theory*, p. 64.]

$$0 = \qquad -\frac{1}{2}a_1c_{-1} \qquad +[g^2-(1+a_0)]c_0 \quad -\frac{1}{2}a_1c_1$$

$$0 = \qquad -\frac{1}{2}a_1c_0+[(g+k)^2-(1+a_0)]c_1-\frac{1}{2}a_1c_2$$

$$0 = \qquad -\frac{1}{2}a_1c_1+[(g+2k)^2-(1+a_0)]c_2\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Or if
$$\frac{1+a_0}{k^2}=\kappa^2,\quad \frac{1}{2}\frac{a_1}{k^2}=a,\quad \frac{g}{k}=\gamma,$$
the equations become

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0=\ldots[(\gamma-2)^2-\kappa^2]c_{-2}-ac_{-1}\ldots$$

$$0= \qquad -ac_{-2} \quad +[(\gamma-1)^2-\kappa^2]c_{-1}-ac_0$$

$$0= \qquad -ac_{-1} \quad +[\gamma^2-\kappa^2]c_0-ac_1$$

$$0= \qquad -ac_0+[(\gamma+1)^2-\kappa^2]c_1-ac_2$$

$$0= \qquad -ac_1+[(\gamma+2)^2-\kappa^2]c_2-\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now the determinant which equated to zero gives the values of γ, is, omitting terms of the order a^4,

$$1-\frac{a^2}{[(\gamma-2)^2-\kappa^2][(\gamma-1)^2-\kappa^2]}-\frac{a^2}{[(\gamma-1)^2-\kappa^2][\gamma^2-\kappa^2]}-\frac{a^2}{[\gamma^2-\kappa^2][(\gamma+1)^2-\kappa^2]}$$
$$-\text{\&c. ad inf.},$$

whence we find on separating each term into partial fractions

$$0=1+\frac{a^2}{\kappa(2\kappa-1)(2\kappa+1)}$$
$$\left[\cdots\frac{1}{\gamma-3-\kappa}+\frac{1}{\gamma-2-\kappa}+\frac{1}{\gamma-1-\kappa}+\frac{1}{\gamma-\kappa}+\frac{1}{\gamma+1-\kappa}+\cdots\right.$$
$$\left.\cdots-\frac{1}{\gamma-3+\kappa}-\frac{1}{\gamma-2+\kappa}-\frac{1}{\gamma-1+\kappa}-\frac{1}{\gamma+\kappa}-\frac{1}{\gamma+1+\kappa}-\cdots\right].$$

But we have

$$\cot\theta=\frac{1}{\theta}+\frac{1}{\theta+\pi}+\frac{1}{\theta+2\pi}+\cdots$$
$$+\frac{1}{\theta-\pi}+\frac{1}{\theta-2\pi}+\cdots$$

Hence the above equation becomes

$$0=1+\frac{a^2}{\kappa(2\kappa-1)(2\kappa+1)}\{\pi\cot(\gamma-\kappa)\pi-\pi\cot(\gamma+\kappa)\pi\},$$

which gives

$$\cos 2\gamma\pi=\cos 2\kappa\pi+\frac{2\pi a^2}{\kappa(2\kappa-1)(2\kappa+1)}\sin 2\kappa\pi.$$

(26 Dec. 68.)

[On comparing the above with the paper "On the Motion of the Moon's Node, &c." *Mon. Not.* Nov. 1877, it will be seen that in addition to the convention there adopted as to the unit of distance, the unit of time is so chosen that $n=1$; also $2\kappa=q$, $4a=q_1$, $2\gamma=g$. In the subsequent work this change of notation will be introduced.

The subject was resumed in 1874, when we find the following entries in the Diary:—

Feb. 4.—Both yesterday and this morning while in bed thought over the mode of treating the linear differential equations which occur in my way of investigating the lunar inequalities. Think I see my way. In the morning worked at one part of the subject.

Feb. 7.—Worked nearly all the morning at formation of terms of 4th order of my determinant. In evening finished calculation of terms of 4th order.

Feb. 10.—Thought over method of treating mean motion of apse similarly to that of node by means of a differential equation of 2nd order. Began operations by transforming fundamental equations into one with $\phi=\theta-n't$ for independent variable.

Feb. 11.—Went on with my investigation so as to form the differential equation of 3rd order in Δv and the theory of its reduction to the 2nd order.

Feb. 23.—Thought of a simpler mode of treating the differential equation for $\Delta\theta$. In evening worked out reduction of equation to 2nd order in another way.

With respect to the method adopted for developing the determinant to higher orders, we notice that we may either proceed entirely along the diagonal, which gives unity, or forming a minor determinant with any number of consecutive constituents of the diagonal of the infinite

determinant for its diagonal, we may replace the unit which the diagonal of this minor would contribute by any other element of the minor.]

If (p) denote the term

$$\frac{\alpha^2}{[(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]}=\frac{q_1^2}{[(g+2p)^2-q^2][(g+2p+2)^2-q^2]},$$

and (p, q) the product of two such terms, in which we suppose $p \mp q$, then the terms of the fourth order in the determinant will be given by

$$\Sigma(p, q)-\Sigma(p, p+1),$$

and it is easy to see that this is

$$\frac{1}{2}\{\Sigma(p)\}^2-\frac{1}{2}\Sigma(p, p)-\Sigma(p, p+1).$$

We then find by separating into partial fractions the different expressions that occur

$$\Sigma(p, p)=\frac{q_1^4}{32q^2}\frac{q^2+1}{(q^2-1)^2}\left\{\Sigma\frac{1}{(g+2p-q)^2}+\Sigma\frac{1}{(g+2p+q)^2}\right\}$$
$$-\frac{q_1^4(5q^2-1)}{32q^3(q^2-1)^3}\left\{\Sigma\frac{1}{g+2p-q}\quad-\Sigma\frac{1}{g+2p+q}\quad\right\},$$

and $\Sigma(p,\ p+1)=\Sigma\dfrac{q_1^4}{[(g+2p-2)^2-q^2][(g+2p)^2-q^2]^2[(g+2p+2)^2-q^2]}$

$$=\qquad-\frac{q_1^4}{64q^2(q^2-1)}\left\{\Sigma\frac{1}{(g+2p-q)^2}+\Sigma\frac{1}{(g+2p+q)^2}\right\}$$
$$-\frac{q_1^4(5q^2-2)}{32q^3(q^2-4)(q^2-1)^2}\left\{\Sigma\frac{1}{g+2p-q}\quad-\Sigma\frac{1}{g+2p+q}\quad\right\}.$$

[The method by which these expressions were obtained involved considerable labour,—probably the reason why the developments were carried no further at this date; for another method, see below.]

Now substitute

$$\Sigma\frac{1}{g+2p-q}\quad=\frac{\pi}{2}\cot(g-q)\frac{\pi}{2};\qquad\qquad \Sigma\frac{1}{g+2p+q}\quad=\frac{\pi}{2}\cot(g+q)\frac{\pi}{2};$$

$$\Sigma\frac{1}{(g+2p-q)^2}=\frac{\pi^2}{4}\operatorname{cosec}^2(g-q)\frac{\pi}{4};\qquad\qquad \Sigma\frac{1}{(g+2p+q)^2}=\frac{\pi^2}{4}\operatorname{cosec}^2(g+q)\frac{\pi}{2},$$

$$=\frac{\pi^2}{4}\left(1+\cot^2(g-q)\frac{\pi}{4}\right);\qquad\qquad =\frac{\pi^2}{4}\left(1+\cot^2(g+q)\frac{\pi}{4}\right).$$

Thus

$$\Sigma(p, p) = \frac{\pi^2 q_1^4 (q^2+1)}{128 q^2 (q^2-1)^2} \left\{\operatorname{cosec}^2 (g-q)\frac{\pi}{2} + \operatorname{cosec}^2 (g+q)\frac{\pi}{2}\right\}$$
$$- \frac{\pi q_1^4 (5q^2-1)}{64 q^3 (q^2-1)^3} \left\{\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right\};$$

$$\Sigma(p, p+1) = -\frac{\pi^2 q_1^4}{256 q^2 (q^2-1)} \left\{\operatorname{cosec}^2 (g-q)\frac{\pi}{2} + \operatorname{cosec}^2 (g+q)\frac{\pi}{2}\right\}$$
$$- \frac{\pi q_1^4 (5q^2-2)}{64 q^3 (q^2-4)(q^2-1)^2} \left\{\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right\};$$

and

$$\{\Sigma(p)\}^2 = \frac{\pi^2 q_1^4}{64 q^2 (q^2-1)^2} \left\{\cot^2 (g-q)\frac{\pi}{2} + \cot^2 (g+q)\frac{\pi}{2} - 2\cot (g-q)\frac{\pi}{2}\cot (g+q)\frac{\pi}{2}\right\}$$
$$= \frac{\pi^2 q_1^4}{64 q^2 (q^2-1)^2} \left\{\operatorname{cosec}^2 (g-q)\frac{\pi}{2} + \operatorname{cosec}^2 (g+q)\frac{\pi}{2}\right.$$
$$\left. - 2\cot q\pi \left(\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right)\right\}.$$

Hence

$$\Sigma(p, q) - \Sigma(p, p+1) = \frac{\pi q_1^4 (15q^4 - 35q^2 + 8)}{128 q^3 (q^2-1)^3 (q^2-4)} \left\{\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right\},$$

the terms in $\operatorname{cosec}^2 (g-q)\frac{\pi}{2} + \operatorname{cosec}^2 (g+q)\frac{\pi}{2}$ disappearing, as might have been expected.

Hence the determinant, to the fourth order in a, or the eighth order in m, equated to zero gives

$$0 = 1 + \frac{\pi q_1^2}{8q(q^2-1)} \left\{\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right\}$$
$$+ \frac{\pi q_1^4 (15q^4 - 35q^2 + 8)}{128 q^3 (q^2-1)^3 (q^2-4)} \left\{\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right\}$$
$$- \frac{\pi^2 q_1^4 \cot q\pi}{64 q^2 (q^2-1)^2} \left\{\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2}\right\}.$$

But

$$\cot (g-q)\frac{\pi}{2} - \cot (g+q)\frac{\pi}{2} = \frac{2\sin q\pi}{\cos q\pi - \cos g\pi};$$

therefore we have

$$\cos g\pi = \cos q\pi \left\{1 - \frac{\pi^2 q_1^4}{32q^2(q^2-1)^2}\right\}$$

$$+ \sin q\pi \left\{\frac{\pi q_1^2}{4q(q^2-1)} + \frac{\pi q_1^4 (15q^4 - 35q^2 + 8)}{64q^3(q^2-1)^3(q^2-4)}\right\}.$$

(9 Feb. 74.)

[The work was resumed in 1875 when we find the following.]

We will now try the simpler mode of finding $\Sigma(p, p)$, &c., which occurred to me on the evening of Wednesday November 3, when on my way to attend the Ray meeting.

We have

$$(p) = \frac{a^2}{[(\gamma+p)^2 - \kappa^2][(\gamma+p+1)^2 - \kappa^2]},$$

so that

$$(p, p) = \frac{a^4}{[(\gamma+p)^2 - \kappa^2]^2[(\gamma+p+1)^2 - \kappa^2]^2}.$$

Let $\gamma + p - \kappa = x$; then

$$(p, p) = a^4 x^{-2}(2\kappa + x)^{-2}(1+x)^{-2}(2\kappa + 1 + x)^{-2}.$$

Develope this in ascending powers of x,

$$\text{coefficient of } x^{-2} = \frac{a^4}{4\kappa^2(2\kappa+1)^2},$$

$$\text{coefficient of } x^{-1} = \frac{a^4}{4\kappa^2(2\kappa+1)^2}\left[-\frac{2}{2\kappa} - 2 - \frac{2}{2\kappa+1}\right] = -\frac{a^4(4\kappa^2 + 6\kappa + 1)}{4\kappa^3(2\kappa+1)^3}.$$

Next let $\gamma + p + 1 + \kappa = x$; then

$$(p, p) = a^4(x-1)^{-2}(x - 2\kappa - 1)^{-2}x^{-2}(x - 2\kappa)^{-2}.$$

In this

$$\text{coefficient of } x^{-2} = \frac{a^4}{4\kappa^2(2\kappa+1)^2},$$

$$\text{coefficient of } x^{-1} = \frac{a^4}{4\kappa^2(2\kappa+1)^2}\left[2 + \frac{2}{2\kappa+1} + \frac{2}{2\kappa}\right] = \frac{a^4(4\kappa^2 + 6\kappa + 1)}{4\kappa^3(2\kappa+1)^3}.$$

Also it is evident that the other two pairs of terms merely differ from the above in having $-\kappa$ in place of κ.

Again

$$\Sigma \frac{1}{\gamma+p+1+\kappa} = \Sigma \frac{1}{\gamma+p+\kappa},$$

$$\Sigma \frac{1}{\gamma+p+1-\kappa} = \Sigma \frac{1}{\gamma+p-\kappa}.$$

Hence we have

$$\begin{aligned}\Sigma(p, p) = &\left\{ \frac{a^4}{4\kappa^2(2\kappa+1)^2} + \frac{a^4}{4\kappa^2(2\kappa-1)^2} \right\} \Sigma \frac{1}{(\gamma+p-\kappa)^2} \\ &+ \left\{ \frac{a^4}{4\kappa^2(2\kappa+1)^2} + \frac{a^4}{4\kappa^2(2\kappa-1)^2} \right\} \Sigma \frac{1}{(\gamma+p+\kappa)^2} \\ &+ \left\{ -\frac{a^4(4\kappa^2+6\kappa+1)}{4\kappa^3(2\kappa+1)^3} + \frac{a^4(4\kappa^2-6\kappa+1)}{4\kappa^3(2\kappa-1)^3} \right\} \Sigma \frac{1}{(\gamma+p-\kappa)} \\ &+ \left\{ \frac{a^4(4\kappa^2+6\kappa+1)}{4\kappa^3(2\kappa+1)^3} - \frac{a^4(4\kappa^2-6\kappa+1)}{4\kappa^3(2\kappa-1)^3} \right\} \Sigma \frac{1}{(\gamma+p+\kappa)},\end{aligned}$$

or

$$\begin{aligned}\Sigma(p, p) = &\frac{a^4(4\kappa^2+1)}{2\kappa^2(4\kappa^2-1)^2} \left\{ \Sigma \frac{1}{(\gamma+p-\kappa)^2} + \Sigma \frac{1}{(\gamma+p+\kappa)^2} \right\} \\ &- \frac{a^4(20\kappa^2-1)}{2\kappa^3(4\kappa^2-1)^3} \left\{ \Sigma \frac{1}{\gamma+p-\kappa} - \Sigma \frac{1}{\gamma+p+\kappa} \right\},\end{aligned}$$

which agrees with the result of p. 89.

[This method was employed throughout. The details of the subsequent work will not generally be given.

To find the terms of the sixth order in a or q_1, or of the twelfth order in m, we must find the sum of the products of the quantities (p) three together, no quantity (p) being multiplied either by itself or by a consecutive quantity, that is to say we require

$$\Sigma(p, q, r) - \Sigma(p, p+1)\Sigma(p) + \Sigma(p, p, p+1) + \Sigma(p-1, p, p) + \Sigma(p-1, p, p+1),$$

where p, q, r are all different, and therefore

$$\Sigma(p, q, r) = \frac{1}{6}\{\Sigma(p)\}^3 - \frac{1}{2}\Sigma(p, p)\Sigma(p) + \frac{1}{3}\Sigma(p, p, p).]$$

We have

$$(p, p, p) = \frac{q_1^6}{[(g+2p)^2-q^2]^3[(g+2p+2)^2-q^2]^3}.$$

Whence as above

$$\Sigma(p,\ p,\ p) = -\frac{q_1^6(3q^2+1)}{256q^3(q^2-1)^3}\left\{\Sigma\frac{1}{(g+2p-q)^3} - \Sigma\frac{1}{(g+2p+q)^3}\right\}$$
$$-\frac{3q_1^6(q^2+1)(q^4-6q^2+1)}{512q^4(q^2-1)^4}\left\{\Sigma\frac{1}{(g+2p-q)^2} + \Sigma\frac{1}{(g+2p+q)^2}\right\}$$
$$-\frac{3q_1^6(21q^4-6q^2+1)}{512q^5(q^2-1)^5}\left\{\Sigma\frac{1}{(g+2p-q)} - \Sigma\frac{1}{(g+2p+q)}\right\}.$$

Let us call

$$\Sigma\frac{1}{(g+2p-q)} - \Sigma\frac{1}{(g+2p+q)} = \frac{\pi}{2}\left(\cot(g-q)\frac{\pi}{2} - \cot(g+q)\frac{\pi}{2}\right) = A,$$

$$\Sigma\frac{1}{(g+2p-q)^2} + \Sigma\frac{1}{(g+2p+q)^2} = \frac{\pi^2}{4}\left(\operatorname{cosec}^2(g-q)\frac{\pi}{2} + \operatorname{cosec}^2(g+q)\frac{\pi}{2}\right) = B,$$

$$\Sigma\frac{1}{(g+2p-q)^3} - \Sigma\frac{1}{(g+2p+q)^3} = \frac{\pi^3}{8}\left(\cot(g-q)\frac{\pi}{2}\operatorname{cosec}^2(g-q)\frac{\pi}{2}\right.$$
$$\left. - \cot(g+q)\frac{\pi}{2}\operatorname{cosec}^2(g+q)\frac{\pi}{2}\right) = C.$$

Then we have

$$\Sigma(p) = -\frac{q_1^2}{4q(q^2-1)}A, \quad \Sigma(p,\ p) = \frac{q_1^4(q^2+1)}{32q^2(q^2-1)^2}B - \frac{q_1^4(5q^2-1)}{32q^3(q^2-1)^3}A;$$

hence

$$\Sigma(p,\ q,\ r) = -\frac{q_1^6}{384q^3(q^2-1)^3}A^3 + \frac{q_1^6(q^2+1)}{256q^3(q^2-1)^3}AB - \frac{q_1^6(5q^2-1)}{256q^4(q^2-1)^4}A^2$$
$$-\frac{q_1^6(3q^2+1)}{768q^3(q^2-1)^3}C - \frac{q_1^6(q^2+1)(q^4-6q^2+1)}{512q^4(q^2-1)^4}B - \frac{q_1^6(21q^4-6q^2+1)}{512q^5(q^2-1)^5}A.$$

And we further find

$$\Sigma(p,\ p,\ p+1) + \Sigma(p-1,\ p,\ p) + \Sigma(p-1,\ p,\ p+1)$$
$$= \frac{q_1^6}{256q^3(q^2-1)^2}C + \frac{q_1^6(q^6-8q^4+37q^2-12)}{512q^4(q^2-1)^3(q^2-4)}B$$
$$-\frac{3q_1^6(63q^8-586q^6+1307q^4-640q^2+144)}{512q^5(q^2-1)^4(q^2-4)^2(q^2-9)}A,$$

and together with

$$\Sigma(p,\ p+1)\,\Sigma(p) = \frac{q_1^6}{256q^3(q^2-1)^2}AB + \frac{q_1^6(5q^2-2)}{128q^4(q^2-1)^3(q^2-4)}A^2$$

we have the materials for forming the expression

$$\Sigma(p,\ q,\ r)-\Sigma(p,\ p+1)\,\Sigma(p)+\Sigma(p,\ p,\ p+1)+\Sigma(p-1,\ p,\ p)$$
$$+\Sigma(p-1,\ p,\ p+1).$$

Before doing so, substitute

$$B=\ A^2+\ \pi\cot q\pi A,$$
$$2C=2A^3+3\pi\cot q\pi A^2-\pi^2 A.$$

Then in the resulting expression,

$$\text{coefficient of } A^3=-\frac{q_1^6}{384q^3(q^2-1)^3}+\frac{q_1^6(q^2+1)}{256q^3(q^2-1)^3}-\frac{q_1^6(3q^2+1)}{768q^3(q^2-1)^3}$$
$$-\frac{q_1^6}{256q^3(q^2-1)^2}+\frac{q_1^6}{256q^3(q^2-1)^2}$$
$$=\ 0,$$

$$\text{coefficient of } A^2=\ \frac{q_1^6(q^2+1)}{256q^3(q^2-1)^3}\pi\cot q\pi-\frac{q_1^6(5q^2-1)}{256q^4(q^2-1)^4}$$
$$-\frac{q_1^6(3q^2+1)}{512q^3(q^2-1)^3}\pi\cot q\pi-\frac{q_1^6(q^2+1)(q^4-6q^2+1)}{512q^4(q^2-1)^4}$$
$$-\frac{q_1^6}{256q^3(q^2-1)^2}\pi\cot q\pi-\frac{q_1^6(5q^2-2)}{128q^4(q^2-1)^3(q^2-4)}$$
$$+\frac{3q_1^6}{512q^3(q^2-1)^2}\pi\cot q\pi-\frac{q_1^6(q^6-8q^4+37q^2-12)}{512q^4(q^2-1)^3(q^2-4)}$$
$$=\ \pi\cot q\pi\frac{q_1^6}{512q^3(q^2-1)^3}[2q^2+2-3q^2-1-2q^2+2+3q^2-3]$$
$$+\frac{q_1^6}{512q^4(q^2-1)^4(q^2-4)}[-2(5q^2-1)(q^2-4)$$
$$-(q^2+1)(q^4-6q^2+1)(q^2-4)$$
$$-4(5q^2-2)(q^2-1)+(q^6-8q^4+37q^2-12)(q^2-1)]$$
$$=\ 0,$$

and these supplementary terms of the determinant reduce to

$$Aq_1^6\left\{\frac{15q^4-35q^2+8}{256q^4(q^2-1)^4(q^2-4)}\pi\cot q\pi+\frac{\pi^2}{384q^3(q^2-1)^3}\right.$$
$$\left.-\frac{105q^{10}-1155q^8+3815q^6-4705q^4+1652q^2-288}{256q^5(q^2-1)^5(q^2-4)^2(q^2-9)}\right\}.$$

Adding these terms, with a negative sign, to the value of the determinant found on p. 90, and completing the solution as it is there completed, we find

$$\cos g\pi = \cos q\pi \left\{1 - \frac{\pi^2 q_1^4}{32q^2(q^2-1)^2} - \frac{15q^4 - 35q^2 + 8}{256q^4(q^2-1)^4(q^2-4)}\pi^2 q_1^6\right\}$$
$$+ \sin q\pi \left\{\frac{\pi q_1^2}{4q(q^2-1)} + \frac{15q^4 - 35q^2 + 8}{64q^3(q^2-1)^3(q^2-4)}\pi q_1^4 - \frac{\pi^3 q_1^6}{384q^3(q^2-1)^3}\right.$$
$$\left. + \frac{105q^{10} - 1155q^8 + 3815q^6 - 4705q^4 + 1652q^2 - 288}{256q^5(q^2-1)^5(q^2-4)^2(q^2-9)}\pi q_1^6\right\}.$$

This value of $\cos g\pi$ may be put under the form

$$\cos g\pi = \cos q\pi + \sin q\pi \left\{\frac{\pi q_1^2}{4q(q^2-1)} + \frac{15q^4 - 35q^2 + 8}{64q^3(q^2-1)^3(q^2-4)}\pi q_1^4 - \frac{\pi^2 q_1^4}{32q^2(q^2-1)^2}\cot q\pi\right.$$
$$- \frac{\pi^3 q_1^6}{384q^3(q^2-1)^3} + \frac{105q^{10} - 1155q^8 + 3815q^6 - 4705q^4 + 1652q^2 - 288}{256q^5(q^2-1)^5(q^2-4)^2(q^2-9)}\pi q_1^6$$
$$\left. - \frac{15q^4 - 35q^2 + 8}{256q^4(q^2-1)^4(q^2-4)}\pi^2 q_1^6 \cot q\pi\right\}.$$

We will now substitute in it the value of $\cot q\pi$ in terms of q. We have, namely,

$$\cot q\pi = -\cot(1-q)\pi = \frac{1}{(q-1)\pi} - \frac{1}{3}(q-1)\pi - \frac{1}{45}(q-1)^3\pi^3 - \frac{2}{945}(q-1)^5\pi^5$$
$$- \frac{1}{4725}(q-1)^7\pi^7 - \frac{2}{93555}(q-1)^9\pi^9 - \dots$$

Then we have, picking out terms of highest negative power of q,

in coefficient of πq_1^4,
$$\frac{1}{64q^3(q^2-1)^3(q^2-4)}\left[15q^4 - 35q^2 + 8 - 2q(q+1)(q^2-4)\right]$$
$$= \frac{1}{64q^3(q^2-1)^3(q^2-4)}\left[13q^4 - 2q^3 - 27q^2 + 8q + 8\right]$$
$$= \frac{(q-1)^2(13q^2 + 24q + 8)}{64q^3(q^2-1)^3(q^2-4)} = \frac{13q^2 + 24q + 8}{64q^3(q-1)(q+1)^3(q^2-4)},$$

in coefficient of πq_1^6,
$$\frac{1}{256q^5(q^2-1)^5(q^2-4)^2(q^2-9)}\left[105q^{10} - 1155q^8 + 3815q^6\right.$$
$$\left. - 4705q^4 + 1652q^2 - 288 - (15q^4 - 35q^2 + 8)q(q+1)(q^2-4)(q^2-9)\right],$$

$$\text{numerator} = 90q^{10} - 15q^9 - 925q^8 + 230q^7 + 2812q^6 - 1003q^5$$
$$- 3341q^4 + 1364q^3 + 1364q^2 - 288q - 288$$
$$= (q-1)^4(90q^6 + 345q^5 - 85q^4 - 1820q^3$$
$$- 2668q^2 - 1440q - 288),$$

and

$$\text{expression} = \frac{90q^6 + 345q^5 - 85q^4 - 1820q^3 - 2668q^2 - 1440q - 288}{256q^5(q-1)(q+1)^5(q^2-4)^2(q^2-9)}.$$

Also,

$$\text{in coefficient of } \pi^3 q_1^6, \quad \frac{1}{768q^4(q^2-1)^3(q+1)(q^2-4)}[-2q(q+1)(q^2-4) + 15q^4 - 35q^2 + 8]$$

$$= \frac{13q^2 + 24q + 8}{768q^4(q-1)(q+1)^4(q^2-4)}.$$

In this way we see that no higher power of $q-1$ than the first occurs in the denominator of the multiplier of $\sin q\pi$. In the case from which we have derived our equations, this is a small quantity. Hence the degree of approximation attained by the above formula is most satisfactory.

[It will be observed that in writing originally (p. 86)

$$\frac{1}{r^3} + \frac{m'}{r'^3} = 1 + a_0 + a_1 \cos kt,$$

we are ignoring orders of m above the second, corresponding to the fourth order in our determinant. We now proceed to include the neglected terms.]

Let the equation for z be

$$0 = \frac{d^2z}{dt^2} + z(1 + a_0 + a_1 \cos kt + a_2 \cos 2kt + a_3 \cos 3kt + \dots).$$

Assume as before

$$z = c_0 \sin gt + c_1 \sin(g+k)t + c_2 \sin(g+2k)t + \dots$$
$$+ c_{-1} \sin(g-k)t + c_{-2} \sin(g-2k)t + \dots$$

Then by equating to zero the coefficient of each sine in the result of substitution in the above differential equation we obtain

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0 = \dots [(g-2k)^2 - (1+a_0)]c_{-2} - \frac{1}{2}a_1c_{-1} - \frac{1}{2}a_2c_0 - \frac{1}{2}a_3c_1 - \frac{1}{2}a_4c_2 - \dots$$

$$0 = \dots \quad -\frac{1}{2}a_1c_{-2} + [(g-k)^2 - (1+a_0)]c_{-1} - \frac{1}{2}a_1c_0 - \frac{1}{2}a_2c_1 - \frac{1}{2}a_3c_2 - \dots$$

$$0 = \dots \quad -\frac{1}{2}a_2c_{-2} - \frac{1}{2}a_1c_{-1} + [g^2 - (1+a_0)]c_0 - \frac{1}{2}a_1c_1 - \frac{1}{2}a_2c_2 - \dots$$

$$0 = \ldots \quad -\frac{1}{2}a_3c_{-2} - \frac{1}{2}a_2c_{-1} - \frac{1}{2}a_1c_0 + [(g+k)^2 - (1+a_0)]\,c_1 - \frac{1}{2}a_1c_2 - \ldots$$

$$0 = \ldots \quad -\frac{1}{2}a_4c_{-2} - \frac{1}{2}a_3c_{-1} - \frac{1}{2}a_2c_0 - \frac{1}{2}a_1c_1 + [(g+2k)^2 - (1+a_0)]\,c_2 - \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Or if, as before, we put

$$\frac{g}{k} = \gamma = \frac{\mathrm{g}}{2}, \quad \frac{1+a_0}{k^2} = \kappa^2 = \frac{q^2}{4}, \quad \frac{1}{2}\frac{a_1}{k^2} = a = \frac{q_1}{4}, \quad \frac{1}{2}\frac{a_2}{k^2} = b = \frac{q_2}{4}, \text{ etc.,}$$

the equations become

$$\begin{array}{llllll}
 & & \ldots\ldots\ldots\ldots\ldots\ldots & & & \\
0 = \ldots & [(\gamma-2)^2-\kappa^2]\,c_{-2} - ac_{-1} - bc_0 - cc_1 - dc_2 - \ldots & & & & \\
0 = \ldots & -ac_{-2} & +[(\gamma-1)^2-\kappa^2]\,c_{-1} - ac_0 - bc_1 - cc_2 - \ldots & & & \\
0 = \ldots & -bc_{-2} & -ac_{-1} & +[\gamma^2-\kappa^2]\,c_0 - ac_1 - bc_2 - \ldots & & \\
0 = \ldots & -cc_{-2} & -bc_{-1} & -ac_0 & +[(\gamma+1)^2-\kappa^2]\,c_1 - ac_2 - \ldots & \\
0 = \ldots & -dc_{-2} & -cc_{-1} & -bc_0 & -ac_1 & +[(\gamma+2)^2-\kappa^2]\,c_2 - \ldots \\
 & & \ldots\ldots\ldots\ldots\ldots\ldots & & &
\end{array}$$

Divide each of these equations by the coefficient of the term in the diagonal line, and form as before the determinant that gives the value of γ.

With a view to determining which terms it is necessary to consider, write down all the elements of the minors up to those of four rows and columns, together with the powers of a, b, c, &c. which they involve. In the notation below the position of each figure represents the row it is drawn from, and its value represents the column, thus 123 represents the diagonal element for three rows and columns. The sign of each term is also given.

	Two			Three			Four			Four	
(1)	12	1	(3)	123	1	(9)	1234	1	(21)	1423	a^2b
(2)	−21	a^2	(4)	−213	a^2	(10)	−2134	a^2	(22)	−2413	a^2b^2
			(5)	−132	a^2	(11)	−1324	a^2	(23)	−1432	b^2
			(6)	231	a^2b	(12)	2314	a^2b	(24)	2431	abc
			(7)	312	a^2b	(13)	3124	a^2b	(25)	3412	b^4
			(8)	−321	b^2	(14)	−3214	b^2	(26)	−3421	ab^2c
						(15)	−1243	a^2	(27)	−4123	a^3c
						(16)	2143	a^4	(28)	4213	abc
						(17)	1342	a^2b	(29)	4132	abc
						(18)	−2341	a^3c	(30)	−4231	c^2
						(19)	−3142	a^2b^2	(31)	−4312	ab^2c
						(20)	3241	abc	(32)	4321	a^2c^2

We have already considered the following

(1) = (3) = (9),

(2) = (4) = (5) = (10) = (11) = (15),

(16).

We must therefore now consider

(6) = (12) = (17)	involving	a^2b,	
(7) = (13) = (21)	,,	a^2b,	
(8) = (14) = (23)	,,	b^2,	
(30)	,,	c^2,	with two constituents of the diagonal,
(20), (28)	,,	abc,	with constituent 2 of the diagonal,
(24), (29)	,,	abc,	,, ,, ,, 3 ,, ,, ,,
(18), (27)	,,	a^3c,	with no constituent of the diagonal,
(19), (22)	,,	a^2b^2,	,, ,, ,, ,, ,, ,,

We thus omit (25), (26), (31), (32) each of which is of the eighth order in a; further it is easily seen that no new element involving five constituents not of the diagonal is of as low order as a^6 except terms compounded out of (6), (7), (8) in the forms

(33)	-23154	involving	a^4b,
(34)	-31254	,,	a^4b,
(35)	32154	,,	a^2b^2,

the constituents 5, 4 being not necessarily drawn from rows consecutive to the rows 1, 2, 3.

Consider the terms (6), 231, and (7), 312.

These will be equal to one another, and their sum

$$= -\frac{2a^2b}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]}.$$

The sum of these terms

$$(6) \text{ and } (7) = -\frac{24a^2bA}{q(q^2-1)(q^2-4)}.$$

Now consider the terms (33), -23154, and (34), -31254.

Each of such terms will be of the form

$$\frac{2a^2b}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]} \cdot \frac{a^2}{[(\gamma+s)^2-\kappa^2][(\gamma+s+1)^2-\kappa^2]},$$

where both s and $s+1$ are integers other than $p-1$, p, $p+1$. Now we have

$$\Sigma \frac{a^2}{[(\gamma+s)^2-\kappa^2][(\gamma+s+1)^2-\kappa^2]} = -\frac{4a^2A}{q(q^2-1)}.$$

Hence the sum will be

$$\frac{24a^2bA}{q(q^2-1)(q^2-4)} \cdot \frac{-4a^2A}{q(q^2-1)}$$

$$-2a^4b \,.\, \Sigma \left\{\frac{1}{[(\gamma+p-2)^2-\kappa^2][(\gamma+p-1)^2-\kappa^2]^2[\gamma^2-\kappa^2][(\gamma+1)^2-\kappa^2]}\right.$$

$$+\frac{1}{[(\gamma+p-1)^2-\kappa^2]^2[(\gamma+p)^2-\kappa^2]^2[(\gamma+p+1)^2-\kappa^2]}$$

$$+\frac{1}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2]^2[(\gamma+p+1)^2-\kappa^2]^2}$$

$$\left.+\frac{1}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]^2[(\gamma+p+2)^2-\kappa^2]}\right\}.$$

We find this latter sum

$$=\frac{96B}{q^2(q^2-1)^2(q^2-4)} - \frac{32(35q^6-280q^4+497q^2-108)}{q^3(q^2-1)^3(q^2-4)^2(q^2-9)}A.$$

Substitute for B its value $A^2+\pi \cot q\pi A$, and we find the terms of the determinant

$$(33) \text{ and } (34) = a^4bA\left\{\frac{96\pi\cot q\pi}{q^2(q^2-1)^2(q^2-4)} - \frac{32(35q^6-280q^4+497q^2-108)}{q^3(q^2-1)^3(q^2-4)^2(q^2-9)}\right\},$$

the terms in A^2 cancelling one another.

In this formula it may be shewn that $q-1$ appears in only the first power in the denominator.

Next consider the term (8), -321.

Each term is of the form

$$-\frac{b^2}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]},$$

and the sum

$$(8)=\frac{4b^2A}{q(q^2-4)}.$$

Now take (35), 32154. Each such element is of the form

$$\frac{b^2}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]}\cdot\frac{a^2}{[(\gamma+s)^2-\kappa^2][(\gamma+s+1)^2-\kappa^2]},$$

where s and $s+1$ are both different from $p-1$ and $p+1$, so that s must not be equal to $p-2$, $p-1$, p, $p+1$.

Hence the sum of all these elements is

$$-\frac{4b^2A}{q(q^2-4)}\cdot\frac{-4a^2A}{q(q^2-1)}$$

$$-a^2b^2\Sigma\left\{\frac{1}{[(\gamma+p-2)^2-\kappa^2][(\gamma+p-1)^2-\kappa^2]^2[(\gamma+p+1)^2-\kappa^2]}\right.$$

$$+\frac{1}{[(\gamma+p-1)^2-\kappa^2]^2[\gamma^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]}$$

$$+\frac{1}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]^2}$$

$$\left.+\frac{1}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]^2[(\gamma+p+2)^2-\kappa^2]}\right\}.$$

This latter sum we find to be

$$\frac{16B}{q^2(q^2-1)(q^2-4)}-\frac{160q^6-1360q^4+2544q^2-576}{q^3(q^2-1)^2(q^2-4)^2(q^2-9)}A.*$$

Substitute for B, and we get

$$(35)=-a^2b^2A\frac{16\pi\cot q\pi}{q^2(q^2-1)(q^2-4)}$$

$$+a^2b^2A\frac{160q^6-1360q^4+2544q^2-576}{q^3(q^2-1)^2(q^2-4)^2(q^2-9)}.$$

* [It was in the reduction of the coefficient of A that the error occurred alluded to *M. N.*, Nov. 1877, p. 45; *Works*, Vol. I., p. 184. The above is the correct value, bearing the date Aug. 30, 1877.]

We will now consider the remaining terms consisting of four elements.

First take the terms (30), -4231, involving two constituents of the diagonal.

These are of the form

$$-\frac{c^2}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p+2)^2-\kappa^2]};$$

hence their sum

$$(30)=\frac{4c^2A}{q(q^2-9)}.$$

Next take the terms (20) and (28), 3241 and 4213, which involve the constituent 2 of the diagonal.

These terms are equal, and their sum is

$$-\frac{2abc}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2][(\gamma+p+2)^2-\kappa^2]}.$$

Along with these consider (24), 2431, and (29), 4132, which involve the constituent 3 of the diagonal. These are likewise equal, and their sum is

$$-\frac{2abc}{[(\gamma+p-2)^2-\kappa^2][(\gamma+p-1)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2]}.$$

Resolving these terms into partial fractions and effecting the summation, we find

$$(20),\ (28),\ (24),\ \text{and}\ (29)=-\frac{16(3q^2-7)}{q(q^2-1)(q^2-4)(q^2-9)}abcA.$$

Now take the terms (18), -2341, and (27), -4123, which involve no constituent of the diagonal. These are equal, and their sum is

$$-\frac{2a^3c}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2][(\gamma+p+2)^2-\kappa^2]}.$$

Along with these take (19), -3142, and (22), -2413, which also contain no constituent of the diagonal. These are also equal and their sum is

$$-\frac{2a^2b^2}{[(\gamma+p-1)^2-\kappa^2][(\gamma+p)^2-\kappa^2][(\gamma+p+1)^2-\kappa^2][(\gamma+p+2)^2-\kappa^2]},$$

3.

NUMERICAL DEVELOPMENTS IN THE LUNAR THEORY.

[In a paper "On the Motion of the Moon's Node," *Mon. Not.* XXXVIII., Nov. 1877, p. 43; *Works*, Vol. 1, p. 181, Adams gave expression to his views on the most advantageous treatment of the lunar problem, as follows:—

"I have long been convinced that the most advantageous way of treating the Lunar Theory is, first, to determine with all desirable accuracy the inequalities which are independent of the eccentricities e and e', and the inclination $2\sin^{-1}\gamma$, and then, in succession, to find the inequalities which are of one dimension, two dimensions, and so on, with respect to those quantities.

"Thus the coefficient of any inequality in the Moon's coordinates would be represented by a series arranged in powers and products of e, e' and γ, and each term in this series would involve a numerical coefficient which is a function of m alone and which may be calculated for any given value of m without the necessity of developing it in powers of m. The variations of these coefficients which would result from a very small change in m might be found either independently or by making the calculation for two values of m differing by a small quantity.

"This method is particularly advantageous when we wish to compare our results with those of an analytical theory such as Delaunay's, in which the eccentricities and the inclination are left indeterminate, since each numerical coefficient so obtained could be compared separately with its analytical development in powers of m.

"It is to be remarked that it is only the series proceeding by powers of m in Delaunay's Theory which have a slow rate of convergence, so that it is probable that all the sensible corrections required by Delaunay's coefficients would be found among the terms of low order in e, e', and γ.

"The differential equations which would require solution in these successive operations after the determination of the inequalities independent of eccentricities and inclination would be all linear and of the same form.

"It is many years since I obtained the values of these last-named inequalities to a great degree of approximation, the coefficients of the longitude expressed in circular measure, and those of the reciprocal of the radius vector, or of the logarithm of the radius vector, being found to ten or eleven places of decimals.

"In the next place I proceeded to consider the inequalities of latitude......

"......I have also succeeded in reducing the determination of the inequalities of longitude and radius vector which involve the first power of the lunar eccentricity to the solution of a differential equation of the second order, but my method is much less elegant than that of Mr Hill."

This pronouncement explains the purpose of the following developments.]

Taking the equations of Lecture IV.,

$$\frac{1}{r}\frac{d^2r}{dt^2} - \left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^3} = \frac{1}{2}n'^2 + \frac{3}{2}n'^2\cos 2(\theta - n't - \epsilon'),$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{3}{2}n'^2\sin 2(\theta - n't - \epsilon').$$

Suppress the epochs ϵ, ϵ', and define a so that

$$\mu = n^2,$$

and take the value of m,

$$m = \frac{n'}{n} = 0{\cdot}0748013,$$

then the following quantities substituted in the equations satisfy them to ten or eleven places of decimals:—

3.

NUMERICAL DEVELOPMENTS IN THE LUNAR THEORY.

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Taking the equations of Lecture IV.,

$$\frac{1}{r}\frac{d^2r}{dt^2}-\left(\frac{d\theta}{dt}\right)^2+\frac{\mu}{r^3}=\frac{1}{2}n'^2+\frac{3}{2}n'^2\cos 2(\theta-n't-\epsilon'),$$

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Suppress the epochs ϵ, ϵ', and define a so that

$$\mu=n^2,$$

and take the value of m,

$$m=\frac{n'}{n}=0\cdot0748013,$$

then the following quantities substituted in the equations satisfy them to ten or eleven places of decimals:—

$$\begin{aligned}\theta = nt &+ 0{\cdot}01021{,}13629{,}5 \ \sin\ 2(n-n')t \\ &+ 0{\cdot}00004{,}23732{,}7 \ \sin\ 4(n-n')t \\ &+ 0{\cdot}00000{,}02375{,}7 \ \sin\ 6(n-n')t \\ &+ 0{\cdot}00000{,}00015{,}1 \ \sin\ 8(n-n')t \\ &+ 0{\cdot}00000{,}00000{,}1 \ \sin\ 10(n-n')t.\end{aligned}$$

$$\begin{aligned}\frac{1}{r} = &\ \ 1{\cdot}00090{,}73880{,}5 \\ &+ 0{\cdot}00718{,}64751{,}6 \ \cos\ 2(n-n')t \\ &+ 0{\cdot}00004{,}58428{,}9 \ \cos\ 4(n-n')t \\ &+ 0{\cdot}00000{,}03268{,}6 \ \cos\ 6(n-n')t \\ &+ 0{\cdot}00000{,}00024{,}3 \ \cos\ 8(n-n')t \\ &- 0{\cdot}00000{,}00000{,}3 \ \cos\ 10(n-n')t.\end{aligned}$$

[These are the values quoted from an older MS in 1877 (*Mon. Not.* XXXVIII., p. 46; *Works*, Vol. I., p. 184). The original calculation has not been found, but it must have had a date earlier than 1860, for the above numbers are referred to in a MS of that year.]

In order to obtain corresponding series in which θ is the independent variable, first transform these two functions or rather the functions nt and $\log\frac{1}{r}$, by means of Lagrange's Theorem; use these results as approximations and emend them by substitution in the equations of Lecture VI., viz.:—

$$\frac{d^2(au)}{d\theta^2} + au\left[1 + \frac{1}{2}\left(n'\frac{dt}{d\theta}\right)^2 + \frac{3}{2}\left(n'\frac{dt}{d\theta}\right)^2\cos 2(\theta-\theta')\right]$$

$$+\frac{d(au)}{d\theta}\left[\qquad\qquad -\frac{3}{2}\left(n'\frac{dt}{d\theta}\right)^2\sin 2(\theta-\theta')\right] = \frac{\mu a}{H^2},$$

$$\frac{1}{H^2}\frac{d(H^2)}{d\theta} = -3n'^2\left(\frac{dt}{d\theta}\right)^2\sin 2(\theta-\theta'),$$

$$\frac{dt}{d\theta} = \frac{1}{Hu^2},$$

in which we take $\theta' = n't$, and a has the same definition as before. We find

$$\begin{aligned}
u = &\ \ 1{\cdot}00097{,}52861{,}50 \\
&+ 0{\cdot}00718{,}66609{,}56 \cos 2(1-m)\theta \\
&- 0{\cdot}00002{,}20516{,}12 \cos 4(1-m)\theta \\
&+ 0{\cdot}00000{,}01410{,}92 \cos 6(1-m)\theta \\
&- 0{\cdot}00000{,}00011{,}34 \cos 8(1-m)\theta \\
&+ 0{\cdot}00000{,}00000{,}10 \cos 10(1-m)\theta.
\end{aligned}$$

$$\begin{aligned}
nt = \theta &- 0{\cdot}01021{,}13075{,}60 \sin 2(1-m)\theta \\
&+ 0{\cdot}00005{,}40981{,}47 \sin 4(1-m)\theta \\
&- 0{\cdot}00000{,}04037{,}26 \sin 6(1-m)\theta \\
&+ 0{\cdot}00000{,}00034{,}98 \sin 8(1-m)\theta \\
&- 0{\cdot}00000{,}00000{,}33 \sin 10(1-m)\theta.
\end{aligned}$$

Forming the functions required for a second approximation, it appears that they all agree with the former values to 11 or 12 places of decimals; hence no corrections are required.

Dec./60 and *Jan.*/61.

[The next investigations were the papers on the latitude, which lead to an infinite determinant, and are abstracted on p. 85 *et seqq.* The formulae, including the determination of the motion of the node as far as it is independent of e, e' and γ, as well as the corresponding parts of the coefficients of the evection in latitude, were reduced numerically and determined, the former to 15 places of decimals and the latter to 11 or 12, in 1877.

In 1880 Adams availed himself of an offer of a friend and former pupil, Miss Fanny Harrison, in order to carry on these calculations. The calculations were made by Miss Harrison from formulae supplied by Adams, and were examined and, where necessary, corrected, refined, or transformed by Adams.]

If we take the numbers given on p. 106 as approximations to the values of θ and $\frac{1}{r}$ in terms of t and correct the solution by the method

given in Lecture VII., we at length obtain the values of the following functions to 15 places of decimals.

$$
\begin{aligned}
\theta = nt + 0{\cdot}01021,13629,54071,22 &\sin\ 2(n-n')t \\
+ \quad 4,23732,68757,73 &\sin\ 4(n-n')t \\
+ \quad 2375,68231,26 &\sin\ 6(n-n')t \\
+ \quad 15,07977,03 &\sin\ 8(n-n')t \\
+ \quad 10246,17 &\sin 10(n-n')t \\
+ \quad 72,68 &\sin 12(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{r} = \quad 1{\cdot}00090,73880,47512,46 & \\
+ \quad 718,64751,59794,38 &\cos\ 2(n-n')t \\
+ \quad 4,58429,07983,33 &\cos\ 4(n-n')t \\
+ \quad 3268,81854,41 &\cos\ 6(n-n')t \\
+ \quad 24,50530,00 &\cos\ 8(n-n')t \\
+ \quad 18904,15 &\cos 10(n-n')t \\
+ \quad 148,58 &\cos 12(n-n')t \\
- \quad ,21 &\cos 14(n-n')t.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\cos(\theta - nt) = 1 - 0{\cdot}00002,60682,62341,65 & \\
- \quad 2163,47566,91 &\cos\ 2(n-n')t \\
+ \quad 2,60665,43862,72 &\cos\ 4(n-n')t \\
+ \quad 2163,33894,18 &\cos\ 6(n-n')t \\
+ \quad 17,18369,39 &\cos\ 8(n-n')t \\
+ \quad 13671,84 &\cos 10(n-n')t \\
+ \quad 109,54 &\cos 12(n-n')t \\
+ \quad ,88 &\cos 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\sin(\theta - nt) = \quad 0{\cdot}01021,12298,58342,60 &\sin\ 2(n-n')t \\
+ \quad 4,23721,64162,51 &\sin\ 4(n-n')t \\
+ \quad 2819,24397,22 &\sin\ 6(n-n')t \\
+ \quad 20,60200,05 &\sin\ 8(n-n')t \\
+ \quad 15691,70 &\sin 10(n-n')t \\
+ \quad 122,40 &\sin 12(n-n')t \\
+ \quad ,99 &\sin 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\cos 2(\theta - nt) = 1 - 0{\cdot}00010,42710,10699,69 & \\
- \quad 8653,67708,37 & \cos \ 2(n-n')t \\
+ \quad 10,42634,57455,66 & \cos \ 4(n-n')t \\
+ \quad 8653,01740,04 & \cos \ 6(n-n')t \\
+ \quad 75,52669,46 & \cos \ 8(n-n')t \\
+ \quad 65963,35 & \cos 10(n-n')t \\
+ \quad 574,57 & \cos 12(n-n')t \\
+ \quad 4,99 & \cos 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\sin 2(\theta - nt) = \quad 0{\cdot}02042,16611,56191,75 & \sin \ 2(n-n')t \\
+ \quad 8,47377,00897,68 & \sin \ 4(n-n')t \\
+ \quad 8299,78852,47 & \sin \ 6(n-n')t \\
+ \quad 74,33623,08 & \sin \ 8(n-n')t \\
+ \quad 65443,01 & \sin 10(n-n')t \\
+ \quad 571,94 & \sin 12(n-n')t \\
+ \quad 5,01 & \sin 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\cos 3(\theta - nt) = 1 - 0{\cdot}00023,46021,29250,49 & \\
- \quad 19469,92748,85 & \cos \ 2(n-n')t \\
+ \quad 23,45825,87058,84 & \cos \ 4(n-n')t \\
+ \quad 19468,02031,45 & \cos \ 6(n-n')t \\
+ \quad 195,40369,67 & \cos \ 8(n-n')t \\
+ \quad 1,90700,27 & \cos 10(n-n')t \\
+ \quad 1821,98 & \cos 12(n-n')t \\
+ \quad 17,13 & \cos 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\sin 3(\theta - nt) = \quad 0{\cdot}03063,04954,02444,10 & \sin \ 2(n-n')t \\
+ \quad 12,70899,83498,03 & \sin \ 4(n-n')t \\
+ \quad 19102,58735,98 & \sin \ 6(n-n')t \\
+ \quad 194,32916,16 & \sin \ 8(n-n')t \\
+ \quad 1,90245,35 & \sin 10(n-n')t \\
+ \quad 1819,71 & \sin 12(n-n')t \\
+ \quad 17,17 & \sin 14(n-n')t.
\end{aligned}
$$

and

$$
\begin{aligned}
r = 1 - 0{\cdot}00088,08126,20166,16 & \\
- \quad 717,33998,43971,66 & \cos 2(n-n')t \\
- \quad 2,00071,48305,31 & \cos 4(n-n')t \\
- \quad 901,86346,92 & \cos 6(n-n')t \\
- \quad 4,92764,82 & \cos 8(n-n')t \\
- \quad 2987,04 & \cos 10(n-n')t \\
- \quad 19,33 & \cos 12(n-n')t \\
- \quad ,13 & \cos 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
r^2 = 1 - 0{\cdot}00173,51203,76652,70 & \\
- \quad 1433,40193,24640,23 & \cos 2(n-n')t \\
- \quad 1,42495,71899,85 & \cos 4(n-n')t \\
- \quad 366,91009,90 & \cos 6(n-n')t \\
- \quad 1,37554,46 & \cos 8(n-n')t \\
- \quad 629,54 & \cos 10(n-n')t \\
- \quad 3,27 & \cos 12(n-n')t \\
- \quad 2 & \cos 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{r^3} = \quad 1{\cdot}00280,21783,19040,9 & \\
+ \quad 2159,98364,46032,2 & \cos 2(n-n')t \\
+ \quad 21,53274,03812,3 & \cos 4(n-n')t \\
+ \quad 20644,79748,4 & \cos 6(n-n')t \\
+ \quad 192,87144,3 & \cos 8(n-n')t \\
+ \quad 31254,1 & \cos 10(n-n')t \\
+ \quad 127,2 & \cos 12(n-n')t \\
- \quad ,7 & \cos 14(n-n')t.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mu}{r^3} = \quad 1{\cdot}17150,79521,90228,408 & \\
+ \quad 2523,36709,16780,19 & \cos 2(n-n')t \\
+ \quad 25,15529,33585,67 & \cos 4(n-n')t \\
+ \quad 24118,73874,61 & \cos 6(n-n')t \\
+ \quad 226,05776,09 & \cos 8(n-n')t \\
+ \quad 2,08748,11 & \cos 10(n-n')t \\
+ \quad 1907,29 & \cos 12(n-n')t \\
+ \quad 17,32 & \cos 14(n-n')t.
\end{aligned}
$$

These numbers have been calculated from the datum $m = 0{\cdot}0748013$. It is important to find how they are modified by a small change in m.

Start from the equations of Lecture VII.,

$$\frac{d^2l}{dt^2} + \left(\frac{dl}{dt}\right)^2 - \left(\frac{d\theta}{dt}\right)^2 + \mu e^{-3l} - n'^2\left[\frac{1}{2} + \frac{3}{2}\cos 2(\theta - n't)\right] = 0,$$

$$\frac{d^2\theta}{dt^2} + 2\frac{dl}{dt}\frac{d\theta}{dt} \qquad + n'^2\left[\quad \frac{3}{2}\sin 2(\theta - n't)\right] = 0,$$

where $l = \log_e r$ and the unit of length is taken as before, and suppose there is a small increase in m, introducing increases δl, $\delta\theta$ in l, θ, the squares of all these quantities being supposed negligible. Let us take as coordinates δl, $\delta\omega$, in place of δl, $\delta\theta$ where

$$\omega = \theta - n't,$$

and therefore

$$\delta\theta = \delta\omega + \frac{\delta n'}{n'} \,.\, n't.$$

[Now we have developed $\frac{1}{r}$ and $\omega = \theta - n't$ in series of the shape $\Sigma f_i(m) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} 2i(n-n')t$, and we wish to find the changes in the coefficients, the arguments being unchanged. Hence we take

$$\delta(n - n') = 0,$$

and therefore

$$\frac{\delta m}{m} = (1 - m)\frac{\delta n'}{n'},$$

and

$$\delta\mu = \delta(n^2) = 2nn'\frac{\delta n'}{n'}.]$$

Hence we find the equations

$$0 = X + \frac{d^2\delta l}{dt^2} + 2\frac{dl}{dt}\frac{d\delta l}{dt} - 2\frac{d\theta}{dt}\frac{d\delta\omega}{dt} - 3\mu e^{-3l}\delta l + 3n'^2 \sin 2\omega\,\delta\omega,$$

$$0 = Y + \frac{d^2\delta\omega}{dt^2} + 2\frac{dl}{dt}\frac{d\delta\omega}{dt} + 2\frac{d\theta}{dt}\frac{d\delta l}{dt} \qquad + 3n'^2\cos 2\omega\,\delta\omega,$$

where

$$X = \frac{\delta n'}{n'}\left\{-2n'\frac{d\theta}{dt} + 2nn'e^{-3l} - 2n'^2\left[\frac{1}{2} + \frac{3}{2}\cos 2\omega\right]\right\},$$

$$Y = \frac{\delta n'}{n'}\left\{\ \ 2n'\frac{dl}{dt} \qquad\qquad + 2n'^2\left[\qquad \frac{3}{2}\sin 2\omega\right]\right\};$$

the coefficients of $\frac{\delta n'}{n'}$, δl, $\delta\omega$, and their differential coefficients being known as functions of $(n-n')\,t$.

Now these equations are of the form discussed in Lecture VII., and we may approach their solution in the way that is shewn there; or still more simply, let P_1 and Q_1 be the most important parts of X and Y, and c the constant part of μe^{-3l}, and determine $\delta_1 l$, $\delta_1\omega$ from the equations

$$0 = P_1 + \frac{d^2\delta_1 l}{dt^2} - 2n\frac{d\delta_1\omega}{dt} - 3c\delta_1 l,$$

$$0 = Q_1 + \frac{d^2\delta_1\omega}{dt^2} + 2n\frac{d\delta_1 l}{dt}.$$

Then let

$$X_1 = X - P_1 + 2\frac{dl}{dt}\frac{d\delta_1 l}{dt} - 2\left(\frac{d\theta}{dt} - n\right)\frac{d\delta_1\omega}{dt} - 3\left(\mu e^{-3l} - c\right)\delta_1 l + 3n'^2\sin 2\omega\,\delta_1\omega,$$

$$Y_1 = Y - Q_1 + 2\frac{dl}{dt}\frac{d\delta_1\omega}{dt} + 2\left(\frac{d\theta}{dt} - n\right)\frac{d\delta_1 l}{dt} \qquad + 3n'^2\cos 2\omega\,\delta_1\omega,$$

and repeat the approximation with X_1, Y_1 in place of X, Y; whence finally if $\delta_1 l$, $\delta_2 l$, $\delta_3 l \ldots \delta_1\omega$, $\delta_2\omega$, $\delta_3\omega \ldots$ are the successive corrections found, the complete corrections are

$$\delta l\ = \delta_1 l + \delta_2 l + \delta_3 l + \ldots$$

$$\delta\omega = \delta_1\omega + \delta_2\omega + \delta_3\omega + \ldots$$

We find

$$\begin{aligned} X = \frac{dn'}{n'}\big[& -0{\cdot}00584,65667,60159,44 \\ & - \quad 1913,50693,01356,20 \ \cos\ 2(n-n')\,t \\ & - \quad 18,99962,11544,85 \ \cos\ 4(n-n')\,t \\ & - \quad 17227,86035,84 \ \cos\ 6(n-n')\,t \\ & - \quad 151,90731,94 \ \cos\ 8(n-n')\,t \\ & - \quad 1,32276,48 \ \cos 10(n-n')\,t \\ & - \quad 1144,15 \ \cos 12(n-n')\,t \\ & - \quad 9,85 \ \cos 14(n-n')\,t\big]. \end{aligned}$$

$$
\begin{aligned}
Y=\frac{\delta n'}{n'}\big[\ & 0{\cdot}02192{,}93435{,}69543{,}96 \sin 2(n-n')t \\
+\ & 22{,}15097{,}16543{,}17 \sin 4(n-n')t \\
+\ & 20403{,}69529{,}98 \sin 6(n-n')t \\
+\ & 182{,}50089{,}05 \sin 8(n-n')t \\
+\ & 1{,}61025{,}56 \sin 10(n-n')t \\
+\ & 1409{,}92 \sin 12(n-n')t \\
+\ & 12{,}29 \sin 14(n-n')t\big],
\end{aligned}
$$

and thence

$$
\begin{aligned}
\delta l=\frac{\delta n'}{n'}\big[& -0{\cdot}00157{,}07440{,}23063{,}65 \\
-\ & 1503{,}04573{,}28666{,}23 \cos 2(n-n')t \\
-\ & 13{,}85178{,}94754{,}61 \cos 4(n-n')t \\
-\ & 12198{,}71210{,}26 \cos 6(n-n')t \\
-\ & 106{,}12235{,}10 \cos 8(n-n')t \\
-\ & 91844{,}94 \cos 10(n-n')t \\
-\ & 792{,}73 \cos 12(n-n')t \\
-\ & 6{,}89 \cos 14(n-n')t\big],
\end{aligned}
$$

$$
\begin{aligned}
\delta\omega=\frac{\delta n'}{n'}\big[\ & 0{\cdot}02172{,}72068{,}72058{,}04 \sin 2(n-n')t \\
+\ & 17{,}93852{,}23118{,}97 \sin 4(n-n')t \\
+\ & 15064{,}30388{,}89 \sin 6(n-n')t \\
+\ & 127{,}41426{,}27 \sin 8(n-n')t \\
+\ & 1{,}08179{,}31 \sin 10(n-n')t \\
+\ & 920{,}61 \sin 12(n-n')t \\
+\ & 7{,}92 \sin 14(n-n')t\big].
\end{aligned}
$$

In the values of δl, $\delta\omega$ the fifteenth figure is in some cases doubtful, especially in the coefficients with the argument $2(n-n')t$.

Dec. 1881.

We next find the parallactic inequalities. We have the equations

$$\frac{d^2l}{dt^2}+\left(\frac{dl}{dt}\right)^2-\left(\frac{d\theta}{dt}\right)^2+\frac{\mu}{a^3}e^{-3l}-n'^2\left[\frac{1}{2}+\frac{3}{2}\cos 2\omega\right]-\lambda n'^2\frac{r}{a}\left[\frac{9}{8}\cos\omega+\frac{15}{8}\cos 3\omega\right]=0,$$

$$\frac{d^2\theta}{dt^2}+2\frac{dl}{dt}\frac{d\theta}{dt}\qquad\qquad+n'^2\left[\quad\frac{3}{2}\sin 2\omega\right]+\lambda n'^2\frac{r}{a}\left[\frac{3}{8}\sin\omega+\frac{15}{8}\sin 3\omega\right]=0,$$

where
$$\lambda=\frac{E-M}{E+M}\frac{a}{a'},$$

restoring a which was taken as unit, and is defined by

$$\mu=n^2a^3.$$

We have found values of l, θ which reduce to zero the sum of all the terms in each equation excluding those multiplied by λ; let the result of substituting these values of l, θ in those terms be X, Y; then our equations for correcting l and θ will be of the form we have just discussed, and we might proceed in a like manner. But we can see that the approximations would be comparatively tedious; for, taking the equations

$$0=P_1+\frac{d^2\delta_1 l}{dt^2}-2n\frac{d\delta_1\omega}{dt}-3c\delta_1 l,$$

$$0=Q_1+\frac{d^2\delta_1\omega}{dt^2}+2n\frac{d\delta_1 l}{dt},$$

suppose

$$P_1=\Sigma p_j\cos j(n-n')t,$$

$$Q_1=\Sigma q_j\sin j(n-n')t,$$

where j takes all positive odd integral values; then let

$$\delta_1 l=\Sigma a_j\cos j(n-n')t,$$

$$\delta_1\omega=\Sigma b_j\sin j(n-n')t,$$

and we have

$$0=p_j-j^2(n-n')^2a_j-2jn(n-n')b_j-3ca_j,$$

$$0=q_j-j^2(n-n')^2b_j-2jn(n-n')a_j.$$

Multiply the second equation by $2\dfrac{n}{j(n-n')}$ and subtract from the first;

then

$$a_j = \frac{p_j - \frac{2}{j}\frac{n}{n-n'} q_j}{j^2 (n-n')^2 - 4n^2 + 3c},$$

$$b_j = \frac{q_j}{j^2 (n-n')^2} - \frac{2}{j}\frac{n}{n-n'} a_j.$$

Now choosing the unit of time so that

$$n - n' = 1,$$

and calling

$$C_j = j^2 - 4n^2 + 3c,$$

we find the following numerical values for $\frac{1}{C_j}$ and $\frac{2n}{j}$:

j	$\frac{1}{C_j}$	$\frac{2n}{j}$
1	−6·31259,13816,12770,45	2·16169,78061,03705,0
3	+0·12752,52152,30991,45	0·72056,59353,67901,7
5	0·04194,35175,60279,42	0·43233,95612,20741,0
7	0·02090,23168,78861,83	0·30881,39723,00529,3
9	0·01252,48012,27559,34	0·24018,86451,22633,9
11	0·00834,43488,15871,11	0·19651,79823,73064,1
13	0·00595,79989,74753,66	0·16628,44466,23361,9
15	0·00446,74451,06389,87	0·14411,31870,73580,3.

Hence the values of a_1, b_1 will be considerably larger than the coefficients p_1, q_1 in consequence of the large values of $\frac{1}{C_j}$ and $\frac{2n}{j}$ for $j=1$; and since the same multipliers will reappear in the successive corrections to the first values found for a_1, b_1, our approximation to those quantities will be slow. It will be better to avoid this inconvenience, as we may by the following device.

Assume

$$\delta l = a_1 \cos\phi + \delta_1 l + \delta_2 l + \ldots = a_1 \cos\phi + [\delta l],$$

$$\delta\omega = b_1 \sin\phi + \delta_1\omega + \delta_2\omega + \ldots = b_1 \sin\phi + [\delta\omega],$$

where $\delta_1 l$, $\delta_2 l$, ..., $\delta_1\omega$, $\delta_2\omega$, ... consist of cosines and sines of higher odd multiples of ϕ than the first, and ϕ is written for $(n-n')t$.

Then writing as in Lecture VII.,

$$\frac{d\theta}{dt} = n + v, \quad \mu e^{-3l} = c + w,$$

v, w consisting of periodic terms alone, we shall obtain the following equations to determine $[\delta l]$ and $[\delta\omega]$,

$$0 = X_1 + \left[\frac{d^2\delta l}{dt^2}\right] + 2\frac{dl}{dt}\left[\frac{d\delta l}{dt}\right] - 2(n+v)\left[\frac{d\delta\omega}{dt}\right] - 3(c+w)[\delta l] + 3n'^2 \sin 2\omega\,[\delta\omega],$$

$$0 = Y_1 + \left[\frac{d^2\delta\omega}{dt^2}\right] + 2\frac{dl}{dt}\left[\frac{d\delta\omega}{dt}\right] + 2(n+v)\left[\frac{d\delta l}{dt}\right] \qquad + 3n'^2 \cos 2\omega\,[\delta\omega],$$

where

$$X_1 = X - a_1 \cos\phi\,(1 + 3c + 3w) \quad - a_1 \sin\phi\left(2\frac{dl}{dt}\right) - 2b_1 \cos\phi\,(n+v)$$
$$+ b_1 \sin\phi\,(3n'^2 \sin 2\omega),$$

$$Y_1 = Y - b_1 \sin\phi\,(1 - 3n'^2 \cos 2\omega) + b_1 \cos\phi\left(2\frac{dl}{dt}\right) - 2a_1 \sin\phi\,(n+v),$$

and the coefficients a_1 and b_1 are to be so determined that $[\delta l]$ and $[\delta\omega]$ may contain no terms involving $\cos\phi$ and $\sin\phi$ respectively. Now let P_1 and Q_1 represent the terms in X_1 and Y_1 respectively that have the largest coefficients, excluding all terms in $\cos\phi$ and $\sin\phi$; then if $\delta_1 l$, $\delta_1\omega$ be determined by the conditions

$$0 = P_1 + \frac{d^2\delta_1 l}{dt^2} - 2n\frac{d\delta_1\omega}{dt} - 3c\delta_1 l,$$

$$0 = Q_1 + \frac{d^2\delta_1\omega}{dt^2} + 2n\frac{d\delta_1 l}{dt},$$

and X_2, Y_2 be the results of substituting $\delta_1 l$ and $\delta_1\omega$ instead of $[\delta l]$ and $[\delta\omega]$ in the right-hand members of the equations that determine $[\delta l]$ and $[\delta\omega]$, we shall have as before

$$X_2 = X_1 - P_1 + 2\frac{dl}{dt}\frac{d\delta_1 l}{dt} - 2v\frac{d\delta_1\omega}{dt} - 3w\delta_1 l + 3n'^2 \sin 2\omega\,\delta_1\omega,$$

$$Y_2 = Y_1 - Q_1 + 2\frac{dl}{dt}\frac{d\delta_1\omega}{dt} + 2v\frac{d\delta_1 l}{dt} \qquad + 3n'^2 \cos 2\omega\,\delta_1\omega.$$

Repeat the step with X_2, Y_2 in place of X_1, Y_1, and continue until the terms in X_n, Y_n which involve odd multiples of ϕ above the first, are insensible.

The coefficients of the several terms in $\delta_1 l$, $\delta_2 l$, ..., $\delta_1\omega$, $\delta_2\omega$, ... thus found will involve linearly the constants a_1 and b_1, and the same will be the case with regard to the coefficients of $\cos\phi$ and $\sin\phi$ in the final values obtained for the quantities X_n and Y_n respectively. Hence by equating these latter coefficients to zero we shall have two simple equations for determining a_1 and b_1, whence by substitution all the other coefficients in the values of $[\delta l]$ and $[\delta\omega]$ may be found.

Nov. 8/81.

[On this plan the calculations were carried out by Miss Harrison, taking the expressions

$$
\begin{aligned}
X=\lambda\,[&-0{\cdot}00705{,}27630{,}94721{,}5 \cos (n-n')\,t\\
&- 1225{,}34903{,}49610{,}6 \cos 3\,(n-n')\,t\\
&- 14{,}36155{,}36056{,}7 \cos 5\,(n-n')\,t\\
&- 14187{,}55306{,}0 \cos 7\,(n-n')\,t\\
&- 132{,}36321{,}5 \cos 9\,(n-n')\,t\\
&- 1{,}20188{,}9 \cos 11\,(n-n')\,t\\
&- 1074{,}7 \cos 13\,(n-n')\,t\\
&- 9{,}4 \cos 15\,(n-n')\,t],
\end{aligned}
$$

$$
\begin{aligned}
Y=\lambda\,[&\;\;0{\cdot}00223{,}75651{,}99439{,}7 \sin (n-n')\,t\\
&+ 1224{,}60664{,}99386{,}0 \sin 3\,(n-n')\,t\\
&+ 14{,}35867{,}21138{,}5 \sin 5\,(n-n')\,t\\
&+ 14186{,}08099{,}5 \sin 7\,(n-n')\,t\\
&+ 132{,}35462{,}5 \sin 9\,(n-n')\,t\\
&+ 1{,}20183{,}4 \sin 11\,(n-n')\,t\\
&+ 1074{,}7 \sin 13\,(n-n')\,t\\
&+ 9{,}4 \sin 15\,(n-n')\,t].
\end{aligned}
$$

Thence we find the expressions

$$\begin{aligned}\delta l = \lambda [& \ 0{\cdot}11388{,}97944{,}95676{,}6 \cos (n-n')t \\ - & \ 134{,}75546{,}22715{,}5 \cos 3(n-n')t \\ - & \ 1{,}31065{,}61724{,}1 \cos 5(n-n')t \\ - & \ 1161{,}73856{,}5 \cos 7(n-n')t \\ - & \ 10{,}10844{,}5 \cos 9(n-n')t \\ - & \ 8740{,}2 \cos 11(n-n')t \\ - & \ 75{,}3 \cos 13(n-n')t \\ - & \ {,}6 \cos 15(n-n')t],\end{aligned}$$

$$\begin{aligned}\delta\omega = \lambda [& -0{\cdot}24265{,}37811{,}19304{,}3 \sin (n-n')t \\ + & \ 142{,}30587{,}09590{,}8 \sin 3(n-n')t \\ + & \ 1{,}47053{,}22766{,}2 \sin 5(n-n')t \\ + & \ 1313{,}75623{,}3 \sin 7(n-n')t \\ + & \ 11{,}41511{,}3 \sin 9(n-n')t \\ + & \ 9832{,}9 \sin 11(n-n')t \\ + & \ 84{,}4 \sin 13(n-n')t \\ + & \ {,}7 \sin 15(n-n')t].\end{aligned}$$

If these values are substituted in the equations which δl, $\delta\omega$ should satisfy they leave small residuals, in one case reaching 12 units of the fifteenth place of decimals. It does not appear that Adams amended these results as he amended the others; there is no MS. reference of his to them except an entry in the Diary of 1884:—

March 27. A large mass of calculations arrived from Miss F. Harrison: apparently very well done.

Thus these discrepancies remain, as well as others arising from the fact that the values of $\frac{1}{r}$, θ, &c. which Miss Harrison employs, differ from the definitive values which we have just given, in general slightly, but in one case by nearly a unit in the twelfth place.

We learn from the words in his paper on the motion of the Moon's node, quoted above, that he had reduced the determination of the inequalities of longitude and radius vector which involve the first power of

e to the solution of a differential equation of the second order; but I cannot find that he has anywhere made a numerical application of his method, which, as he says, is much less elegant than that of Mr Hill. He proposed to continue the foregoing calculations with Mr Hill's method of finding these terms, and his next and last communication to Miss Harrison consists in directions for computing the function P, where

$$\begin{aligned} P = \frac{\mu}{r^3} + 4n'^2 + 3 & \left\{\frac{1}{V^2}\left[\frac{\mu}{r^3}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) - 3n'^2 x\frac{dy}{dt}\right]\right\}^2 \\ & - 6n'\left\{\frac{1}{V^2}\left[\frac{\mu}{r^3}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) - 3n'^2 x\frac{dy}{dt}\right]\right\} \\ & - \frac{3\mu}{r^3}\frac{1}{r^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)^2 - 3n'^2\frac{1}{V^2}\left(\frac{dy}{dt}\right)^2, \end{aligned}$$

this being the quantity to which the coefficient of w reduces (Lecture XVIII. p. 84) if we ignore the parallactic terms, and the unspecified disturbances X, Y.

These calculations were not completed.]

July 24/84.

4.

THE SECULAR ACCELERATION OF THE MOON'S MEAN MOTION.

[In the paper "Reply to Various Objections," *Monthly Notices*, April, 1860; *Works*, Vol. I., p. 174, Adams mentions that he has determined the secular acceleration of the Moon's mean motion without recourse to developments in series. The following is the method employed.]

If we ignore the parallactic inequalities and the inclination of the orbit, the equations of motion become

$$\frac{d^2l}{dt^2}+\left(\frac{dl}{dt}\right)^2-\left(\frac{d\theta}{dt}\right)^2+\mu e^{-3l}-\frac{m'}{r'^3}\left[\frac{1}{2}+\frac{3}{2}\cos 2(\theta-\theta')\right]=0,$$

$$\frac{d^2\theta}{dt^2}+2\frac{d\theta}{dt}\frac{dl}{dt}\qquad\qquad-\frac{m'}{r'^3}\left[\quad-\frac{3}{2}\sin 2(\theta-\theta')\right]=0,$$

where $$l=\log r.$$

These may be satisfied, if e' be constant, by assumptions of the form

$$\log\frac{a}{r}=\quad a_0+a_2\cos\phi'+a_5\cos 2\xi+a_8\cos(2\xi-\phi')+a_9\cos(2\xi+\phi'),$$

$$\theta=nt+\epsilon+b_2\sin\phi'+b_5\sin 2\xi+b_8\sin(2\xi-\phi')+b_9\sin(2\xi+\phi),$$

where

$$\mu=n^2a^3,\quad \phi'=n't+\epsilon'-\varpi',\quad \xi=nt+\epsilon-(n't+\epsilon'),$$

and a_2, b_2, a_8, b_8, a_9, b_9 involve e' in the first power,

and a_0, n, a_5, b_5 involve e'^2.

Thus if the variation of e' be taken into account, we must assume

$$\begin{aligned}\log\frac{a}{r} = {} & a_0 + a_2\cos\phi' + a_5\cos 2\xi + a_8\cos(2\xi-\phi') + a_9\cos(2\xi+\phi') \\ & + \delta a_2\sin\phi' + \delta a_5\sin 2\xi + \delta a_8\sin(2\xi-\phi') + \delta a_9\sin(2\xi+\phi'), \\ \theta = {} & \int n\,dt + b_2\sin\phi' + b_5\sin 2\xi + b_8\sin(2\xi-\phi') + b_9\sin(2\xi+\phi') \\ & + \delta b_2\cos\phi' + \delta b_5\cos 2\xi + \delta b_8\cos(2\xi-\phi') + \delta b_9\cos(2\xi+\phi'),\end{aligned}$$

—where ξ now denotes $\int n\,dt - (n't+\epsilon')$,—in order to cancel the terms which are introduced by differentiating the coefficients a, b. Here δa_2, δb_2, δa_8, δb_8, δa_9, δb_9 are of the first order in e', but δa_5, δb_5, $\dfrac{dn}{dt}$ are of the second order. We shall suppose $\dfrac{d^2e'}{dt^2}=0$, and we shall ignore terms of the order $\left(\dfrac{de'}{dt}\right)^2$.

Hence we get

$$\begin{aligned}\frac{d}{dt}\left(\log\frac{a}{r}\right) = {} & -mna_2\sin\phi' - (2-2m)\,na_5\sin 2\xi - (2-3m)\,na_8\sin(2\xi-\phi') \\ & -(2-m)\,na_9\sin(2\xi+\phi') \\ & + \frac{da_0}{dt} + \left[mn\delta a_2 + \frac{da_2}{dt}\right]\cos\phi' + \left[(2-2m)\,n\delta a_5 + \frac{da_5}{dt}\right]\cos 2\xi \\ & + \left[(2-3m)\,n\delta a_8 + \frac{da_8}{dt}\right]\cos(2\xi-\phi') \\ & + \left[(2-m)\,n\delta a_9 + \frac{da_9}{dt}\right]\cos(2\xi+\phi');\end{aligned}$$

$$\begin{aligned}\frac{d\theta}{dt} = {} & n + mnb_2\cos\phi' + (2-2m)\,nb_5\cos 2\xi + (2-3m)\,nb_8\cos(2\xi-\phi') \\ & + (2-m)\,nb_9\cos(2\xi+\phi') \\ & + \left[-mn\delta b_2 + \frac{db_2}{dt}\right]\sin\phi' + \left[-(2-2m)\,n\delta b_5 + \frac{db_5}{dt}\right]\sin 2\xi \\ & + \left[-(2-3m)\,n\delta b_8 + \frac{db_8}{dt}\right]\sin(2\xi-\phi') \\ & + \left[-(2-m)\,n\delta b_9 + \frac{db_9}{dt}\right]\sin(2\xi+\phi');\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dt^2}\left(\log\frac{a}{r}\right) = {} & -m^2n^2a_2\cos\phi' - (2-2m)^2 n^2a_5\cos 2\xi - (2-3m)^2 n^2a_8\cos(2\xi-\phi') \\
& -(2-m)^2 n^2 a_9 \cos(2\xi+\phi') \\
& + \left[-m^2n^2\delta a_2 - 2mn\frac{da_2}{dt}\right]\sin\phi' \\
& + \left[-(2-2m)^2 n^2\delta a_5 - 2(2-2m)n\frac{da_5}{dt} - 2a_5\frac{dn}{dt}\right]\sin 2\xi \\
& + \left[-(2-3m)^2 n^2\delta a_8 - 2(2-3m)n\frac{da_8}{dt} - 2a_8\frac{dn}{dt}\right]\sin(2\xi-\phi') \\
& + \left[-(2-m)^2 n^2\delta a_9 - 2(2-m)n\frac{da_9}{dt} - 2a_9\frac{dn}{dt}\right]\sin(2\xi+\phi');
\end{aligned}$$

$$\begin{aligned}
\frac{d^2\theta}{dt^2} = {} & -m^2n^2b_2\sin\phi' - (2-2m)^2 n^2b_5\sin 2\xi - (2-3m)^2 n^2b_8\sin(2\xi-\phi') \\
& -(2-m)^2 n^2 b_9 \sin(2\xi+\phi') \\
& + \frac{dn}{dt} + \left[-m^2n^2\delta b_2 + 2mn\frac{db_2}{dt}\right]\cos\phi' \\
& + \left[-(2-2m)^2 n^2\delta b_5 + 2(2-2m)n\frac{db_5}{dt} + 2b_5\frac{dn}{dt}\right]\cos 2\xi \\
& + \left[-(2-3m)^2 n^2\delta b_8 + 2(2-3m)n\frac{db_8}{dt} + 2b_8\frac{dn}{dt}\right]\cos(2\xi-\phi') \\
& + \left[-(2-m)^2 n^2\delta b_9 + 2(2-m)n\frac{db_9}{dt} + 2b_9\frac{dn}{dt}\right]\cos(2\xi+\phi').
\end{aligned}$$

Hence observing that

$$\frac{1}{a}\frac{da}{dt} = -\frac{2}{3n}\frac{dn}{dt},$$

and substituting in the equations, and further writing

$$\begin{aligned}
\mu e^{-3l} &= u_0 + u_2\cos\phi' + u_5\cos 2\xi + u_8\cos(2\xi-\phi') + u_9\cos(2\xi+\phi'), \\
\frac{3}{2}\frac{m'}{r'^3}\cos 2(\theta-\theta') &= p_0 + p_2\cos\phi' + p_5\cos 2\xi + p_8\cos(2\xi-\phi') + p_9\cos(2\xi+\phi'), \\
\frac{3}{2}\frac{m'}{r'^3}\sin 2(\theta-\theta') &= q_2\sin\phi' + q_5\sin 2\xi + q_8\sin(2\xi-\phi') + q_9\sin(2\xi+\phi'),
\end{aligned}$$

we obtain the following equations by equating to zero the coefficients of the different new terms introduced.

Non-Periodic Term in Second Equation,

$$0 = \frac{dn}{dt}$$

$$-\frac{4}{3}\frac{dn}{dt} - 2n\frac{da_0}{dt} - mb_2\left[mn^2\delta a_2 + n\frac{da_2}{dt}\right] - (2-2m)\,b_5\left[(2-2m)\,n^2\delta a_5 + n\frac{da_5}{dt}\right]$$

$$-(2-3m)\,b_8\left[(2-3m)\,n^2\delta a_8 + n\frac{da_8}{dt}\right] - (2-m)\,b_9\left[(2-m)\,n^2\delta a_9 + n\frac{da_9}{dt}\right]$$

$$+ma_2\left[-mn^2\delta b_2 + n\frac{db_2}{dt}\right] + (2-2m)\,a_5\left[-(2-2m)\,n^2\delta b_5 + n\frac{db_5}{dt}\right]$$

$$+(2-3m)\,a_8\left[-(2-3m)\,n^2\delta b_8 + n\frac{db_8}{dt}\right] + (2-m)\,a_9\left[-(2-m)\,n^2\delta b_9 + n\frac{db_9}{dt}\right]$$

$$+n^2p_2\delta b_2 + n^2p_5\delta b_5 + n^2p_8\delta b_8 + n^2p_9\delta b_9.$$

Coefficient of $\sin\phi'$ in First Equation,

$$0 = m^2n^2\delta a_2 + 2mn\frac{da_2}{dt}$$

$$-2ma_2\left[\frac{2}{3}\frac{dn}{dt} + n\frac{da_0}{dt}\right] + \{(2-3m)\,a_8 - (2-m)\,a_9\}\left[(2-2m)\,n^2\delta a_5 + n\frac{da_5}{dt}\right]$$

$$-(2-2m)\,a_5\left[(2-3m)\,n^2\delta a_8 + n\frac{da_8}{dt}\right] + (2-2m)\,a_5\left[(2-2m)\,n^2\delta a_9 + n\frac{da_9}{dt}\right]$$

$$+2mn^2\delta b_2 - 2n\frac{db_2}{dt} + \{(2-3m)\,b_8 - (2-m)\,b_9\}\left[(2-m)\,n^2\delta b_5 - n\frac{db_5}{dt}\right]$$

$$-(2-2m)\,b_5\left[(2-3m)\,n^2\delta b_8 - n\frac{db_8}{dt}\right] + (2-2m)\,b_5\left[(2-m)\,n^2\delta b_9 - n\frac{db_9}{dt}\right]$$

$$+3n^2u_0\delta a_2 + \frac{3}{2}n^2(u_8 - u_9)\,\delta a_5 - \frac{3}{2}n^2u_5\delta a_8 + \frac{3}{2}n^2u_5\delta a_9$$

$$+n^2(-q_8 + q_9)\,\delta b_5 + n^2q_5\delta b_8 - n^2q_5\delta b_9.$$

Coefficient of $\cos\phi'$ in Second Equation,

$$0 = -m^2n^2\delta b_2 + 2mn\frac{db_2}{dt}$$

$$-2mb_2\left[\frac{2}{3}\frac{dn}{dt} + n\frac{da_0}{dt}\right] - 2\left[mn^2\delta a_2 + n\frac{da_2}{dt}\right]$$

$$-\{(2-3m)\,b_8+(2-m)\,b_9\}\left[(2-2m)\,n^2\delta a_5+n\frac{da_5}{dt}\right]$$
$$-(2-2m)\,b_5\left[(2-3m)\,n^2\delta a_8+n\frac{da_8}{dt}\right]$$
$$-(2-2m)\,b_5\left[(2-m)\,n^2\delta a_9+n\frac{da_9}{dt}\right]$$
$$+\{(2-3m)\,a_8+(2-m)\,a_9\}\left[-(2-2m)\,n^2\delta b_5+n\frac{db_5}{dt}\right]$$
$$+(2-2m)\,a_5\left[-(2-3m)\,n^2\delta b_8+n\frac{db_8}{dt}\right]$$
$$+(2-2m)\,a_5\left[-(2-m)\,n^2\delta b_9+n\frac{db_9}{dt}\right]$$
$$+2n^2p_0\delta b_2+n^2(p_8+p_9)\,\delta b_5+n^2p_5\delta b_8+n^2p_5\delta b_9.$$

Coefficient of $\sin 2\xi$ in First Equation,

$$0=\quad(2-2m)^2\,n^2\delta a_5+2\,(2-2m)\,n\frac{da_5}{dt}+2a_5\frac{dn}{dt}$$
$$-2\,(2-2m)\,na_5\left[\frac{2}{3}\frac{dn}{dt}+n\frac{da_0}{dt}\right]-\{(2-3m)\,a_8+(2-m)\,a_9\}\left[mn^2\delta a_2+n\frac{da_2}{dt}\right]$$
$$-ma_2\left[(2-3m)\,n^2\delta a_8+n\frac{da_8}{dt}\right]+ma_2\left[(2-m)\,n^2\delta a_9+n\frac{da_9}{dt}\right]$$
$$+\{(2-3m)\,b_8-(2-m)\,b_9\}\left[mn^2\delta b_2-n\frac{db_2}{dt}\right]+2\left[(2-2m)\,n^2\delta b_5-n\frac{db_5}{dt}\right]$$
$$+mb_2\left[(2-3m)\,n^2\delta b_8-n\frac{db_8}{dt}\right]+mb_2\left[(2-m)\,n^2\delta b_9-n\frac{db_9}{dt}\right]$$
$$+\frac{3}{2}n^2(u_8-u_9)\,\delta a_2+3n^2u_0\delta a_5+\frac{3}{2}n^2u_2\delta a_8+\frac{3}{2}n^2u_2\delta a_9$$
$$+n^2(q_8+q_9)\,\delta b_2+n^2q_2\delta b_8-n^2q_2\delta b_9.$$

Coefficient of $\cos 2\xi$ in Second Equation,

$$0=-(2-2m)^2\,n^2\delta b_5+2\,(2-2m)\,n\frac{db_5}{dt}+2b_5\frac{dn}{dt}$$
$$-2\,(2-2m)\,b_5\left[\frac{2}{3}\frac{dn}{dt}+n\frac{da_0}{dt}\right]-\{(2-3m)\,b_8+(2-m)\,b_9\}\left[mn^2\delta a_2+n\frac{da_2}{dt}\right]$$
$$-2\left[(2-2m)\,n^2\delta a_5+n\frac{da_5}{dt}\right]-mb_2\left[(2-3m)\,n^2\delta a_8+n\frac{da_8}{dt}\right]$$

$$-mb_2\left[(2-m)\,n^2\delta a_9+n\frac{da_9}{dt}\right]$$

$$+\{-(2-3m)\,a_8+(2-m)\,a_9\}\left[-mn^2\delta b_2+n\frac{db_2}{dt}\right]$$

$$-ma_2\left[-(2-3m)\,n^2\delta b_8+n\frac{db_8}{dt}\right]$$

$$+ma_2\left[-(2-m)\,n^2\delta b_9+n\frac{db_9}{dt}\right]$$

$$+n^2(p_8+p_9)\,\delta b_2+2n^2p_0\delta b_5+n^2p_2\delta b_8+n^2p_2\delta b_9.$$

Coefficient of $\sin(2\xi-\phi')$ in First Equation,

$$0=\quad(2-3m)^2\,n^2\delta a_8+2\,(2-3m)\,n\frac{da_8}{dt}+2a_8\frac{dn}{dt}$$

$$-2\,(2-3m)\,a_8\left[\frac{2}{3}\frac{dn}{dt}+n\frac{da_0}{dt}\right]-(2-2m)\,a_5\left[mn^2\delta a_2+n\frac{da_2}{dt}\right]$$

$$-ma_2\left[(2-2m)\,n^2\delta a_5+n\frac{da_5}{dt}\right]$$

$$-(2-2m)\,b_5\left[mn^2\delta b_2-n\frac{db_2}{dt}\right]+mb_2\left[(2-2m)\,n^2\delta b_5-n\frac{db_5}{dt}\right]$$

$$+2\left[(2-3m)\,n^2\delta b_8-n\frac{db_8}{dt}\right]$$

$$-\frac{3}{2}n^2u_5\delta a_2+\frac{3}{2}n^2u_2\delta a_5+3n^2u_0\delta a_8$$

$$+n^2q_5\delta b_2-n^2q_2\delta b_5.$$

Coefficient of $\cos(2\xi-\phi')$ in Second Equation,

$$0=-(2-3m)^2\,n^2\delta b_8+2\,(2-3m)\,n\frac{db_8}{dt}+2b_8\frac{dn}{dt}$$

$$-2\,(2-3m)\,b_8\left[\frac{2}{3}\frac{dn}{dt}+n\frac{da_0}{dt}\right]-(2-2m)\,b_5\left[mn^2\delta a_2+n\frac{da_2}{dt}\right]$$

$$-mb_2\left[(2-2m)\,n^2\delta a_5+n\frac{da_5}{dt}\right]-2\left[(2-3m)\,n^2\delta a_8+n\frac{da_8}{dt}\right]$$

$$+(2-2m)\,a_5\left[-mn^2\delta b_2+n\frac{db_2}{dt}\right]-ma_2\left[-(2-2m)\,n^2\delta b_5+n\frac{db_5}{dt}\right]$$

$$+n^2p_5\delta b_2+n^2p_2\delta b_5+2n^2p_0\delta b_8.$$

Coefficient of $\sin(2\xi+\phi')$ in First Equation,

$$
\begin{aligned}
0 = \; & (2-m)^2 n^2 \delta a_9 + 2(2-m) n \frac{da_9}{dt} + 2a_9 \frac{dn}{dt} \\
& - 2(2-m) a_0 \left[\frac{2}{3}\frac{dn}{dt} + n \frac{da_0}{dt}\right] - (2-2m) a_5 \left[mn^2 \delta a_2 + n \frac{da_2}{dt}\right] \\
& - ma_2 \left[(2-2m) n^2 \delta a_5 + n \frac{da_5}{dt}\right] \\
& + (2-2m) b_5 \left[mn^2 \delta b_2 - n \frac{db_2}{dt}\right] + mb_2 \left[(2-2m) n^2 \delta b_5 - n \frac{db_5}{dt}\right] \\
& + 2\left[(2-m) n^2 \delta b_9 - n \frac{db_9}{dt}\right] \\
& + \frac{3}{2} n^2 u_5 \delta a_2 + \frac{3}{2} n^2 u_2 \delta a_5 + 3n^2 u_0 \delta a_9 \\
& + n^2 q_5 \delta b_2 + n^2 q_2 \delta b_5 .
\end{aligned}
$$

Coefficient of $\cos(2\xi+\phi')$ in Second Equation,

$$
\begin{aligned}
0 = & - (2-m)^2 n^2 \delta b_9 + 2(2-m) n \frac{db_9}{dt} + 2b_9 \frac{dn}{dt} \\
& - 2(2-m) b_9 \left[\frac{2}{3}\frac{dn}{dt} + 2n \frac{da_0}{dt}\right] - (2-2m) b_5 \left[mn^2 \delta a_2 + n \frac{da_2}{dt}\right] \\
& - mb_2 \left[(2-2m) n^2 \delta a_5 + n \frac{da_5}{dt}\right] - 2\left[(2-m) n^2 \delta a_9 + n \frac{da_9}{dt}\right] \\
& - (2-2m) a_5 \left[-mn^2 \delta b_2 + n \frac{db_2}{dt}\right] - ma_2 \left[-(2-2m) n^2 \delta b_5 + n \frac{db_5}{dt}\right] \\
& + n^2 p_5 \delta b_2 + n^2 p_2 \delta b_5 + 2n^2 p_0 \delta b_9 .
\end{aligned}
$$

We now proceed to form these equations numerically with the following values for the known quantities:—

$a_0 = \cdot 00089,40892$	
$a_2 = e'(-\cdot 00692,95)$	$b_2 = e'(-\cdot 19057,67)$
$a_5 = \cdot 00717,9892$	$b_5 = \cdot 01021,1346$
$a_8 = e'(\cdot 03036,04)$	$b_8 = e'(\cdot 04396,33)$
$a_9 = e'(-\cdot 00445,12)$	$b_9 = e'(-\cdot 00624,23)$

$$\frac{da_0}{dt} = (-\cdot 00169{,}774)\frac{dn}{ndt} + (\cdot 00101{,}356)\frac{d(e'^2)}{dt}$$

$$\frac{da_5}{dt} = (-\ \cdot 01624{,}57)\frac{dn}{ndt} + (-\cdot 02411{,}57)\frac{d(e'^2)}{dt}$$

$$\frac{db_5}{dt} = (-\ \cdot 02348{,}36)\frac{dn}{ndt} + (-\cdot 03435{,}74)\frac{d(e'^2)}{dt}$$

$$\begin{array}{lll} u_0 = 1\cdot 00280{,}21804 & p_0 = -\cdot 00008{,}57021 & \\ u_2 = e'(-\cdot 02000{,}72) & p_2 = e'(-\cdot 00057{,}37) & q_2 = e'(-\cdot 00004{,}59) \\ u_5 = \ \ \cdot 02159{,}9810 & p_5 = \ \ \cdot 00839{,}2060 & q_5 = \ \ \cdot 00839{,}1893 \\ u_8 = e'(\cdot 09111{,}95) & p_8 = e'(\cdot 03096{,}86) & q_8 = e'(\cdot 03096{,}70) \\ u_9 = e'(-\cdot 01360{,}74) & p_9 = e'(-\cdot 00579{,}81) & q_9 = e'(-\cdot 00579{,}85). \end{array}$$

[The MS. in which these numbers were derived has not been found; the Variation terms will be found to agree closely with the more accurate values of p. 106; the coefficient b_5 is comparable directly; a_0 and a_5 may be found by forming $\log 1/r$; u_0 and u_5 shew that μ is taken equal to unity; the errors are in the eighth place of decimals. The terms in e' may have been found by the method of Lecture X.; they are somewhat more correct than the values there given; the terms in e'^2 would result from a second application of that method; they appear to be far more correct than can be found by transformation of *e.g.* Delaunay's expression for the longitude. The coefficients of $\frac{dn}{ndt}$ in $\frac{da_0}{dt}$, &c. may be derived from the results of p. 113; *e.g.* we have

$$\frac{da_0}{dt} = \frac{dn'}{n'dt}(0\cdot 00157{,}0744) = \frac{1}{1-m}\frac{dm}{mdt}(0\cdot 00157{,}0744)$$

$$= -\frac{dn}{ndt}(0\cdot 00169{,}7737).]$$

We now observe that the coefficients given by the non-periodic term and the coefficients of $\cos 2\xi$, $\sin 2\xi$ involve e'^2 as the lowest order, while all the rest involve e' in the first power and after that the cube and other odd powers. Clearly we may omit from our equations all terms except those of lowest order.

Hence our equations are the following:—

Non-periodic Term,

$$0=\frac{dn}{ndt}(-\cdot329943)+\frac{e'de'}{dt}(-\cdot00405583)$$

$$+n\delta a_2e'(\cdot0010663)+n\delta a_5(-\cdot0349633)+n\delta a_8e'(-\cdot1386050)$$
$$+n\delta a_9e'(\cdot0231365)$$

$$+n\delta b_2e'(-\cdot0005349)+n\delta b_5(-\cdot0161916)+n\delta b_8e'(-\cdot0647496)$$
$$+n\delta b_9e'(\cdot0106998).$$

Coefficient of $\sin\phi'$,

$$0=\frac{de'}{dt}(\cdot3806029)+n\delta a_2(\ 3\cdot01400)+n\delta a_8(-\cdot0559897)+n\delta a_9(\cdot0579772)$$
$$+n\delta b_2(\cdot1496026)+n\delta b_8(-\cdot0251580)+n\delta b_9(\cdot0279848).$$

Coefficient of $\cos\phi'$,

$$0=\frac{de'}{dt}(-\cdot0146402)+n\delta a_2(-\ \cdot1496026)+n\delta a_8(-\cdot0335499)+n\delta a_9(-\cdot0363767)$$
$$+n\delta b_2(-\cdot00576663)+n\delta b_8(-\cdot0151979)+n\delta b_9(-\cdot0171854).$$

Coefficient of $\sin 2\xi$,

$$0=\frac{dn}{ndt}(-\cdot0164641)+\frac{e'de'}{dt}(-\cdot0230823)$$

$$+n\delta a_2e'(\cdot1536989)+n\delta a_5(6\cdot43238)+n\delta a_8e'(-\cdot0290904)+n\delta a_9e'(-\cdot0310087)$$
$$+n\delta b_2e'(\cdot0319065)+n\delta b_5(3\cdot70079)+n\delta b_8e'(-\cdot0253577)+n\delta b_9e'(-\cdot0273986).$$

Coefficient of $\cos 2\xi$,

$$0=\frac{dn}{ndt}(-\cdot0591231)+\frac{e'de'}{dt}(-\cdot1451536)$$

$$+n\delta a_2e'(-\cdot0049402)+n\delta a_5(-3\cdot70079)+n\delta a_8e'(\cdot0253118)+n\delta a_9e'(\cdot0274445)$$
$$+n\delta b_2e'(\cdot0298439)+n\delta b_5(-3\cdot42414)+n\delta b_8e'(-\cdot0014941)+n\delta b_9\,e'(\cdot0004242).$$

Coefficient of $\sin(2\xi-\phi')$,

$$0=\frac{de'}{dt}(\cdot0163800)+n\delta a_2(-\cdot0332935)+n\delta a_8(6\cdot16115)$$
$$+n\delta b_2(\cdot0069785)+n\delta b_8(3\cdot55119).$$

Coefficient of $\cos(2\xi - \phi')$,

$$0 = \frac{de'}{dt}(\cdot 0930002) + n\delta a_2(-\cdot 0014134) + n\delta a_8(-3\cdot 55119)$$
$$+ n\delta b_2(\cdot 0073983) + n\delta b_8(-3\cdot 15291).$$

Coefficient of $\sin(2\xi + \phi')$,

$$0 = \frac{de'}{dt}(-\cdot 0009612) + n\delta a_2(\cdot 0314059) + n\delta a_9(6\cdot 71480)$$
$$+ n\delta b_2(\cdot 0098053) + n\delta b_9(3\cdot 85040).$$

Coefficient of $\cos(2\xi + \phi')$,

$$0 = \frac{de'}{dt}(-\cdot 0124702) + n\delta a_2(-\cdot 0014134) + n\delta a_9(-3\cdot 85040)$$
$$+ n\delta b_2(\cdot 0093859) + n\delta b_9(-3\cdot 70656).$$

These equations give

$$\delta a_2 = \frac{de'}{ndt}(\cdot 0017349), \qquad \delta b_2 = \frac{de'}{ndt}(-\ 2\cdot 58480),$$

$$\delta a_8 = \frac{de'}{ndt}(-\cdot 0377033), \qquad \delta b_8 = \frac{de'}{ndt}(\cdot 0658969),$$

$$\delta a_9 = \frac{de'}{ndt}(\cdot 0237240), \qquad \delta b_9 = \frac{de'}{ndt}(-\cdot 0345550),$$

and

$$\delta a_5 = \frac{e'de'}{ndt}(\cdot 1420895), \qquad \delta b_5 = \frac{e'de'}{ndt}(-\cdot 2184587),$$

$$\frac{dn}{ndt} = \frac{e'de'}{dt}(-\cdot 00898284).$$

If we take

$$\int (e'^2 - E'^2)\, ndt = -1270'' \left(\frac{t}{100}\right)^2,$$

the last gives the secular acceleration

$$5''\cdot 7041.$$

The value found by including terms of the series which represents the same up to m^7 or m^8, and estimating the remainder is $5''\cdot 70$.

Dec. 1859.

5.

NEISON'S LUNAR INEQUALITY.

DIARY, 1877, August 13. "In the course of the day while in the train and also while walking about in London thought out a way of deriving a very approximate value of the coefficient of Neison's inequality due to Jupiter from the coefficient of the Lunar Evection."

In a comparison of Hansen's "Tables de la Lune" with observations made at Greenwich and Washington, Professor Newcomb discovered an inequality of eccentricity and longitude of perigee which may be expressed*

$$e d\varpi = +0''\cdot 75 \sin N,$$

$$de = -0\ \cdot 75 \cos N,$$

where

$$N = 253^\circ\cdot 2 + 21^\circ\cdot 6\,(t - 1868\cdot 1).$$

The explanation of this was found by Mr Neison, whose researches revealed an inequality due to the action of Jupiter

$$\delta l = \quad 2''\cdot 20 \sin(2\varpi - 2J),$$

$$e\delta\varpi = \quad 0\ \cdot 58 \sin(2\varpi - 2J),$$

$$\delta e = -0\ \cdot 58 \cos(2\varpi - 2J),$$

and

$$2\varpi - 2J = 261^\circ\cdot 4 + 20^\circ\cdot 85\,(t - 1868\cdot 1).$$

* See *Mon. Not.* XXXVII., p. 428.

It follows from the equation

$$\theta = nt + \epsilon + 2e \sin(nt + \epsilon - \varpi) + \dots,$$

that the inequality in longitude

$$= -1''\cdot16 \sin(2\varpi - 2J + A)$$

$$= -1''\cdot16 \sin(2\overline{M-J} - A),$$

where M, J are the mean longitudes of the Moon and Jupiter, and A is the Moon's mean anomaly.

Thus the argument of the inequality is analogous to that of the Evection, and the inequality may be considered an Evection produced by Jupiter. From this point of view a very approximate value of its coefficient may be deduced from the coefficient of the solar evection, as follows.

The Sun makes a complete revolution with respect to the Moon's perigee in about 1·127 years; and the Moon's perigee makes a complete revolution with respect to Jupiter in about 34·532 years. The ratio of these is 30·631. The mass of Jupiter is $\frac{1}{1050}$ of that of the Sun; if Jupiter were at his mean distance, the inverse cube of the distance would be about $\frac{1}{140\cdot6}$ that of the Sun. The average inverse cube of his distance is greater than this in the ratio 1·088 to 1, and is therefore about $\frac{1}{129\cdot2}$ that of the Sun.

Therefore the coefficient of the evection being about 4600″, that of the analogous inequality due to Jupiter will be

$$\frac{30\cdot63}{129\cdot2} \cdot \frac{4600}{1050} = \frac{4600}{4430} = 1''\cdot04 \text{ nearly.}$$

Also the sign will be opposite to that of the Evection since the Moon's perigee advances faster than Jupiter.

[A paper probably including among other remarks upon recent advances in Lunar Theory some such matter as the above, was read at the British Association Meeting, 1877. Its title only is published in the Report.]

6.

A METHOD OF SOLVING THE EQUATION $\frac{d^2w}{dt^2}+Qw=0$, WHERE $Q=q_0^2+2q_1\cos 2t+2q_2\cos 4t+2q_3\cos 6t+\ldots$

[THIS is the equation which Adams employed to discuss the Lunar Inequalities depending upon the first power of the inclination, and to which Hill reduced the problem of finding those depending on the first power of the eccentricity. From this double claim to the first rank of importance the following method of solving it derives its interest.]

Two methods of solution have been given already; firstly it may be solved by evaluation of an infinite determinant as in p. 86 *et seqq.*; or again by the method employed in the Lectures on the Lunar Theory, XIV. But in both the important cases to which the equation applies, q_0 is not very different from unity. Hence if we write

$$w=c\cos(kt+\beta)+\&c.,$$

k will also be nearly equal to unity, and it is desirable to find a method which avoids the introduction of the two small quantities $q_0^2-k^2$, $q_0^2-(k-2)^2$, as divisors.

Let us assume, omitting c,

$$\begin{aligned} w &= \cos(kt+\beta)+c_{-1}\cos(\overline{k-2}\,t+\beta)+c_{-2}\cos(\overline{k-4}\,t+\beta)+\ldots \\ &\qquad +c_1\cos(\overline{k+2}\,t+\beta)+c_2\cos(\overline{k+4}\,t+\beta)+\ldots \\ &= w_0+\delta_1 w+\delta_2 w+\ldots \end{aligned}$$

where, in the first place,

$$w_0=\cos(kt+\beta)+c_{-1}\cos(\overline{k-2}\,t+\beta).$$

Let the result of substituting w_0 on the left-hand of the equation for w be

$$Q_0\cos(kt+\beta)+Q_{-1}\cos(\overline{k-2}\,t+\beta)+\ldots$$
$$+Q_1\cos(\overline{k+2}\,t+\beta)+\ldots$$

and let $\delta_1 w$ be determined from the equation

$$0=\frac{d^2\delta_1 w}{dt^2}+q_0^2\delta_1 w+Q_{-2}\cos(\overline{k-4}\,t+\beta)$$
$$+Q_1\cos(\overline{k+2}\,t+\beta)$$

so that $\delta_1 w$ contains only terms involving the angles

$$\overline{k-4}\,t+\beta,\quad \overline{k+2}\,t+\beta.$$

Now let the result of substituting $\delta_1 w$ in the product

$$[2q_1\cos 2t+2q_2\cos 4t+\ldots]\,\delta_1 w$$

be

$$\delta_1 Q_0\cos(kt+\beta)+\delta_1 Q_{-1}\cos(\overline{k-2}\,t+\beta)+\ldots$$
$$+\delta_1 Q_1\cos(\overline{k+2}\,t+\beta)+\ldots.$$

And again let $\delta_2 w$ be determined by means of the equation

$$0=\frac{d^2\delta_2 w}{dt^2}+q_0^2\delta_2 w+\delta_1 Q_{-2}\cos(\overline{k-4}\,t+\beta)+(Q_{-3}+\delta_1 Q_{-3})\cos(\overline{k-6}\,t+\beta)$$
$$+\delta_1 Q_1\cos(\overline{k+2}\,t+\beta)+(Q_2+\delta_1 Q_2)\cos(\overline{k+4}\,t+\beta),$$

so that $\delta_2 w$ contains only terms involving the angles

$$\overline{k-4}\,t+\beta,\quad \overline{k-6}\,t+\beta,$$
$$\overline{k+2}\,t+\beta,\quad \overline{k+4}\,t+\beta.$$

Similarly let the result of substituting $\delta_2 w$ in the product

$$[2q_1\cos 2t+2q_2\cos 4t+\ldots]\,\delta_2 w$$

be

$$\delta_2 Q_0\cos(kt+\beta)+\delta_2 Q_{-1}\cos(\overline{k-2}\,t+\beta)+\ldots$$
$$+\delta_2 Q_1\cos(\overline{k+2}\,t+\beta)+\ldots,$$

and determine $\delta_3 w$ by means of the equation

$$0=\frac{d^2\delta_3 w}{dt^2}+q_0^2\delta_3 w+\delta_2 Q_{-2}\cos(\overline{k-4}\,t+\beta)+\ldots+(Q_{-4}+\delta_1 Q_{-4}+\delta_2 Q_{-4})\cos(\overline{k-8}\,t+\beta)$$
$$+\delta_2 Q_1\cos(\overline{k+2}\,t+\beta)+\ldots+(Q_3+\delta_1 Q_3+\delta_2 Q_3)\cos(\overline{k+6}\,t+\beta),$$

the angles involved being

$$\overline{k-4}\,t+\beta,\quad \overline{k-6}\,t+\beta,\quad \overline{k-8}\,t+\beta,$$

$$\overline{k+2}\,t+\beta,\quad \overline{k+4}\,t+\beta,\quad \overline{k+6}\,t+\beta.$$

Proceed in this way as far as may be necessary for the degree of approximation desired. Then the result of substituting

$$w=w_0+\delta_1 w+\delta_2 w+\ldots$$

in the equation

$$\frac{d^2w}{dt^2}+Qw=0,$$

will be

$$(Q_0+\delta_1 Q_0+\delta_2 Q_0+\ldots)\cos(kt+\beta)+(Q_{-1}+\delta_1 Q_{-1}+\delta_2 Q_{-1}+\ldots)\cos(\overline{k-2}\,t+\beta),$$

where the coefficients $\cos(kt+\beta)$, $\cos(\overline{k-2}\,t+\beta)$ involve k and c_{-1}. If these coefficients be equated to zero we have means of determining k and c_{-1}.

[30 *May* 1884.]

[To illustrate this method somewhat further; substitute

$$w_0=\cos kt+c_{-1}\cos(k-2)\,t$$

in the differential equation; then we get

$$Q_0=q_0^2-k^2+q_1c_{-1},\quad Q_{-1}=q_1+c_{-1}q_0^2-c_{-1}(k-2)^2,$$

$$Q_1=q_1+c_{-1}q_2,\qquad Q_{-2}=q_2+c_{-1}q_1;$$

....

then take the equation

$$\frac{d^2\delta_1 w}{dt^2}+q_0^2\delta_1 w+Q_{-2}\cos(k-4)\,t+Q_1\cos(k+2)\,t,$$

whence

$$\delta_1 w=\frac{Q_{-2}}{(k-4)^2-q_0^2}\cos(k-4)\,t+\frac{Q_1}{(k+2)^2-q_0^2}\cos(k+2)\,t;$$

substitute this in the expression

$$(2q_1\cos 2t+2q_2\cos 4t+\ldots)\,\delta_1 w,$$

and we get

$$\delta_1 Q_0=\frac{q_1Q_1}{(k+2)^2-q_0^2}+\frac{q_2Q_{-2}}{(k-4)^2-q_0^2},\quad \delta_1 Q_{-1}=\frac{q_1Q_{-2}}{(k-4)^2-q_0^2}+\frac{q_2Q_1}{(k+2)^2-q_0^2},$$

....

and so on. If we stop at this stage the equations to determine k and c_1 are

$$Q_0+\delta_1 Q_0=0,\quad Q_{-1}+\delta_1 Q_{-1}=0.$$

For example, take the equation of Lecture XIV. for finding the Moon's latitude; here with our present notation

$$q_0^2=1{\cdot}17803{,}9,\quad q_1={\cdot}01261{,}5,\quad q_2={\cdot}00012{,}6.$$

Hence

$$Q_1={\cdot}01261{,}5+{\cdot}00012{,}6c_{-1},\quad Q_{-2}={\cdot}00012{,}6+{\cdot}01261{,}5c_{-1},$$

and taking $k=q_0$ as sufficient approximation in $\delta_1 Q_0$, $\delta_1 Q_1$ we have

$$\delta_1 Q_0={\cdot}00001{,}9,\quad \delta_1 Q_{-1}={\cdot}00002{,}2c_{-1},$$

and the equations for k and c_{-1} are

$$k^2-{\cdot}01261{,}5c_{-1}=1{\cdot}17805{,}8,$$

$$c_1\left[(k-2)^2-1{\cdot}17806{,}1\right]={\cdot}01261{,}5,$$

whence

$$k=1{\cdot}08517{,}1,\quad c_{-1}=-{\cdot}03698{,}3,$$

which may be compared with the results of p. 103.]

7.

THEORY OF JUPITER'S SATELLITES.

[LECTURES on the Theory of Jupiter's Satellites were given in 1878 and again in 1880. They included matter which did not differ from Laplace, and this has been omitted. Moreover as it did not seem to add to clearness to preserve the division into lectures, what remained, that was original and characteristic, has been cast into the form of three essays.

As an account of the whole problem these are incomplete in detail, and would require much development before they could be applied to such questions as the determination of the masses and other constants from observation; but they are of interest because they seem to indicate some outlines of the plan upon which Adams would have attacked the entire problem.]

I.

MOTION OF A SATELLITE ABOUT AN OBLATE PRIMARY, IN AN APPROXIMATELY ELLIPTICAL ORBIT INCLINED AT A FINITE ANGLE TO THE EQUATOR OF THE PRIMARY.

The potential at any external point of an oblate spheroid of slight ellipticity, whose free surface is a level surface under the attractions of its body and the centrifugal forces due to a rotation about its axis of symmetry, is of the form

$$\frac{\mu}{r}+\frac{\nu}{r^3}\left(\frac{1}{3}-\frac{z^2}{r^2}\right),$$

where r, z are the distances of the point from the centre and the equatoreal plane of the spheroid, respectively. In this expression, the

square of the ellipticity is neglected, μ is equal to the mass of the spheroid, and

$$\nu = \mu A^2 \left(\rho - \frac{1}{2}\phi\right),$$

where A is the equatoreal radius, ρ the ellipticity, and ϕ the ratio of centrifugal force to gravity at the equator.

The equations of motion of a small body moving under the attraction of this spheroid are the following:—

$$\frac{d^2x}{dt^2} = -\frac{\mu}{r^2}\frac{x}{r} - \frac{\nu}{r^4}\frac{x}{r} + 5\frac{\nu}{r^4}\frac{xz^2}{r^3},$$

$$\frac{d^2y}{dt^2} = -\frac{\mu}{r^2}\frac{y}{r} - \frac{\nu}{r^4}\frac{y}{r} + 5\frac{\nu}{r^4}\frac{yz^2}{r^3},$$

$$\frac{d^2z}{dt^2} = -\frac{\mu}{r^2}\frac{z}{r} - \frac{\nu}{r^4}\frac{z}{r} + 5\frac{\nu}{r^4}\frac{z^3}{r^3} - 2\frac{\nu}{r^4}\frac{z}{r}.$$

From these we derive immediately

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\frac{\mu}{r} + \frac{2}{3}\frac{\nu}{r^3} - 2\frac{\nu z^2}{r^5} - C,$$

where C is a constant, and

$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} = -\frac{\mu}{r} - \frac{\nu}{r^3} + 3\frac{\nu z^2}{r^5}.$$

Add these together:—

$$\frac{d}{dt}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) = \frac{\mu}{r} - \frac{1}{3}\frac{\nu}{r^3} + \frac{\nu z^2}{r^5} - C,$$

or

$$\frac{d}{dt}\left(r\frac{dr}{dt}\right) = \frac{\mu}{r} - \frac{1}{3}\frac{\nu}{r^3} + \frac{\nu z^2}{r^5} - C,$$

or again

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2) = \frac{\mu}{r} - \frac{1}{3}\frac{\nu}{r^3} + \frac{\nu z^2}{r^5} - C.$$

Also

$$\frac{d^2z}{dt^2} = -\frac{\mu z}{r^3} - 3\frac{\nu z}{r^5} + 5\frac{\nu z^3}{r^7}.$$

Thus we have eliminated one coordinate, and have obtained a pair of simultaneous equations between r and z; we proceed to integrate these equations. To integrate completely we must introduce four new arbitrary constants. We shall first consider the case in which two only are present, namely the inclination, and the longitude of the node.

Assume

$$z = ac[\sin(qt+\gamma) + c_1 \sin 3(qt+\gamma) + c_2 \sin 5(qt+\gamma)],$$

$$\frac{1}{r} = \frac{1}{a}[\quad 1 \quad + a_1 \cos 2(qt+\gamma) + a_2 \cos 4(qt+\gamma)],$$

where c, γ determine the inclination and the position of the node, while a is a third arbitrary which, it will appear, is involved in C already introduced. For brevity we shall omit the constant γ, and write

$$\frac{\nu}{\mu a^2} = f.$$

Considering f as a small quantity of the first order, we shall find that a_1, c_1, are of the first order and a_2, c_2, of the second.

Substitute for r and z, and we find to the second order

$$r^2 = a^2\left[1 - 2a_1 \cos 2qt - 2a_2 \cos 4qt + \frac{3}{2} a_1^2 (1 + \cos 4qt)\right],$$

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2) = a^2[4q^2 a_1 \cos 2qt + (16q^2 a_2 - 12a_1^2 q^2) \cos 4qt],$$

$$\frac{\mu}{r} = \frac{\mu}{a}[1 + a_1 \cos 2qt + a_2 \cos 4qt],$$

$$-\frac{1}{3}\frac{\nu}{r^3} = -\frac{1}{3}\frac{\nu}{a^3}[1 + 3a_1 \cos 2qt],$$

$$\frac{\nu z^2}{r^5} = \frac{\nu c^2}{a^3}\left[\frac{1}{2} - \frac{5}{4} a_1 + \left(-\frac{1}{2} + c_1 + \frac{5}{2} a_1\right) \cos 2qt + \left(-c_1 - \frac{5}{4} a_1\right) \cos 4qt\right].$$

The constant term merely gives the relation between a and C; equate the coefficients of the periodic terms and we get, after dividing throughout by a^2,

$$4q^2 a_1 = \frac{\mu}{a^3}\left[a_1 - fa_1 - \frac{1}{2} fc^2 + \frac{5}{2} fc^2 a_1 + fc^2 c_1\right].$$

This gives a_1 to the second order when c_1 is known to the first order,

$$(16a_2 - 12a_1^2)\, q^2 = \frac{\mu}{a^3}\left[a_2 - \frac{5}{4} fc^2 a_1 - fc^2 c_1\right].$$

This gives a_2 when a_1 and c_1 are known to the first order.

Again

$$-\frac{d^2z}{dt^2} = acq^2\left[\sin qt + 9c_1 \sin 3qt + 25c_2 \sin 5qt\right],$$

$$\begin{aligned}
\frac{\mu z}{r^3} &= \frac{\mu c}{a^2}\left[1 + 3a_1 \cos 2qt + 3a_2 \cos 4qt + \frac{3}{2} a_1^2 (1 + \cos 4qt)\right] \\
&\qquad \times \left[\sin qt + c_1 \sin 3qt + c_2 \sin 5qt\right] \\
&= \frac{\mu c}{a^2}\Bigg[\sin qt + c_1 \sin 3qt + c_2 \sin 5qt \\
&\qquad + \frac{3}{2} a_1 (-\sin qt + \sin 3qt) + \frac{3}{2} a_1 c_1 (\sin qt + \sin 5qt) \\
&\qquad + \frac{3}{2} a_2 (-\sin 3qt + \sin 5qt) + \frac{3}{2} a_1^2 \sin qt + \frac{3}{4} a_1^2 (-\sin 3qt + \sin 5qt)\Bigg].
\end{aligned}$$

$$\begin{aligned}
\frac{3\nu z}{r^5} &= \frac{3\nu c}{a^4}\left[1 + 5a_1 \cos 2qt\right]\left[\sin qt + c_1 \sin 3qt\right] \\
&= \frac{3\nu c}{a^4}\left[\sin qt + c_1 \sin 3qt + \frac{5}{2} a_1 (-\sin qt + \sin 3qt)\right],
\end{aligned}$$

$$\begin{aligned}
-\frac{5\nu z^3}{r^7} &= -\frac{5\nu c^3}{a^4}\left[1 + 7a_1 \cos 2qt\right]\left[\frac{1}{4}(3 \sin qt - \sin 3qt) + \frac{3}{2}(1 - \cos 2qt)\, c_1 \sin 3qt\right] \\
&= -\frac{5\nu c^3}{a^4}\left[1 + 7a_1 \cos 2qt\right]\Bigg[\frac{3}{4} \sin qt - \frac{1}{4} \sin 3qt + \frac{3}{2} c_1 \sin 3qt \\
&\qquad - \frac{3}{4} c_1 \sin qt - \frac{3}{4} c_1 \sin 5qt\Bigg] \\
&= -\frac{5\nu c^3}{a^4}\Bigg[\left(\frac{3}{4} - \frac{3}{4} c_1\right) \sin qt - \left(\frac{1}{4} - \frac{3}{2} c_1\right) \sin 3qt - \frac{3}{4} c_1 \sin 5qt \\
&\qquad + \frac{21}{8} a_1 (-\sin qt + \sin 3qt) - \frac{7}{8} a_1 (\sin qt + \sin 5qt)\Bigg] \\
&= -\frac{5\nu c^3}{a^4}\Bigg[\left(\frac{3}{4} - \frac{3}{4} c_1 - \frac{7}{2} a_1\right) \sin qt - \left(\frac{1}{4} - \frac{3}{2} c_1 - \frac{21}{8} a_1\right) \sin 3qt \\
&\qquad - \left(\frac{3}{4} c_1 + \frac{7}{8} a_1 \sin 5qt\right)\Bigg].
\end{aligned}$$

Now equate coefficients of corresponding terms, dividing throughout by ac

$$q^2 = \frac{\mu}{a^3}\left[1 - \frac{3}{2}a_1 + \frac{3}{2}a_1c_1 + \frac{3}{2}a_1^2 + 3f\left(1 - \frac{5}{2}a_1\right) - 5fc^2\left(\frac{3}{4} - \frac{3}{4}c_1 - \frac{7}{2}a_1\right)\right],$$

$$9q^2c_1 = \frac{\mu}{a^3}\left[c_1 + \frac{3}{2}a_1 - \frac{3}{2}a_2 - \frac{3}{4}a_1^2 + 3f\left(c_1 + \frac{5}{2}a_1\right) + 5fc^2\left(\frac{1}{4} - \frac{3}{2}c_1 - \frac{21}{8}a_1\right)\right],$$

$$25q^2c_2 = \frac{\mu}{a^3}\left[c_2 + \frac{3}{2}a_1c_1 + \frac{3}{2}a_2 + \frac{3}{4}a_1^2 + 5fc^2\left(\frac{3}{4}c_1 + \frac{7}{8}a_1\right)\right].$$

The first of these gives the relation between q^2 and $\frac{\mu}{a^3}$ when a_1 and c_1 are known;
the second gives c_1 in terms of a_1 and a_2;
the third gives c_2 in terms of c_1, a_1, a_2.

Now substitute the value of q^2 given by the first of these latter equations in all the rest and divide out by the common factor μ/a^3, and we have the following equations for the determination of the coefficients a_1, a_2, c_1, c_2, taking into account terms of the second order,

$$4a_1 - 6a_1^2 + 12fa_1 - 15fc^2a_1 = a_1 - fa_1 - \frac{1}{2}fc^2 + \frac{5}{2}fc^2a_1 + fc^2c_1,$$

or

$$3a_1 - 6a_1^2 + 13fa_1 - \frac{35}{2}fc^2a_1 = -\frac{1}{2}fc^2 + fc^2c_1 \quad\text{.................................... (1)},$$

$$16a_2 - 12a_1^2 = a_2 - \frac{5}{4}fc^2a_1 - fc^2c_1,$$

or

$$15a_2 = 12a_1^2 - \frac{5}{4}fc^2a_1 - fc^2c_1 \quad\text{.............................. (2)},$$

$$9c_1 - \frac{27}{2}a_1c_1 + 27fc_1 - \frac{135}{4}fc^2c_1 = c_1 + \frac{3}{2}a_1 - \frac{3}{2}a_2 - \frac{3}{4}a_1^2 + 3f\left(c_1 + \frac{5}{2}a_1\right)$$
$$+ 5fc^2\left(\frac{1}{4} - \frac{3}{2}c_1 - \frac{21}{8}a_1\right),$$

or

$$8c_1 = \frac{3}{2}a_1 - \frac{3}{2}a_2 - \frac{3}{4}a_1^2 + \frac{27}{2}a_1c_1$$
$$- 24fc_1 + \frac{15}{2}fa_1 + \frac{5}{4}fc^2 + \frac{105}{4}fc^2c_1 - \frac{105}{8}fc^2a_1 \ldots (3),$$

$$25c_2 = c_3 + \frac{3}{2}a_1c_1 + \frac{3}{2}a_2 + \frac{3}{4}a_1^2 + 5fc^2\left(\frac{3}{4}c_1 + \frac{7}{8}a_1\right)$$

or

$$24c_2 = \quad \frac{3}{2}a_1c_1 + \frac{3}{2}a_2 + \frac{3}{4}a_1^2 + 5fc^2\left(\frac{3}{4}c_1 + \frac{7}{8}a_1\right) \quad \ldots\ldots\ldots\ldots (4).$$

From (1) neglecting terms of the second order

$$a_1 = -\frac{1}{6}fc^2.$$

Substitute this in (3), still neglecting the second order

$$c_1 = \frac{1}{8}fc^2.$$

Substitute these values of a_1, c_1 in the terms of the second order in (1);

$$3a_1 = -\frac{1}{2}fc^2 + \frac{1}{6}f^2c^4 + \frac{13}{6}f^2c^2 - \frac{35}{12}f^2c^4 + \frac{1}{8}f^2c^4$$

$$= -\frac{1}{2}fc^2 + \frac{13}{6}f^2c^2 - \frac{21}{8}f^2c^4,$$

or

$$a_1 = -\frac{1}{6}fc^2 + \frac{13}{18}f^2c^2 - \frac{7}{8}f^2c^4.$$

Substitute the same values in (2),

$$15a_2 = \frac{1}{3}f^2c^4 + \frac{5}{24}f^2c^4 - \frac{1}{8}f^2c^4 = \frac{5}{12}f^2c^4,$$

$$a_2 = \frac{1}{36}f^2c^4.$$

Now substitute the above found values of a_1 and a_2 in (3) and also the approximate values of a_1, c_1 in terms of the second order;

$$8c_1 = -\frac{1}{4}f\,c^2 + \frac{13}{12}f^2c^2 - \frac{21}{16}f^2c^4 - \frac{1}{24}f^2c^4 - \frac{1}{48}f^2c^4 - \frac{9}{32}f^2c^4 - 3f^2c^2$$

$$-\frac{5}{4}f^2c^2 + \frac{5}{4}f\,c^2 + \frac{105}{32}f^2c^4 + \frac{35}{16}f^2c^4 = fc^2 - \frac{19}{6}f^2c^2 + \frac{61}{16}f^2c^4,$$

$$c_1 = \quad \frac{1}{8}f\,c^2 - \frac{19}{48}f^2c^2 + \frac{61}{128}f^2c^4.$$

Also

$$24c_2 = -\frac{1}{32}f^2c^2 + \frac{1}{24}f^2c^4 + \frac{1}{48}f^2c^4 + \frac{15}{32}f^2c^4 - \frac{35}{48}f^2c^4 = -\frac{11}{48}f^2c^4,$$

$$c_2 = -\frac{11}{1152}f^2c^4.$$

Finally substituting for a_1, c_1 in the equation which gives the relation between q^2 and $\frac{\mu}{a^3}$ we have

$$q^2 = \frac{\mu}{a^3}\left[1 + \frac{1}{4}fc^2 - \frac{13}{12}f^2c^2 + \frac{21}{16}f^2c^4 - \frac{1}{32}f^2c^4 + \frac{1}{24}f^2c^4 + 3f + \frac{5}{4}f^2c^2\right.$$
$$\left. - \frac{15}{4}fc^2 + \frac{15}{32}f^2c^4 - \frac{35}{12}f^2c^4\right],$$

or

$$q^2 = \frac{\mu}{a^3}\left[1 + 3f - \frac{7}{2}fc^2 + \frac{1}{6}f^2c^2 - \frac{9}{8}f^2c^4\right].$$

Hence the particular case of our problem is solved to the second order in f.

Now suppose new terms be added to $\frac{1}{r}$ and to z, which involve two new arbitrary constants. If the principal periodic term in $\frac{1}{r}$ be taken as $\frac{1}{a}e\cos(pt+\beta)$ these constants may be supposed to be e, β. We will suppose e small, so that its square may be neglected, and we will omit β in writing, remembering that it always accompanies pt.

Let δr and δz be the increments of the former values of r and z, due to these terms involving e.

Then since the new values must satisfy the same differential equations, we must have

$$\frac{d^2}{dt^2}(r\delta r) = -\frac{\mu}{r^2}\delta r + \frac{\nu}{r^4}\delta r - \frac{5\nu z^2}{r^6}\delta r + \frac{2\nu z}{r^5}\delta z,$$

$$\frac{d^2}{dt^2}(\delta z) = -\frac{\mu}{r^3}\delta z - \frac{3\nu}{r^5}\delta z + \frac{15\nu z^2}{r^7}\delta z$$
$$+ \frac{3\mu z}{r^4}\delta r + \frac{15\nu z}{r^6}\delta r - \frac{35\nu z^3}{r^8}\delta r,$$

or making $r\delta r$ and δz our new variables, the equations are

$$\frac{d^2}{dt^2}(r\delta r)=\left(-\frac{\mu}{r^3}+\frac{\nu}{r^5}-\frac{5\nu z^2}{r^7}\right)r\delta r+\frac{2\nu z}{r^5}\delta z,$$

$$\frac{d^2}{dt^2}(\delta z)=\left(-\frac{\mu}{r^3}-\frac{3\nu}{r^5}+\frac{15\nu z^2}{r^7}\right)\delta z+\left(\frac{3\mu z}{r^5}+\frac{15\nu z}{r^7}-\frac{35\nu z^3}{r^9}\right)r\delta r,$$

in which the values of r and z already determined are to be substituted in the coefficients of $r\delta r$ and δz.

To find the coefficients in our equations :—

$$\frac{\mu}{r^3}=\frac{\mu}{a^3}\left[1+3a_1\cos 2qt+3a_2\cos 4qt+\frac{3}{2}a_1^2(1+\cos 4qt)\right]$$

$$=\frac{\mu}{a^3}\left[1+\frac{1}{24}f^2c^4+\left(-\frac{1}{2}f\,c^2+\frac{13}{6}f^2c^2-\frac{21}{8}f^2c^4\right)\cos 2qt\right.$$
$$\left.+\left(\frac{1}{12}f^2c^4+\frac{1}{24}f^2c^4\right)\cos 4qt\right]$$

$$=\frac{\mu}{a^3}\left[1+\frac{1}{24}f^2c^4+\left(-\frac{1}{2}f\,c^2+\frac{13}{6}f^2c^2-\frac{21}{8}f^2c^4\right)\cos 2qt+\frac{1}{8}f^2c^4\cos 4qt\right],$$

$$\frac{\nu}{r^5}=\frac{\nu}{a^5}[1+5a_1\cos 2qt]$$

$$=\frac{\mu}{a^3}\left[f-\frac{5}{6}f^2c^2\cos 2qt\right],$$

$$\frac{5\nu z^2}{r^7}=\frac{5\nu c^2}{a^5}[1+7a_1\cos 2qt]\left[\frac{1}{2}(1-\cos 2qt)+c_1(\cos 2qt-\cos 4qt)\right]$$

$$=\frac{5\nu c^2}{a^5}\left[\frac{1}{2}(1-\cos 2qt)+c_1(\cos 2qt-\cos 4qt)+\frac{7}{2}a_1\cos 2qt-\frac{7}{4}a_1\right.$$
$$\left.-\frac{7}{4}a_1\cos 4qt\right]$$

$$=\frac{5\nu c^2}{a^5}\left[\frac{1}{2}-\frac{7}{4}a_1+\left(-\frac{1}{2}+c_1+\frac{7}{2}a_1\right)\cos 2qt+\left(-c_1-\frac{7}{4}a_1\right)\cos 4qt\right]$$

$$=\frac{5\nu c^2}{a^5}\left[\frac{1}{2}+\frac{7}{24}fc^2+\left(-\frac{1}{2}+\frac{1}{8}fc^2-\frac{7}{12}fc^2\right)\cos 2qt\right.$$
$$\left.+\left(-\frac{1}{8}fc^2+\frac{7}{24}fc^2\right)\cos 4qt\right]$$

$$=\frac{5\mu fc^2}{a^3}\left[\frac{1}{2}+\frac{7}{24}fc^2+\left(-\frac{1}{2}-\frac{11}{24}fc^2\right)\cos 2qt+\frac{1}{6}fc^2\cos 4qt\right],$$

$$\frac{\mu z}{r^5} = \frac{\mu c}{a^4}\left[1 + 5a_1 \cos 2qt + 5a_2 \cos 4qt + 5a_1^2 (1 + \cos 4qt)\right]$$

$$\times \left[\sin qt + c_1 \sin 3qt + c_2 \sin 5qt\right]$$

$$= \frac{\mu c}{a^4}\left[(1 + 5a_1^2) \sin qt + c_1 \sin 3qt + c_2 \sin 5qt\right.$$

$$+ \frac{5}{2} a_1 (-\sin qt + \sin 3qt) + \frac{5}{2} a_1 c_1 (\sin qt + \sin 5qt)$$

$$\left. + \left(\frac{5}{2} a_2 + \frac{5}{2} a_1^2\right) (-\sin 3qt + \sin 5qt)\right]$$

$$= \frac{\mu c}{a^4}\left[\left(1 - \frac{5}{2} a_1 + 5a_1^2 + \frac{5}{2} a_1 c_1\right) \sin qt + \left(c_1 + \frac{5}{2} a_1 - \frac{5}{2} a_2 - \frac{5}{2} a_1^2\right) \sin 3qt\right.$$

$$\left. + \left(c_2 + \frac{5}{2} a_1 c_1 + \frac{5}{2} a_2 + \frac{5}{2} a_1^2\right) \sin 5qt\right]$$

$$= \frac{\mu c}{a^4}\left[\left(1 + \frac{5}{12} fc^2 - \frac{65}{36} f^2c^2 + \frac{35}{16} f^2c^4 + \frac{5}{36} f^2c^4 - \frac{5}{96} f^2c^4\right) \sin qt\right.$$

$$+ \left(\frac{1}{8} fc^2 - \frac{19}{48} f^2c^2 + \frac{61}{128} f^2c^4 - \frac{5}{12} fc^2 + \frac{65}{36} f^2c^2 - \frac{35}{16} f^2c^4 - \frac{5}{72} f^2c^4\right.$$

$$\left. - \frac{5}{72} f^2c^4\right) \sin 3qt$$

$$\left. + \left(-\frac{11}{1152} f^2c^4 - \frac{5}{96} f^2c^4 + \frac{5}{72} f^2c^4 + \frac{5}{72} f^2c^4\right) \sin 5qt\right]$$

$$= \frac{\mu c}{a^4}\left[\left(1 + \frac{5}{12} fc^2 - \frac{65}{36} f^2c^2 + \frac{655}{288} f^2c^4\right) \sin qt\right.$$

$$\left. + \left(-\frac{7}{24} fc^2 + \frac{203}{144} f^2c^2 - \frac{2131}{1152} f^2c^4\right) \sin 3qt + \frac{89}{1152} f^2c^4 \sin 5qt\right];$$

$\frac{\nu z}{r^5}$ is found by multiplying this by fa^2.

$$\frac{15\nu z}{r^7} = \frac{15\nu c}{a^6}\left[1 + 7a_1 \cos 2qt\right]\left[\sin qt + c_1 \sin 3qt\right]$$

$$= \frac{15\nu c}{a^6}\left[\sin qt + c_1 \sin 3qt + \frac{7}{2} a_1 (-\sin qt + \sin 3qt)\right]$$

$$= \frac{15\nu c}{a^6}\left[\left(1-\frac{7}{2}\,a_1\right)\sin qt + \left(c_1+\frac{7}{2}\,a_1\right)\sin 3qt\right]$$

$$= \frac{15\nu c}{a^6}\left[\left(1+\frac{7}{12}fc^2\right)\sin qt + \left(\frac{1}{8}fc^2-\frac{7}{12}fc^2\right)\sin 3qt\right]$$

$$= \frac{15\mu cf}{a^4}\left[\left(1+\frac{7}{12}fc^2\right)\sin qt - \frac{11}{24}fc^2\sin 3qt\right],$$

$$\frac{35\nu z^3}{r^9} = \frac{35\nu c^3}{a^6}\left[1+9a_1\cos 2qt\right]\left[\frac{1}{4}(3\sin qt-\sin 3qt)+\frac{3}{2}(1-\cos 2qt)\,c_1\sin 3qt\right]$$

$$= \frac{35\nu c^3}{a^6}\left[1+9a_1\cos 2qt\right]\left[\left(\frac{3}{4}-\frac{3}{4}\,c_1\right)\sin qt+\left(-\frac{1}{4}+\frac{3}{2}\,c_1\right)\sin 3qt-\frac{3}{4}c_1\sin 5qt\right]$$

$$= \frac{35\nu c^3}{a^6}\left[\left(\frac{3}{4}-\frac{3}{4}\,c_1\right)\sin qt+\left(-\frac{1}{4}+\frac{3}{2}\,c_1\right)\sin 3qt-\frac{3}{4}\,c_1\sin 5qt\right.$$
$$\left.+\frac{27}{8}\,a_1(-\sin qt+\sin 3qt)-\frac{9}{8}\,a_1(\sin qt+\sin 5qt)\right]$$

$$= \frac{35\mu fc^3}{a^4}\left[\left(\frac{3}{4}-\frac{9}{2}\,a_1-\frac{3}{4}\,c_1\right)\sin qt+\left(-\frac{1}{4}+\frac{27}{8}\,a_1+\frac{3}{2}\,c_1\right)\sin 3qt\right.$$
$$\left.+\left(-\frac{3}{4}\,c_1-\frac{9}{8}\,a_1\right)\sin 5qt\right].$$

Hence the coefficient of $r\delta r$ in the first differential equation is μ/a^3 multiplied by

$$-1-\frac{1}{24}f^2c^4+f-5fc^2\left(\frac{1}{2}+\frac{7}{24}fc^2\right)$$
$$+\cos 2qt\left[\frac{1}{2}fc^2-\frac{13}{6}f^2c^2+\frac{21}{8}f^2c^4-\frac{5}{6}f^2c^2+5fc^2\left(\frac{1}{2}+\frac{11}{24}fc^2\right)\right]$$
$$+\cos 4qt\left[-\frac{1}{8}f^2c^4-5fc^2\cdot\frac{1}{6}fc^2\right],$$

or

$$-1+f-\frac{5}{2}fc^2-\frac{3}{2}f^2c^4+\cos 2qt\left[3fc^2-3f^2c^2+\frac{59}{12}f^2c^4\right]+\cos 4qt\left[-\frac{23}{24}f^2c^4\right];$$

the coefficient of δz in the same equation is μ/a^2 multiplied by

$$\sin qt\left[2fc+\frac{5}{6}f^2c^3\right]+\sin 3qt\left[-\frac{7}{12}f^2c^3\right].$$

Also the coefficient of δz in the second differential equation is μ/a^3 multiplied by

$$-1-\frac{1}{24}f^2c^4-3f+15fc^2\left(\frac{1}{2}+\frac{7}{24}fc^2\right)$$

$$+\cos 2qt\left[\frac{1}{2}fc^2-\frac{13}{6}f^2c^2+\frac{21}{8}f^2c^4+\frac{5}{2}f^2c^2-15fc^2\left(\frac{1}{2}+\frac{11}{24}fc^2\right)\right]$$

$$+\cos 4qt\left[-\frac{1}{8}f^2c^4+15fc^2\,.\,\frac{1}{6}fc^2\right],$$

or

$$-1-3f+\frac{15}{2}fc^2+\frac{13}{3}f^2c^4+\cos 2qt\left[-7fc^2+\frac{1}{3}f^2c^2-\frac{17}{4}f^2c^4\right]+\cos 4qt\left[\frac{19}{8}f^2c^4\right];$$

the coefficient of $r\delta r$ in the second equation is μ/a^4 multiplied by

$$\sin\ qt\left[3c+\frac{5}{4}fc^3-\frac{65}{12}f^2c^3+\frac{655}{96}f^2c^5+15fc+\frac{35}{4}f^2c^3-35fc^3\left(\frac{3}{4}+\frac{3}{4}fc^2-\frac{3}{32}fc^2\right)\right]$$

$$+\sin 3qt\left[-\frac{7}{8}fc^3+\frac{203}{48}f^2c^3-\frac{2131}{384}f^2c^5-\frac{55}{8}f^2c^3-35fc^3\left(-\frac{1}{4}-\frac{9}{16}fc^2+\frac{3}{16}fc^2\right)\right]$$

$$+\sin 5qt\left[\frac{89}{384}f^2c^5-35fc^3\left(\frac{3}{16}fc^2-\frac{3}{32}fc^2\right)\right],$$

or

$$\sin qt\left[3c+15fc-25fc^3+\frac{10}{3}f^2c^3-\frac{775}{48}f^2c^5\right]$$

$$+\sin 3qt\left[\frac{63}{8}fc^3-\frac{127}{48}f^2c^3+\frac{2909}{384}f^2c^5\right]+\sin 5qt\left[-\frac{1171}{384}f^2c^5\right].$$

Now we have seen that

$$q^2=\frac{\mu}{a^3}\left[1+3f-\frac{7}{2}fc^2+\frac{1}{6}f^2c^2-\frac{9}{8}f^2c^4\right],$$

whence

$$\frac{\mu}{a^3}=q^2\left[1-3f+\frac{7}{2}fc^2+9f^2-\frac{127}{6}f^2c^2+\frac{107}{8}f^2c^4\right].$$

Substitute the above coefficients, and divide the first equation by q^2a^2 and the second by q^2a; the equations then become

$$\frac{1}{q^2}\frac{d^2}{dt^2}\left(\frac{r\delta r}{a^2}\right) = -\left(\frac{r\delta r}{a^2}\right)\left[1 - 4f + 6fc^2 + 12f^2 - \frac{193}{6}f^2c^2 + \frac{189}{8}f^2c^4\right.$$
$$\left. + \cos 2qt\left(-3fc^2 + 12f^2c^2 - \frac{185}{12}f^2c^4\right) + \cos 4qt\left(\frac{23}{24}f^2c^4\right)\right]$$
$$+\left(\frac{\delta z}{a}\right)\left[\sin qt\left(2fc - 6f^2c + \frac{47}{6}f^2c^3\right) + \sin 3qt\left(-\frac{7}{12}f^2c^3\right)\right],$$

$$\frac{1}{q^2}\frac{d^2}{dt^2}\left(\frac{\delta z}{a}\right) = -\left(\frac{\delta z}{a}\right)\left[1 - 4fc^2 + \frac{71}{6}f^2c^2 - \frac{413}{24}f^2c^4\right.$$
$$\left. + \cos 2qt\left(7fc^2 - \frac{64}{3}f^2c^2 + \frac{115}{4}f^2c^4\right) + \cos 4qt\left(-\frac{19}{8}f^2c^4\right)\right]$$
$$+\left(\frac{r\delta r}{a^2}\right)\left[\sin qt\left(3c + 6fc - \frac{29}{2}fc^3 - 18f^2c + \frac{202}{3}f^2c^3 - \frac{3049}{48}f^2c^5\right)\right.$$
$$\left. + \sin 3qt\left(\frac{63}{8}fc^3 - \frac{1261}{48}f^2c^3 + \frac{13493}{384}f^2c^5\right) + \sin 5qt\left(-\frac{1171}{384}f^2c^5\right)\right].$$

Now suppose a term in $\dfrac{r\delta r}{a^2}$ to be $-e\cos(pt+\beta)$, for which we shall simply write $-e\cos pt$.

If we substitute this term for $\dfrac{r\delta r}{a^2}$ in the second equation and at first omit all the terms of the equation containing f as a factor, we have

$$\frac{1}{q^2}\frac{d^2}{dt^2}\left(\frac{\delta z}{a}\right) + \frac{\delta z}{a} = -3ce\sin qt\cos pt = -\frac{3}{2}ce\left[\sin(q-p)t + \sin(q+p)t\right].$$

Hence

$$\frac{\delta z}{a} = -\frac{3}{2}ce\left[\frac{q^2}{p(2q-p)}\sin(q-p)t - \frac{q^2}{p(2q+p)}\sin(q+p)t\right].$$

Now again substitute this value of $\delta z/a$ in the terms which contain

the first power of f, and also take into account the first powers of f in the multiplier of $r\delta r/a^2$, and we shall have

$$\frac{1}{q^2}\frac{d^2}{dt^2}\left(\frac{\delta z}{a}\right)+\left(\frac{\delta z}{a}\right)=-\frac{3}{2}ce\left(1+2f-\frac{29}{6}fc^2\right)[\sin(q-p)t+\sin(q+p)t]$$

$$-\frac{63}{16}c^3ef[\sin(3q-p)t+\sin(3q+p)t]$$

$$-6c^3ef\left[\frac{q^2}{p(2q-p)}\sin(q-p)t-\frac{q^2}{p(2q+p)}\sin(q+p)t\right]$$

$$+\frac{21}{4}c^3ef\frac{q^2}{p(2q-p)}[-\sin(q+p)t+\sin(3q-p)t]$$

$$-\frac{21}{4}c^3ef\frac{q^2}{p(2q+p)}[-\sin(q-p)t+\sin(3q+p)t]$$

$$=-\frac{3}{2}ce\left[1+2f-\frac{29}{6}fc^2+4\frac{q^2}{p(2q-p)}fc^2\right.$$
$$\left.-\frac{7}{2}\frac{q^2}{p(2q+p)}fc^2\right]\sin(q-p)t$$

$$-\frac{3}{2}ce\left[1+2f-\frac{29}{6}fc^2-4\frac{q^2}{p(2q+p)}fc^2\right.$$
$$\left.+\frac{7}{2}\frac{q^2}{p(2q-p)}fc^2\right]\sin(q+p)t$$

$$-\frac{21}{4}c^3ef\left[\frac{3}{4}-\frac{q^2}{p(2q-p)}\right]\sin(3q-p)t$$

$$-\frac{21}{4}c^3ef\left[\frac{3}{4}+\frac{q^2}{p(2q+p)}\right]\sin(3q+p)t.$$

Whence again,

$$\frac{\delta z}{a}=-\frac{3}{2}ce\frac{q^2}{p(2q-p)}\left[1+2f-\frac{29}{6}fc^2+4\frac{q^2}{p(2q-p)}fc^2\right.$$
$$\left.-\frac{7}{2}\frac{q^2}{p(2q+p)}fc^2\right]\sin(q-p)t$$

$$+\frac{3}{2}ce\frac{q^2}{p(2q+p)}\left[1+2f-\frac{29}{6}fc^2-4\frac{q^2}{p(2q+p)}fc^2\right.$$
$$\left.+\frac{7}{2}\frac{q^2}{p(2q-p)}fc^2\right]\sin(q+p)t$$

$$+\frac{21}{4}c^3ef\frac{q^2}{(2q-p)(4q-p)}\left[\frac{3}{4}-\frac{q^2}{p(2q-p)}\right]\sin(3q-p)t$$
$$+\frac{21}{4}c^3ef\frac{q^2}{(2q+p)(4q+p)}\left[\frac{3}{4}+\frac{q^2}{p(2q+p)}\right]\sin(3q+p)t.$$

Now substitute this value of $\delta z/a$ in the first equation and also $-e\cos pt$ for $r\delta r/a^2$ and we shall thus include all the terms which are of the second order in f which arise from the assumed term in $r\delta r/a^2$.

Hence if we transpose all the terms of the equation

$$\frac{d^2}{dt^2}\left(\frac{r\delta r}{a^2}\right)=-q^2\left(\frac{r\delta r}{a^2}\right)[1+\&c.]+q^2\frac{\delta z}{a}[\sin qt(2fc+\&c.)\ldots]$$

to the left-hand side, the terms arising from the assumed term in $r\delta r/a^2$ will be

$$e\left[p^2-q^2\left(1-4f+6fc^2+12f^2-\frac{193}{6}f^2c^2+\frac{189}{8}f^2c^4\right)\right]\cos pt$$
$$+eq^2\left[\frac{3}{2}fc^2-6f^2c^2+\frac{185}{24}f^2c^4\right][\cos(2q-p)t+\cos(2q+p)t]$$
$$+eq^2\left[\qquad\qquad-\frac{23}{48}f^2c^4\right][\cos(4q-p)t+\cos(4q+p)t]$$
$$+\frac{3}{2}c^2ef\frac{q^2}{p(2q-p)}q^2\left[1+2f-\frac{29}{6}fc^2+4\frac{q^2}{p(2q-p)}fc^2-\frac{7}{2}\frac{q^2}{p(2q+p)}fc^2\right.$$
$$\left.-3f+\frac{47}{12}fc^2\right][\cos pt-\cos(2q-p)t]$$
$$-\frac{7}{16}c^4ef^2\frac{q^2}{p(2q-p)}q^2[\cos(2q+p)t-\cos(4q-p)t]$$
$$-\frac{3}{2}c^2ef\frac{q^2}{p(2q+p)}q^2\left[1+2f-\frac{29}{6}fc^2-4\frac{q^2}{p(2q+p)}fc^2+\frac{7}{2}\frac{q^2}{p(2q-p)}fc^2\right.$$
$$\left.-3f+\frac{47}{12}fc^2\right][\cos pt-\cos(2q+p)t]$$
$$+\frac{7}{16}c^4ef^2\frac{q^2}{p(2q+p)}q^2[\cos(2q-p)t-\cos(4q+p)t]$$
$$-\frac{21}{4}c^4ef^2\frac{q^2}{(2q-p)(4q-p)}q^2\left[\frac{3}{4}-\frac{q^2}{p(2q-p)}\right][\cos(2q-p)t-\cos(4q-p)t]$$
$$-\frac{21}{4}c^4ef^2\frac{q^2}{(2q+p)(4q+p)}q^2\left[\frac{3}{4}+\frac{q^2}{p(2q+p)}\right][\cos(2q+p)t-\cos(4q+p)t];$$

the coefficient of $e \cos pt$ in this is

$$p^2 - q^2\left(1 - 4f + 6fc^2 + 12f^2 - \frac{193}{6}f^2c^2 + \frac{189}{6}f^2c^4\right)$$

$$+\frac{3}{2}fc^2\frac{q^2}{p(2q-p)}q^2\left[1 - f - \frac{11}{12}fc^2 + 4\frac{q^2}{p(2q-p)}fc^2 - \frac{7}{2}\frac{q^2}{p(2q+p)}fc^2\right]$$

$$-\frac{3}{2}fc^2\frac{q^2}{p(2q+p)}q^2\left[1 - f - \frac{11}{12}fc^2 - 4\frac{q^2}{p(2q+p)}fc^2 + \frac{7}{2}\frac{q^2}{p(2q-p)}fc^2\right];$$

the coefficient of $e \cos(2q-p)t$ in the same is

$$q^2\left[\frac{3}{2}fc^2 - 6f^2c^2 + \frac{185}{24}f^2c^4\right]$$

$$-\frac{3}{2}c^2f\frac{q^2}{p(2q-p)}q^2\left[1 - f - \frac{11}{12}fc^2 + 4\frac{q^2}{p(2q-p)}fc^2 - \frac{7}{2}\frac{q^2}{p(2q+p)}fc^2\right]$$

$$+\frac{7}{16}c^4f^2\frac{q^2}{p(2q+p)}q^2 - \frac{21}{4}c^4f^2\frac{q^2}{(2q-p)(4q-p)}q^2\left[\frac{3}{4} - \frac{q^2}{p(2q-p)}\right];$$

the coefficient of $e \cos(2q+p)t$ is

$$q^2\left[\frac{3}{2}fc^2 - 6f^2c^2 + \frac{185}{24}f^2c^4\right]$$

$$+\frac{3}{2}c^2f\frac{q^2}{p(2q+p)}q^2\left[1 - f - \frac{11}{12}fc^2 - 4\frac{q^2}{p(2q+p)}fc^2 + \frac{7}{2}\frac{q^2}{p(2q-p)}fc^2\right]$$

$$-\frac{7}{16}c^4f^2\frac{q^2}{p(2q-p)}q^2 - \frac{21}{4}c^4f^2\frac{q^2}{(2q+p)(4q+p)}q^2\left[\frac{3}{4} + \frac{q^2}{p(2q+p)}\right];$$

the coefficient of $e \cos(4q-p)t$ is

$$q^2\left[-\frac{23}{48}f^2c^4\right] + \frac{21}{4}f^2c^4\frac{q^2}{(2q-p)(4q-p)}q^2\left[\frac{3}{4} - \frac{q^2}{p(2q-p)}\right] + \frac{7}{16}f^2c^4\frac{q^2}{p(2q-p)}q^2,$$

and the coefficient of $e \cos(4q+p)t$ is

$$q^2\left[-\frac{23}{48}f^2c^4\right] - \frac{7}{16}f^2c^4\frac{q^2}{p(2q+p)}q^2 + \frac{21}{4}f^2c^4\frac{q^2}{(2q+p)(4q+p)}q^2\left[\frac{3}{4} + \frac{q^2}{p(2q+p)}\right].$$

If for the sake of simplification we first confine our attention to the term in the above found coefficients which involves the first power of f,

we see that the result of substituting the term $-e\cos pt$ for $r\delta r/a^2$ is to produce the following terms on the left-hand side of the final equation

$$e\cos pt\left[p^2-q^2(1-4f+6fc^2)+3fc^2\frac{q^4}{4q^2-p^2}\right]$$

$$+e\cos(2q-p)\,t\left[-\frac{3}{2}fc^2\frac{(q-p)^2}{p(2q-p)}q^2\right]$$

$$+e\cos(2q+p)\,t\left[\ \frac{3}{2}fc^2\frac{(q+p)^2}{p(2q+p)}q^2\right].$$

It is to be especially remarked that if p is nearly equal to q the coefficient of the term $e\cos(2q-p)\,t$ arising from the term $-e\cos pt$ in $r\delta r/a^2$ will be very small.

Now suppose another term in $r\delta r/a^2$ to be $-e_1\cos(2q-p)\,t$, then the result of substituting this term will be found at once by putting e_1 in place of e, and $2q-p$ in place of p; hence will arise the following additional terms on the left-hand side of the final equation, viz.:—

$$e_1\cos(2q-p)\,t\left[(2q-p)^2-q^2(1-4f+6fc^2)+3fc^2\frac{q^4}{p(4q-p)}\right]$$

$$+e_1\cos pt\qquad\left[-\frac{3}{2}fc^2\frac{(q-p)^2}{p(2q-p)}q^2\right]$$

$$+e_1\cos(4q-p)\,t\left[\ \frac{3}{2}fc^2\frac{(3q-p)^2}{(2q-p)(4q-p)}q^2\right].$$

Again suppose another term in $r\delta r/a^2$ to be $-e_2\cos(2q+p)\,t$, then the terms on the left-hand side of the final equation arising from this will be

$$e_2\cos(2q+p)\,t\left[(2q+p)^2-q^2(1-4f+6fc^2)-3fc^2\frac{q^4}{p(4q+p)}\right]$$

$$+e_2\cos pt\qquad\left[\frac{3}{2}fc^2\frac{(q+p)^2}{p(2q+p)}q^2\right]$$

$$+e_2\cos(4q+p)\,t\left[\frac{3}{2}fc^2\frac{(3q+p)^2}{(2q+p)(4q+p)}q^2\right]$$

which may also be derived from the last by changing p into $-p$.

Now if p and the ratios of e, e_1, e_2 be so chosen as to make the coefficients of $\cos pt$, $\cos(2q-p)t$, $\cos(2q+p)t$ vanish, we have

$$0=e\left[p^2-q^2\left(1-4f+6fc^2+12f^2-\frac{193}{6}f^2c^2+\frac{189}{8}f^2c^4\right)\right.$$
$$\left.+3fc^2\frac{q^4}{4q^2-p^2}\left(1-f-\frac{11}{12}fc^2\right)+12f^2c^4\frac{q^6(4q^2+p^2)}{p^2(4q^2-p^2)^2}-\frac{21}{2}f^2c^4\frac{q^6}{p^2(4q^2-p^2)}\right]$$
$$+e_1\left[-\frac{3}{2}fc^2\frac{(q-p)^2q^2}{p(2q-p)}\right]+e_2\left[\frac{3}{2}fc^2\frac{(q+p)^2q^2}{p(2q+p)}\right],$$

$$0=e\left[q^2\left(\frac{3}{2}fc^2-6f^2c^2+\frac{185}{24}f^2c^4\right)-\frac{3}{2}c^2f\frac{q^4}{p(2q-p)}\left(1-f-\frac{11}{12}fc^2\right)\right.$$
$$-6f^2c^4\frac{q^6}{p^2(2q-p)^2}+\frac{21}{4}f^2c^4\frac{q^6}{p^2(4q^2-p^2)}+\frac{7}{16}f^2c^4\frac{q^4}{p(2q+p)}$$
$$\left.+\frac{21}{4}f^2c^4\frac{q^6}{p(2q-p)^2(4q-p)}-\frac{63}{16}f^2c^4\frac{q^4}{(2q-p)(4q-p)}\right]$$
$$+e_1\left[(2q-p)^2-q^2(1-4f+6fc^2)+3fc^2\frac{q^4}{p(4q-p)}\right],$$

and

$$0=e\left[q^2\left(\frac{3}{2}fc^2-6f^2c^2+\frac{185}{24}f^2c^4\right)+\frac{3}{2}c^2f\frac{q^4}{p(2q+p)}\left(1-f-\frac{11}{12}fc^2\right)\right.$$
$$-6f^2c^4\frac{q^6}{p^2(2q+p)^2}+\frac{21}{4}f^2c^4\frac{q^6}{p^2(4q^2-p^2)}-\frac{7}{16}f^2c^4\frac{q^4}{p(2q-p)}$$
$$\left.-\frac{21}{4}f^2c^4\frac{q^6}{p(2q+p)^2(4q+p)}-\frac{63}{16}f^2c^4\frac{q^4}{(2q+p)(4q+p)}\right]$$
$$+e_2\left[(2q+p)^2-q^2(1-4f+6fc^2)-3fc^2\frac{q^4}{p(4q+p)}\right].$$

These two last equations give the ratios of e_1 and e_2 respectively to e in terms of p, and the substitution of the results in the first equation gives the final equation for determining p.

It is readily seen from a remark made before that the term in e_1 in the first equation contributes nothing of the order of f^2; and the approximate value of e_2/e given by the third equation is

$$\frac{e_2}{e}=-\frac{3}{2}fc^2\frac{(q+p)q^2}{p(2q+p)(3q+p)},$$

which is nearly equal to $-\frac{1}{4}fc^2$, and this is to be substituted in the first equation in order to obtain the final equation for p to the second order in f.

We may see that with the values thus found for e_1/e, e_2/e terms of the second order in f will be left outstanding involving cosines of the angles $(4q-p)t$ and $(4q+p)t$. In order to get rid of these terms suppose $-e_3 \cos(4q-p)t$ and $-e_4 \cos(4q+p)t$ to be two additional terms in $r\delta r/a^2$; in forming the left-hand side of the final equation due to these terms we may omit all quantities involving f, so that the equations for determining e_3, e_4 will be

$$\begin{aligned}0 = e\Big[q^2\left(-\frac{23}{48}f^2c^4\right) &- \frac{21}{4}f^2c^4\frac{q^6}{p(2q-p)^2(4q-p)} + \frac{63}{16}f^2c^4\frac{q^4}{(2q-p)(4q-p)} \\ &+ \frac{7}{16}f^2c^4\frac{q^4}{p(2q-p)}\Big] \\ + e_1\left[\frac{3}{2}fc^2\frac{(3q-p)^2q^2}{(2q-p)(4q-p)}\right] &+ e_3[(4q-p)^2-q^2],\end{aligned}$$

$$\begin{aligned}0 = e\Big[q^2\left(-\frac{23}{48}f^2c^4\right) &+ \frac{21}{4}f^2c^4\frac{q^6}{p(2q+p)^2(4q+p)} + \frac{63}{16}f^2c^4\frac{q^4}{(2q+p)(4q+p)} \\ &- \frac{7}{16}f^2c^4\frac{q^4}{p(2q+p)}\Big] \\ + e_2\left[\frac{3}{2}fc^2\frac{(3q+p)^2q^2}{(2q+p)(4q+p)}\right] &+ e_4[(4q+p)^2-q^2],\end{aligned}$$

and the determination of these quantities will complete the solution of our problem.

At first neglect the terms in f^2 in the first equation (divided throughout by e), and put $p=q$ in the terms containing f in the first power, and we have approximately

$$p^2 = q^2(1-4f+6fc^2) - q^2(fc^2) = q^2(1-4f+5fc^2).$$

Next take into account terms in f^2, putting $p=q$ in these terms, and in the terms which involve f in the first power, putting

$$\frac{q^2}{4q^2-p^2} = \frac{q^2}{q^2(3+4f-5fc^2)} = \frac{1}{3} - \frac{1}{9}(4f-5fc^2).$$

Hence to the second order in f

$$\frac{p^2}{q^2} = 1 - 4f + 6fc^2 + 12f^2 - \frac{193}{6} f^2c^2 + \frac{189}{8} f^2c^4$$

$$- fc^2 + \frac{1}{3} fc^2 (4f - 5fc^2) + f^2c^2 + \frac{11}{12} f^2c^4$$

$$- \frac{20}{3} f^2c^4 + \frac{7}{2} f^2c^4 + \frac{1}{2} f^2c^4,$$

or

$$\frac{p^2}{q^2} = 1 - 4f + 5fc^2 + 12f^2 - \frac{179}{6} f^2c^2 + \frac{485}{24} f^2c^4.$$

Next in the second equation make the same substitutions, and in the coefficient of e_1 put

$$(2q - p)^2 - q^2 = (q - p)(3q - p) = (q^2 - p^2) \frac{3q - p}{q + p} = q^2 (4f - 5fc^2).$$

Hence

$$\frac{e_1}{e} [4f - 5fc^2 + 4f - 6fc^2 + fc^2]$$

$$= -\frac{3}{2} fc^2 + 6f^2c^2 - \frac{185}{24} f^2c^4 + \frac{3}{2} fc^2 - \frac{3}{2} f^2c^2 - \frac{33}{24} f^2c^4$$

$$+ 6f^2c^4 - \frac{7}{4} f^2c^4 - \frac{7}{48} f^2c^4 - \frac{7}{4} f^2c^4 + \frac{21}{16} f^2c^4$$

$$= \frac{9}{2} f^2c^2 - \frac{65}{12} f^2c^4,$$

so that

$$\frac{e_1}{e} = \frac{9}{16} fc^2 \frac{1 - \frac{65}{54} c^2}{1 - \frac{5}{4} c^2};$$

if $1 - \frac{5}{4} c^2 = 0$, the approximation becomes insufficient, and the terms in the denominator of e_1/e must be carried to one order higher in f; and e_1/e becomes finite, that is, contains a term which is independent of f.

We have already found $e_2/e = -\frac{1}{4} fc^2$ to the first order in f, and by again substituting in the third equation we may find e_2/e to the second order.

Substituting the above values in the equation for e_3 we have

$$8\frac{e_3}{e}=\frac{23}{48}f^2c^4+\frac{7}{4}f^2c^4-\frac{21}{16}f^2c^4-\frac{7}{16}f^2c^4-2fc^2\,.\,\frac{9}{16}fc^2\,\frac{1-\frac{65}{54}c^2}{1-\frac{5}{4}c^2}$$

$$=\frac{23}{48}f^2c^4-\frac{9}{8}f^2c^4\,\frac{1-\frac{65}{54}c^2}{1-\frac{5}{4}c^2},$$

or

$$\frac{e_3}{e}=\frac{23}{384}f^2c^4-\frac{9}{64}f^2c^4\,\frac{1-\frac{65}{54}c^2}{1-\frac{5}{4}c^2},$$

and from the equation for e_4,

$$24\frac{e_4}{e}=\frac{23}{48}f^2c^4-\frac{7}{60}f^2c^4-\frac{21}{80}f^2c^4+\frac{7}{48}f^2c^4+\frac{8}{5}fc^2\,.\,\frac{1}{4}fc^2$$

$$=\frac{31}{48}f^2c^4,$$

or

$$\frac{e_4}{e}=\frac{31}{1152}f^2c^4.$$

We have already found the terms in $\delta z/a$ which depend on the term $-e\cos pt$ in $r\delta r/a^2$; similarly the terms which depend on the term

$$-e_1\cos(2q-p)\,t$$

may be found by writing $2q-p$ instead of p and e_1 instead of e. If we neglect terms multiplied by f in the coefficients involving e_1, we have

$$\frac{\delta z}{a}=\frac{3}{2}ce_1\sin(q-p)\,t+\frac{1}{2}ce_1\sin(3q-p)\,t,$$

and similarly the terms depending upon e_2 will be found by writing $2q+p$ for p,

$$\frac{\delta z}{a}=-\frac{1}{2}ce_2\sin(q+p)\,t+\frac{1}{10}ce_2\sin(3q+p)\,t,$$

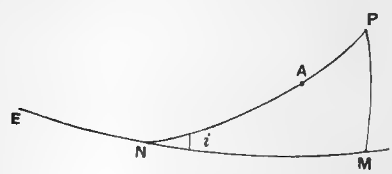
and by substituting for e_1 and e_2 in terms of e, and adding the new terms to the terms in $\delta z/a$ previously found, we find the complete value of $\delta z/a$ to the first order in e and the first order in f.

Thus we have determined our assumed expressions so that they satisfy the differential equations to a specified degree of approximation; and since they contain four arbitrary constants, viz. c, e, β, γ, they are competent to express any initial conditions, subject only to the proviso that e is small.

[We ought now to proceed and discuss the third coordinate which was eliminated from our equations at the beginning, and deduce the motions of node and apse. Adams has left no indication of the method he would have adopted. The following method may be indicated:—

We have $$x\ddot{y} - y\ddot{x} = 0,$$

hence $$x\dot{y} - y\dot{x} = (r^2 - z^2)\,\dot{\phi} = \text{constant},$$



where $$\phi = EM,$$

ENM is the equator,

NP the orbit,

PM perpendicular to EM,

E a fixed point.

Hence we find EM.

Again $$\frac{z}{r} = \sin MP = \sin i \sin NP,$$

where i is the angle PNM.

We may take $\sin i = c$. Hence we find NP, and

$$\tan NM = \cos i \tan NP;$$

and EM being known, this gives the position of the node at any time.

Again p is the mean rate of separation of P from A, where A is the apse of the orbit; and the mean motion of P itself is the non-periodic part of $\frac{d}{dt}(NP) + \cos i \frac{d}{dt}(EN)$. Hence we find the motion of the apse.]

II.

DEVELOPMENT OF THE DISTURBING FUNCTION.

In the mutual disturbances of the Satellites hereafter considered, the disturbing forces are obtained from a function of which the expression

$$[a^2 - 2aa' \cos(nt - n't + \epsilon - \epsilon') + a'^2]^{-\frac{1}{2}}$$

is the chief part. Write

$$a/a' = \alpha, \quad nt - n't + \epsilon - \epsilon' = \phi,$$

and let us consider the development of the expression

$$(1 - 2\alpha \cos \phi + \alpha^2)^{-s}$$

according to cosines of multiples of ϕ. We may evidently take α to be less than unity.

Write

$$S^{-s} \equiv (1 - 2\alpha \cos \phi + \alpha^2)^{-s} \equiv \frac{1}{2} b_0 + b_1 \cos \phi + b_2 \cos 2\phi + \dots,$$

then our investigation deals with the coefficients b.

Now

$$S = (1 - \alpha e^{\iota\phi})(1 - \alpha e^{-\iota\phi});$$

develope S^{-s} by the Binomial Theorem and pick out the coefficient of $\cos i\phi \equiv \frac{1}{2}(e^{i\iota\phi} + e^{-i\iota\phi})$, and we get

$$b_i = 2\frac{s(s+1)\dots(s+i-1)}{1 \,.\, 2 \dots i} \alpha^i \left[1 + \frac{s}{1}\frac{s+i}{i+1}\alpha^2 + \frac{s(s+1)}{1 \,.\, 2}\frac{(s+i)(s+i+1)}{(i+1)(i+2)}\alpha^4 + \dots\right].$$

Such a series as this may be transformed with advantage in certain cases, so as to proceed by powers of a different quantity to α. For example if

$$f(\alpha) = A_1\alpha^2 + A_2\alpha^4 + \dots + A_n\alpha^{2n} + \dots,$$

then

$$f(\alpha) = \frac{\alpha^2}{1-\alpha^2}[A_1 + (A_2 - A_1)\alpha^2 + (A_3 - A_2)\alpha^4 + \dots],$$

or writing

$$\beta^2 = \frac{\alpha^2}{1-\alpha^2},$$

$$A_{n+1} - A_n = \delta A_n,$$

$$f(\alpha) = A_1\beta^2 + \beta^2 [\delta A_1 \alpha^2 + \delta A_2 \alpha^4 + \ldots].$$

In the same way if we write

$$\delta A_{n+1} - \delta A_n = \delta^2 A_n,$$

$$\delta^2 A_{n+1} - \delta^2 A_n = \delta^3 A_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

we obtain

$$\begin{aligned} f(\alpha) &= A_1\beta^2 + \delta A_1\beta^4 + \beta^4 [\delta^2 A_1 \alpha^2 + \delta^2 A_2 \alpha^4 + \ldots] \\ &= A_1\beta^2 + \delta A_1 \beta^4 + \delta^2 A_1 \beta^6 + \beta^6 [\delta^3 A_1 \alpha^2 + \ldots] \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= A_1\beta^2 + \delta A_1\beta^4 + \delta^2 A_1 \beta^6 + \delta^3 A_1 \beta^8 + \ldots\ldots \end{aligned}$$

This series may prove more advantageous to deal with than the original. In the case of the quantities b_i, Legendre has given a transformation which facilitates some calculations.

Denote the series of coefficients

$$1, \quad \frac{s+i}{i+1}, \quad \frac{(s+i)(s+i+1)}{(i+1)(i+2)}, \quad \frac{(s+i)(s+i+1)(s+i+2)}{(i+1)(i+2)(i+3)}, \ldots$$

by the symbols

$$1, \; 1+\Delta_1, \quad 1 + 2\Delta_1 + \Delta_2, \quad 1 + 3\Delta_1 + 3\Delta_2 + \Delta_3, \ldots$$

respectively, so that in fact Δ_1, Δ_2, etc., are the same as δA_1, $\delta^2 A_1$, etc., A_1 being unity. Substitute in the expression for b_i. Then within the square brackets, we have the following terms:—

independent of Δ

$$1 + \frac{s}{1}\alpha^2 + \frac{s(s+1)}{1\,.\,2}\alpha^4 + \ldots \qquad = \frac{1}{(1-\alpha^2)^s};$$

multiplied by Δ_1

$$\frac{s}{1}\alpha^2 + \frac{s(s+1)}{1\,.\,2} 2\alpha^4 + \frac{s(s+1)(s+2)}{1\,.\,2\,.\,3} 3\alpha^6 + \ldots = \frac{1}{(1-\alpha^2)^s}\,\frac{s\alpha^2}{1-\alpha^2};$$

multiplied by Δ_2

$$\frac{s(s+1)}{1\,.\,2}\alpha^4+\frac{s(s+1)(s+2)}{1\,.\,2\,.\,3}\,\frac{3\,.\,2}{1\,.\,2}\alpha^6+\frac{s(s+1)(s+2)(s+3)}{1\,.\,2\,.\,3\,.\,4}\,\frac{4\,.\,3}{1\,.\,2}\alpha^8+\ldots$$

$$=\frac{1}{(1-\alpha^2)^s}\,\frac{s(s+1)}{1\,.\,2}\,\frac{\alpha^4}{(1-\alpha^2)^2};$$

and so on, the transformed expression being

$$\frac{1}{(1-\alpha^2)^s}\left[1+\frac{s}{1}\,\frac{\alpha^2}{1-\alpha^2}\Delta_1+\frac{s(s+1)}{1\,.\,2}\,\frac{\alpha^4}{(1-\alpha^2)^2}\Delta_2+\frac{s(s+1)(s+2)}{1\,.\,2\,.\,3}\,\frac{\alpha^6}{(1-\alpha^2)^3}\Delta_3+\ldots\right].$$

But taking the first differences of the coefficients

$$1,\ \frac{s+i}{i+1},\ \frac{(s+i)(s+i+1)}{(i+1)(i+2)},\ \frac{(s+i)(s+i+1)(s+i+2)}{(i+1)(i+2)(i+3)},\ \ldots,$$

we have

$$\frac{s-1}{i+1},\quad \frac{s-1}{i+1}\,\frac{s+i}{i+2},\quad \frac{s-1}{i+1}\,\frac{(s+i)(s+i+1)}{(i+2)(i+3)},\ \ldots.$$

Taking second differences

$$\frac{(s-1)(s-2)}{(i+1)(i+2)},\ \frac{(s-1)(s-2)}{(i+1)(i+2)}\,\frac{s+i}{i+3},\ \frac{(s-1)(s-2)}{(i+1)(i+2)}\,\frac{(s+i)(s+i+1)}{(i+3)(i+4)},\ \ldots.$$

The law of succession is evident; we have

$$\Delta_1=\frac{s-1}{i+1},\quad \Delta_2=\frac{(s-1)(s-2)}{(i+1)(i+2)},\quad \Delta_3=\frac{(s-1)(s-2)(s-3)}{(i+1)(i+2)(i+3)},\ \ldots,$$

and

$$b_i=2\,\frac{s(s+1)\ldots(s+i-1)}{1\,.\,2\ldots i}\,\frac{\alpha^i}{(1-\alpha^2)^s}\left[1+\frac{(s-1)s}{1\,.\,(i+1)}\,\frac{\alpha^2}{1-\alpha^2}\right.$$
$$\left.+\frac{(s-2)(s-1)s(s+1)}{1\,.\,2\,(i+1)(i+2)}\,\frac{\alpha^4}{(1-\alpha^2)^2}+\ldots\right].$$

This expression is very useful for computing b_i for large values of i.

We observe that the expression within square brackets is unchanged if we write $1-s$ for s. Hence if

$$S^{-1+s}=\frac{1}{2}\beta_0+\beta_1\cos\phi+\beta_2\cos 2\phi+\ldots,$$

we have

$$\beta_i=\frac{(1-s)(2-s)\ldots(i-s)}{s(s+1)\ldots(s+i-1)}(1-\alpha^2)^{2s-1}b_i.$$

We can exhibit b_i as the solution of a differential equation; thus it may be verified that

$$\alpha^2 \frac{d^2}{d\alpha^2}(S^{-s}) + \frac{d^2}{d\phi^2}(S^{-s}) + \frac{1-(4s+1)\alpha^2}{1-\alpha^2}\alpha\frac{d}{d\alpha}(S^{-s}) - \frac{4s^2\alpha^2}{1-\alpha^2}S^{-s} = 0;$$

substitute for S^{-s} its development, and equate to zero the coefficient of $\cos i\phi$; then

$$\alpha^2\frac{d^2b_i}{d\alpha^2} + \frac{1-(4s+1)\alpha^2}{1-\alpha^2}\alpha\frac{db_i}{d\alpha} - \frac{i^2+(4s^2-i^2)\alpha^2}{1-\alpha^2}b_i = 0.$$

An expression for b_i in the form of a definite integral is of frequent use; we have

$$S^{-s} = \frac{1}{2}b_0 + b_1\cos\phi + b_2\cos 2\phi + \ldots + b_i\cos i\phi + \ldots.$$

Multiply both members of this equality by $\cos i\phi$ and integrate with respect to ϕ between the limits 0 and 2π; then

$$b_i = \frac{1}{\pi}\int_0^{2\pi}\frac{\cos i\phi\, d\phi}{S^s}.$$

This leads to a Sequence Equation connecting the quantities b_i for consecutive values of i. We have

$$\begin{aligned}\frac{d}{d\phi}\left(\frac{\sin i\phi}{S^{s-1}}\right) &= \frac{i\cos i\phi}{S^{s-1}} - 2\frac{(s-1)\alpha\sin i\phi\sin\phi}{S^s}\\ &= \frac{i(1+\alpha^2)\cos i\phi}{S^s} - 2\frac{i\alpha\cos\phi\cos i\phi}{S^s} - 2\frac{(s-1)\alpha\sin\phi\sin i\phi}{S^s}\\ &= \frac{i(1+\alpha^2)\cos i\phi}{S^s} - \frac{\alpha(i+s-1)\cos(i-1)\phi}{S^s} - \frac{\alpha(i-s+1)\cos(i+1)\phi}{S^s}.\end{aligned}$$

Integrate with respect to ϕ between the limits 0 and 2π; observing that the right-hand member vanishes,

$$0 = (1+\alpha^2)ib_i - \alpha(i+s-1)b_{i-1} - \alpha(i-s+1)b_{i+1}.$$

This equation enables us to deduce the values of all the quantities b_i when the values are known for any two consecutive values of i.

Consider next the relations between the coefficients that arise by giving different related values to the quantity s. Let us write

$$S^{-s-1}=\frac{1}{2}c_0+c_1\cos\phi+c_2\cos 2\phi+\ldots,$$

then it is required to investigate the relations between the quantities b and c.

We have
$$c_i=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos i\phi\, d\phi}{S^{s+1}};$$

hence from the equation

$$\frac{d}{d\phi}\left(\frac{\sin i\phi}{S^s}\right)=\frac{i\cos i\phi}{S^s}-2\frac{sa\sin i\phi\sin\phi}{S^{s+1}}$$
$$=\frac{i\cos i\phi}{S^s}-\frac{sa\cos(i-1)\phi}{S^{s+1}}+\frac{sa\cos(i+1)\phi}{S^{s+1}},$$

we deduce
$$ib_i=sa\left[c_{i-1}-c_{i+1}\right].$$

For the case $i=0$, we must replace this by

$$b_0=(1+a^2)c_0-2ac_1.$$

In virtue of the sequence equation connecting the quantities c,

$$(1+a^2)ic_i-a(i+s)c_{i-1}-a(i-s)c_{i+1},$$

we may write this result in the two forms

$$(i+s)b_i=s\left[(1+a^2)c_i-2ac_{i+1}\right],$$
$$(i-s)b_i=s\left[2ac_{i-1}-(1+a^2)c_i\right].$$

Change i into $(i+1)$ in the latter expression; then

$$s(1-a)^2\left[c_i+c_{i+1}\right]=(i+s)b_i-(i-s+1)b_{i+1},$$
$$s(1+a)^2\left[c_i-c_{i+1}\right]=(i+s)b_i+(i-s+1)b_{i+1}.$$

Thus from two consecutive members of either of the series of quantities b, c, we can obtain the values of all the members of the other series.

Let us now investigate the relations between the functions b, and their derived functions with respect to a.

We have
$$b_i=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos i\phi\, d\phi}{S^s};$$

hence
$$\frac{db_i}{da} = \frac{2s}{\pi}\int_0^{2\pi} \frac{(\cos\phi - a)\cos i\phi}{S^{s+1}}\,d\phi,$$

$$= \frac{s}{\pi}\int_0^{2\pi} \frac{\cos(i-1)\phi + \cos(i+1)\phi - 2a\cos i\phi}{S^{s+1}}\,d\phi,$$

or
$$\frac{db_i}{da} = s\left[c_{i-1} + c_{i+1} - 2ac_i\right].$$

Substitute for the quantities c in terms of b;

$$(1-a^2)\frac{db_i}{da} = (i+s-1)\,b_{i-1} - (i-s+1)\,b_{i+1} + 2asb_i$$

$$= \left\{2as - \left(a + \frac{1}{a}\right)i\right\} b_i + 2\,(i+s-1)\,b_{i-1}$$

$$= \left\{2as + \left(a + \frac{1}{a}\right)i\right\} b_i - 2\,(i-s+1)\,b_{i+1}.$$

These equations are sufficient for determining $\frac{db_i}{da}$, and thence derived functions of higher order; but we can find others which it will generally be preferable to employ. We have

$$ib_i = sa\left[c_{i-1} - c_{i+1}\right];$$

therefore

$$i\frac{db_i}{da} = s\left[c_{i-1} - c_{i+1}\right] + sa\left[\frac{dc_{i-1}}{da} - \frac{dc_{i+1}}{da}\right].$$

Substitute for $\frac{db_i}{da}$ the expression in terms of the quantities c found above; then

$$a\left[\frac{dc_{i-1}}{da} - \frac{dc_{i+1}}{da}\right] = (i-1)\,c_{i-1} + (i+1)\,c_{i+1} - 2aic_i,$$

and in exactly the same way,

$$a\left[\frac{db_{i-1}}{da} - \frac{db_{i+1}}{da}\right] = (i-1)\,b_{i-1} + (i+1)\,b_{i+1} - 2aib_i.$$

Differentiate this, and we find

$$a\left[\frac{d^2b_{i-1}}{da^2} - \frac{d^2b_{i+1}}{da^2}\right] = (i-2)\frac{db_{i-1}}{da} + (i+2)\frac{db_{i+1}}{da} - 2ia\frac{db_i}{da} - 2ib_i;$$

$$a\left[\frac{d^3b_{i-1}}{da^3} - \frac{d^3b_{i+1}}{da^3}\right] = (i-3)\frac{d^2b_{i-1}}{da^2} + (i+3)\frac{d^2b_{i+1}}{da^2} - 2ia\frac{d^2b_i}{da^2} - 4i\frac{db_i}{da},$$

and so on. By these formulae we find the derived functions of any order from those of orders next below. We require to calculate independently two of these functions, say the derived functions of b_0 and b_1.

Thus in the case of first derived functions we find $\frac{db_0}{da}$ and $\frac{db_1}{da}$ from the formulae

$$(1-a^2)\frac{db_0}{da} = 2sab_0 + 2(s-1)b_1,$$

$$(1-a^2)\frac{db_1}{da} = 2sb_0 \quad + \left(2as - a - \frac{1}{a}\right)b_1,$$

which may be written more simply

$$\frac{db_0}{da} - a\frac{db_1}{da} = (2s-1)b_1,$$

$$\frac{db_1}{da} - a\frac{db_0}{da} = 2sb_0 - \frac{1}{a}b_1;$$

and by differentiating these we can find the values of $\frac{d^2b_0}{da^2}$, $\frac{d^2b_1}{da^2}$, etc.

Let us now return to the case from which we started, that is to say, the case when $s = \frac{1}{2}$. The quantities b_0 and b_1 are then expressible by means of elliptic functions.

We have

$$b_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{d\phi}{(1-2a\cos\phi + a^2)^{\frac{1}{2}}}, \quad b_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\cos\phi\, d\phi}{(1-2a\cos\phi + a^2)^{\frac{1}{2}}}.$$

Assume $$\sin(\theta - \phi) = a\sin\theta,$$

so that $$\cos(\theta-\phi) = (1 - a^2\sin^2\theta)^{\frac{1}{2}}, \quad = \Delta, \text{ suppose.}$$

Then $$\cos(\theta-\phi)(d\theta - d\phi) = a\cos\theta\, d\theta,$$

or $$\Delta d\phi = (\Delta - a\cos\theta)\, d\theta.$$

Also $$\cos\phi = \cos(\theta-\phi)\cos\theta + \sin(\theta-\phi)\sin\theta$$

$$= \Delta\cos\theta + a\sin^2\theta,$$

so that

$$(1-2a\cos\phi+a^2)^{\frac{1}{2}}=(1-2\Delta a\cos\theta-2a^2\sin^2\theta+a^2)^{\frac{1}{2}}$$
$$=(\Delta^2-2\Delta a\cos\theta+a^2\cos^2\theta)^{\frac{1}{2}}$$
$$=\Delta-a\cos\theta.$$

Hence
$$\frac{d\phi}{(1-2a\cos\phi+a^2)^{\frac{1}{2}}}=\frac{d\theta}{\Delta},$$
$$\frac{\cos\phi d\phi}{(1-2a\cos\phi+a^2)^{\frac{1}{2}}}=\left(\cos\theta+\frac{a\sin^2\theta}{\Delta}\right)d\theta.$$

Now let θ vary from 0 to 2π; then since $\theta-\phi$ can never exceed the angle whose sine is a, it varies from 0 continuously to 0 again, and therefore ϕ will vary with θ from 0 to 2π. Hence, remarking that

$$\int_0^{2\pi}\cos\theta d\theta=0,$$

we have
$$b_0=\frac{1}{\pi}\int_0^{2\pi}\frac{d\theta}{\Delta},\quad b_1=\frac{1}{a\pi}\int_0^{2\pi}\left(\frac{d\theta}{\Delta}-\Delta d\theta\right),$$

or writing as is usual
$$F(a)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-a^2\sin^2\theta)^{\frac{1}{2}}},\quad E(a)=\int_0^{\frac{\pi}{2}}(1-a^2\sin^2\theta)^{\frac{1}{2}}d\theta,$$

we have
$$b_0=\frac{4}{\pi}F(a),\quad b_1=\frac{4}{a\pi}[F(a)-E(a)].$$

From these expressions b_0, b_1 may be computed; thus it is known that if a, a', a'', ... be a set of moduli derived in succession by the formula

$$a'=\frac{1-\beta}{1+\beta};$$

where
$$a^2+\beta^2=1,$$

then
$$F(a)=\frac{\pi}{2}(1+a')(1+a'')\ldots,$$

and
$$\frac{F(a)-E(a)}{a}=F(a)\left[\frac{1}{2}a+\frac{1}{4}aa'+\frac{1}{8}aa'a''+\ldots\right].$$

An alternative method of reaching these results is given by Gauss (*Determinatio Attractionis etc.*, §§ 16, 17).

Write

$$\frac{1}{\mu}=\frac{1}{2\pi}\int_0^{2\pi}\frac{dT}{(m^2\cos^2 T+n^2\sin^2 T)^{\frac{1}{2}}},\quad -\frac{\nu}{\mu}=\frac{1}{2\pi}\int_0^{2\pi}\frac{(\cos^2 T-\sin^2 T)\,dT}{(m^2\cos^2 T+n^2\sin^2 T)^{\frac{1}{2}}};$$

then if

$$m'=\frac{1}{2}(m+n)\ ,\ \ n'=\sqrt{mn}\ ,$$

$$m''=\frac{1}{2}(m'+n'),\ \ n''=\sqrt{m'n'},$$

and we proceed till we find a common limit to the quantities m and n, we shall see that this limit is μ. μ is called the arithmetico-geometric mean of m and n. Again if

$$\frac{1}{4}(m^2-n^2)^{\frac{1}{2}}=\lambda,\ \ \frac{1}{4}(m'^2-n'^2)^{\frac{1}{2}}=\lambda',\ \ldots,$$

so that

$$\lambda'=\frac{\lambda^2}{m'},\quad \lambda''=\frac{\lambda'^2}{m''},\ \ldots$$

then it will appear further that

$$\nu=\frac{2\lambda'^2+4\lambda''^2+8\lambda'''^2+\ldots}{\lambda^2}.$$

The first of these may be proved by making the substitution

$$\sin T=\frac{2m\sin T'}{(m+n)\cos^2 T'+2m\sin^2 T'},$$

when we find

$$\frac{dT}{(m^2\cos^2 T+n^2\sin^2 T)^{\frac{1}{2}}}=\frac{dT'}{(m'^2\cos^2 T'+n'^2\sin^2 T')^{\frac{1}{2}}};$$

making a second similar substitution

$$\frac{dT'}{(m'^2\cos^2 T'+n'^2\sin^2 T')^{\frac{1}{2}}}=\frac{dT''}{(m''^2\cos^2 T''+n''^2\sin^2 T'')^{\frac{1}{2}}}$$

$$=\ldots\ldots\ldots\ldots\ldots\ldots=\frac{d\Theta}{(\mu^2\cos^2\Theta+\mu^2\sin^2\Theta)^{\frac{1}{2}}}=\frac{d\Theta}{\mu},$$

where μ is the arithmetico-geometric mean between m and n; and in all cases 0 and 2π are corresponding limits for the quantities $T, T', \ldots \Theta$. Hence integrating between these limits, we get the first theorem. Again by the same transformation we find

$$\frac{m'(\cos^2 T - \sin^2 T)\,dT}{(m^2\cos^2 T + n^2\sin^2 T)^{\frac{1}{2}}} = -\frac{1}{2}\,\frac{(m-n)\sin^2 T' dT'}{(m'^2\cos^2 T' + n'^2\sin^2 T')^{\frac{1}{2}}} + d\,(\sin T'\cos T);$$

substituting $\sin^2 T' = \frac{1}{2} - \frac{1}{2}(\cos^2 T' - \sin^2 T')$, and integrating from 0 to 2π, we have

$$\frac{m^2-n^2}{2\pi}\int_0^{2\pi}\frac{(\cos^2 T - \sin^2 T)\,dT}{(m^2\cos^2 T + n^2\sin^2 T)^{\frac{1}{2}}} = -\frac{2\,(m'^2-n'^2)}{\mu}$$
$$+2\,\frac{(m'^2-n'^2)}{2\pi}\int_0^{2\pi}\frac{(\cos^2 T' - \sin^2 T')\,dT'}{(m'^2\cos^2 T' + n'^2\sin^2 T')^{\frac{1}{2}}},$$

whence

$$-\frac{\nu}{\mu} = -\frac{2\,(m'^2-n'^2) + 4\,(m''^2-n''^2) + \ldots}{(m^2-n^2)\,\mu},$$

which is the second theorem.

To apply these results to the calculation of the quantities b_0 and b_1, for the case when $s=\frac{1}{2}$, we notice that if

$$\phi = \pi - 2T,$$

then

$$b_0 = \frac{2}{\pi}\int_0^{\pi}\frac{d\phi}{(1-2a\cos\phi + a^2)^{\frac{1}{2}}} = \frac{1}{\pi}\int_0^{2\pi}\frac{dT}{\{(1+a)^2\cos^2 T + (1-a)^2\sin^2 T\}^{\frac{1}{2}}},$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\frac{\cos\phi\, d\phi}{(1-2a\cos\phi + a^2)^{\frac{1}{2}}} = -\frac{2}{\pi}\int_0^{2\pi}\frac{(\cos^2 T - \sin^2 T)\,dT}{\{(1+a)^2\cos^2 T + (1-a)^2\sin^2 T\}^{\frac{1}{2}}},$$

or

$$b_0 = \frac{2}{\mu},\quad b_1 = \frac{2\nu}{\mu},$$

where

$$m = 1 + a,\quad n = 1 - a.$$

Let us illustrate these formulae by computing a few of the quantities b for the first and second Satellites of Jupiter, for the case $s=\frac{1}{2}$.

Here

$$a = 0{\cdot}6285152.$$

Let us first compute b_0 and b_1 by Gauss's method of the arithmetico-geometric mean.

m	1·6285152	$\log m$	0·2117918
n	0·3714848	$\log n$	9·5699410
	2) 2·0000000		2) 9·7817328
m'	1·0000000	$\log n'$	9·8908664
n'	0·7777973	$\log m'$	0·0000000
	2) 1·7777973		2) 9·8908664
m''	0·8888986	$\log n''$	9·9454332
n''	0·8819282	$\log m''$	9·9488522
	2) 1·7708268		2) 9·8942854
m'''	0·8854134	$\log n'''$	9·9471427
n'''	0·8854065	$\log m'''$	9·9471461
	2) 1·7708199		2) 9·8942888
m^{iv}	0·8854100	$\log n^{iv}$	9·9471444
$\mu = n^{iv}$	0·8854100		

Hence

$$b_0 = 2{\cdot}2588406 \qquad \log b_0 = 0{\cdot}3538856.$$

Again for ν,

$$\lambda^2 = \frac{1}{4}a = 0{\cdot}1571288,$$

$\log \lambda^2$	9·1962558	$2\lambda'^2$	0·04937892
$\log m'$	0·0000000	$4\lambda''^2$	0·00308588
	9·1962558	$8\lambda'''^2$	0·00000607
$\log \lambda'^2$	8·3925116		0·05247087
$\log m''$	9·9488522	$\log$	8·7199182
	8·4436594	$\log \lambda^2$	9·1962558
$\log \lambda''^2$	6·8873188	$\log \nu =$	9·5236624
$\log m'''$	9·9471461	$\log \mu =$	9·9471444
	6·9401727		9·5765180
$\log \lambda'''^2$	3·8803454		
$\log m^{iv}$	9·9471444		
$\log \lambda^{iv}$	3·9332010		

Hence

$$b_1 = 0{\cdot}7543069 \qquad \log b_1 = 9{\cdot}3775480.$$

Next compute b_9 and b_{10} for the same value of s from Legendre's formula.

We have

$$b_i = 2\,\frac{1\,.\,3\,.\,5\,\ldots\,2i-1}{2\,.\,4\,.\,6\,\ldots\,2i}\,\frac{\alpha^i}{(1-\alpha^2)^{\frac{1}{2}}}\left[1-\frac{1^2}{4\,(i+1)}\,\frac{\alpha^2}{1-\alpha^2}+\frac{1^2\,.\,3^2}{4\,.\,8\,(i+1)(i+2)}\,\frac{\alpha^4}{(1-\alpha^2)^2}-\ldots\right],$$

and

$$\alpha = 0{\cdot}6285152$$

$$\beta^2 = \frac{\alpha^2}{1-\alpha^2} = 0{\cdot}6529783 \quad (9{\cdot}8148988).$$

We find $$\frac{1\,.\,3\,\ldots\,17}{2\,.\,4\,\ldots\,18}\,\frac{\alpha^9}{(1-\alpha^2)^{\frac{1}{2}}} = (7{\cdot}5622508).$$

Also computing the terms of the series within brackets, in succession

$\frac{1}{40}$	8·3979400		4·5552311
β^2	9·8148988	121	2·0827854
	8·2128388 n	$\frac{1}{360}$	7·4436975
9	0·9542425	β^2	9·8148988
$\frac{1}{88}$	8·0555173		3·8966128
β^2	9·8148988	169	2·2278867
	7·0374974	$\frac{1}{448}$	7·3487220
25	1·3979400	β^2	9·8148988
$\frac{1}{144}$	7·8416375		3·2881203 n
β^2	9·8148988	225	2·3521825
	6·0919737 n	$\frac{1}{544}$	7·2644011
49	1·6901961	β^2	9·8148988
$\frac{1}{208}$	7·6819367		2·7196027
β^2	9·8148988	289	2·4608978
	5·2790053	$\frac{1}{648}$	7·1884250
81	1·9084850	β^2	9·8148988
$\frac{1}{280}$	7·5528420		2·1838243 n
β^2	9·8148988		
	4·5552311 n		

Collecting separately the positive and negative terms

	1·	−0·01632446
	·00109018	0·00012359
	·00001901	0·00000359
	·00000079	0·00000019
	·00000005	0·00000001
	1·00111003	−0·01645184
	−0·01645184	
	0·98465819	
log	9·9932855	
	7·5622508	
	·3010300	
$\log b_9$	7·8565663	$b_9 = 0·007187308.$

From the above we may deduce the corresponding terms in b_{10} by multiplying the successive terms we have found by $\frac{10}{11}, \frac{10}{12}, \ldots, \frac{10}{19}$ respectively and the multiplier outside the bracket by $\frac{19}{20}\alpha$.

The terms become

	1·	−0·01484042
	0·00090848	0·00009507
	0·00001358	0·00000239
	0·00000049	0·00000011
	0·00000003	0·00000001
	1·00092258	−0·01493800
	−0·01493800	
	0·98598458	
log	9·9938701	
log (multiplier)	7·6393202	
$\log b_{10} =$	7·6331903	$b_{10} = 0·004297246,$

and
$$\frac{b_{10}}{b_9} = (9·7766240).$$

Now compute b_2, $b_3 \dots b_8$ by successive steps from the sequence equation, which when $s = \frac{1}{2}$ assumes the form

$$(2i+1)\frac{b_{i+1}}{b_i} - 2i\left(a + \frac{1}{a}\right) + (2i-1)\frac{b_{i-1}}{b_i} = 0,$$

and we have the choice of proceeding backwards from b_{10} and b_9, or forwards from b_0 and b_1. If we try the latter method,

given $b_0 = 2\cdot2588406$
$b_1 = 0\cdot7543069$,

we derive $b_2 = 0\cdot3632098$
$b_3 = 0\cdot1923505$
$b_4 = 0\cdot1065085$
$b_5 = 0\cdot0605299$
$b_6 = 0\cdot03499318$
$b_7 = 0\cdot02047747$
$b_8 = 0\cdot01209361$
$b_9 = 0\cdot00719524$
$b_{10} = 0\cdot00430918$.

The agreement with the values of b_9 and b_{10} already calculated is not very good, and we can trace the reason for this; for if in the sequence equation

$$0 = (i - s + 1)\frac{b_{i+1}}{b_i} - i\left(a + \frac{1}{a}\right) + (i + s - 1)\frac{b_{i-1}}{b_i},$$

we denote b_i/b_{i-1} by p_i, and a small error in this ratio by Δp_i, we have the relation

$$\Delta p_{i+1} = \frac{i+s-1}{i-s+1}\frac{\Delta p_i}{p_i^2},$$

so that

$$\Delta p_i = \frac{s(s+1)\dots(s+i-2)}{(2-s)(3-s)\dots(i-s)}\frac{\Delta p_1}{p_1^2 p_2^2 \dots p_{i-1}^2},$$

but $$p_1 p_2 \dots p_{i-1} = \frac{b_{i-1}}{b_0};$$

therefore $$b_{i-1}^2 \Delta p_i = \frac{s(s+1)\dots(s+i-2)}{(2-s)(3-s)\dots(i-s)} b_0^2 \Delta p_1.$$

Hence if we take i large enough, an original error is increased if we derive p_i from p_1, but it is diminished if we derive p_1 from p_i. Let us then derive b_8, b_7, etc. from b_{10} and b_9;

given
$$b_{10} = 0{\cdot}004297246$$
$$b_9 = 0{\cdot}007187308,$$

we find
$$b_8 = 0{\cdot}01208930$$
$$b_7 = 0{\cdot}02047389$$
$$b_6 = 0{\cdot}03499074$$
$$b_5 = 0{\cdot}06052825$$
$$b_4 = 0{\cdot}1065074$$
$$b_3 = 0{\cdot}1923497$$
$$b_2 = 0{\cdot}3632093$$
$$b_1 = 0{\cdot}7543069$$
$$b_0 = 2{\cdot}2588406.$$

In this case the agreement with the values of b_1 and b_0 already calculated is perfect.

Consider now the application of the foregoing results to the disturbing function

$$R = \frac{m'}{\{r^2 + r'^2 - 2rr' \cos(\theta - \theta')\}^{\frac{1}{2}}} - \frac{m'r \cos(\theta - \theta')}{r'^2},$$

for the disturbances of a body m at the position (r, θ) by a body m' at (r', θ') in the same plane.

Let
$$r = a\ (1 + x), \qquad \theta = nt + \epsilon + y = l + y,$$
$$r' = a'\,(1 + x'), \qquad \theta' = n't + \epsilon' + y' = l' + y',$$

where x, y, x', y' are supposed small; also let

$$Q = \frac{1}{\{a^2 + a'^2 - 2aa' \cos(l - l')\}^{\frac{1}{2}}} - \frac{a \cos(l - l')}{a'^2}.$$

Then if

$$Q = \frac{1}{2} A_0 + A_1 \cos(l - l') + A_2 \cos 2(l - l') + \dots$$

where a refers to the inferior satellite and a' to the superior we have

$$A_i = \frac{1}{a'} b_i$$

in general, but

$$A_1 = \frac{1}{a'} b_1 - \frac{a}{a'^2},$$

where b_i is that function of $\mathfrak{a}$ or a/a' which we have been considering;

and where a refers to the superior satellite and a' to the inferior, taking $\mathfrak{a} = a'/a$ which is less than unity, we have

$$A_i = \frac{1}{a} b_i$$

in general, but

$$A_1 = \frac{1}{a} b_1 - \frac{a}{a'^2}.$$

Hence for the perturbations of an inferior satellite, we have

$$a \frac{dA_i}{da} = \frac{1}{a'} \mathfrak{a} \frac{db_i}{d\mathfrak{a}},$$

$$a' \frac{dA_i}{da'} = -\frac{1}{a'} \left(\mathfrak{a} \frac{db_i}{d\mathfrak{a}} + b_i \right);$$

we notice that A_i is a homogeneous function of a and a', of -1 dimension.

Also
$$a^2 \frac{d^2 A_i}{da^2} = \frac{1}{a'} \mathfrak{a}^2 \frac{d^2 b_i}{d\mathfrak{a}^2},$$

$$aa' \frac{d^2 A_i}{da\, da'} = -\frac{1}{a'} \left(\mathfrak{a}^2 \frac{d^2 b_i}{d\mathfrak{a}^2} + 2\mathfrak{a} \frac{db_i}{d\mathfrak{a}} \right),$$

$$a'^2 \frac{d^2 A_i}{da'^2} = \frac{1}{a'} \left(\mathfrak{a}^2 \frac{d^2 b_i}{d\mathfrak{a}^2} + 4\mathfrak{a} \frac{db_i}{d\mathfrak{a}} + 2b_i \right).$$

In the case $i = 1$, we must add to the differential coefficients of A_i, given by the above formulae, the corresponding differential coefficients of $-a/a'^2$.

For the perturbations of a superior satellite disturbed by an inferior we have, where $\mathfrak{a} = a'/a$,

$$a \frac{dA_i}{da} = -\frac{1}{a} \left(\mathfrak{a} \frac{db_i}{d\mathfrak{a}} + b_i \right),$$

$$a' \frac{dA_i}{da'} = \frac{1}{a} \mathfrak{a} \frac{db_i}{d\mathfrak{a}},$$

and so on.

Now the functions with which we shall have to deal are $\frac{1}{r}\frac{dR}{dr}$, $\frac{1}{r^2}\frac{dR}{d\theta}$, that is to say

$$\frac{1}{a}\frac{dQ}{da}+xa\frac{d}{da}\left(\frac{1}{a}\frac{dQ}{da}\right)+x'a'\frac{d}{da'}\left(\frac{1}{a}\frac{dQ}{da}\right)+(y-y')\frac{d}{dl}\left(\frac{1}{a}\frac{dQ}{da}\right)+\dots,$$

and

$$\frac{1}{a^2}\frac{dQ}{dl}+xa\frac{d}{da}\left(\frac{1}{a^2}\frac{dQ}{dl}\right)+x'a'\frac{d}{da'}\left(\frac{1}{a^2}\frac{dQ}{dl}\right)+(y-y')\frac{d}{dl}\left(\frac{1}{a^2}\frac{dQ}{dl}\right)+\dots.$$

Now

$$\frac{1}{a}\frac{dQ}{da}=\quad\Sigma\frac{1}{a}\frac{dA_i}{da}\cos i(l-l'),$$

$$a\frac{d}{da}\left(\frac{1}{a}\frac{dQ}{da}\right)=\quad\Sigma\left(\frac{d^2A_i}{da^2}-\frac{1}{a}\frac{dA_i}{da}\right)\cos i(l-l'),$$

$$a'\frac{d}{da'}\left(\frac{1}{a}\frac{dQ}{da}\right)=\quad\Sigma\frac{a'}{a}\frac{d^2A_i}{dada'}\cos i(l-l'),$$

$$\frac{d}{dl}\left(\frac{1}{a}\frac{dQ}{da}\right)=-\Sigma\frac{1}{a}\frac{dA_i}{da}i\sin i(l-l').$$

Again

$$\frac{1}{a^2}\frac{dQ}{dl}=\Sigma\frac{1}{a^2}A_i\,i\sin i(l-l'),$$

$$a\frac{d}{da}\left(\frac{1}{a^2}\frac{dQ}{dl}\right)=\Sigma\left(\frac{2}{a^2}A_i-\frac{1}{a^2}\frac{dA_i}{da}\right)i\sin i(l-l'),$$

$$a'\frac{d}{da'}\left(\frac{1}{a^2}\frac{dQ}{dl}\right)=\Sigma-\frac{a'}{a^2}\frac{dA_i}{da'}i\sin i(l-l'),$$

$$\frac{d}{dl}\left(\frac{1}{a^2}\frac{dQ}{dl}\right)=\Sigma-\frac{1}{a^2}A_i i^2\cos i(l-l').$$

In these formulae, under the sign Σ, $\frac{1}{2}A_0$ stands in place of A_0.

We have thus shewn completely how to express numerically the coefficient of any periodic term in the disturbing function.

III.

ON THE INEQUALITIES OF JUPITER'S SATELLITES WHICH ARE INDEPENDENT OF THE ECCENTRICITIES AND INCLINATIONS OF THEIR ORBITS.

Let μ denote the mass of Jupiter, or more strictly the sum of the masses of Jupiter and the satellite whose motion we are investigating;

ρ the ellipticity of Jupiter,

ϕ the ratio of centrifugal force to gravity at his equator,

ν the quantity $\mu A^2\left(\rho - \frac{1}{2}\phi\right)$ where A is his equatoreal radius;

then if the motion be supposed to take place in the plane of his equator, the potential due to the attraction of the planet will be

$$V = \frac{\mu}{r} + \frac{1}{3}\frac{\nu}{r^3}.$$

Let m, r, θ denote the mass and the coordinates of the satellite under consideration,

m', r', θ', &c., corresponding quantities for the satellites by which it is disturbed,

S, D, L, the like quantities for the Sun, which is supposed to move in the plane of Jupiter's equator;

then the disturbing function due to the action of the other satellites and the Sun is

$$R = \Sigma\left[\frac{m'}{\{r^2 + r'^2 - 2rr'\cos(\theta-\theta')\}^{\frac{1}{2}}} - \frac{m'r\cos(\theta-\theta')}{r'^2}\right] + \frac{Sr^2}{4D^3}[1 + 3\cos 2(\theta - L)],$$

where the sign Σ includes all the disturbing satellites. And the equations of motion of the satellite are

$$\frac{1}{r}\frac{d^2r}{dt^2} - \left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^3} + \frac{\nu}{r^5} = \frac{1}{r}\frac{dR}{dr},$$

$$\frac{d^2\theta}{dt^2} + 2\frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt} = \frac{1}{r^2}\frac{dR}{d\theta}.$$

But if we omit the disturbances due to eccentricity and also those due to the square of the disturbing force, we may write

$$\frac{1}{r}\frac{dR}{dr}=\frac{1}{2}\frac{S}{D^3}[1+3\cos 2(\theta-L)]+\Sigma\frac{m'}{a}\frac{dQ}{da},$$

$$\frac{1}{r^2}\frac{dR}{d\theta}=\frac{3}{2}\frac{S}{D^3}[\quad-3\sin 2(\theta-L)]+\Sigma\frac{m'}{a^2}\frac{dQ}{dl},$$

where $$Q=\frac{1}{\{a^2+a'^2-2aa'\cos(l-l')\}^{\frac{1}{2}}}-\frac{a\cos(l-l')}{a'^2},$$

$$l=nt+\epsilon,\qquad l'=n't+\epsilon'.$$

To satisfy these equations, assume

$$r=a[1-c\cos 2(l-L)-\Sigma a_i\cos i(l-l')],$$
$$\theta=\quad l+k\sin 2(l-L)+\Sigma g_i\sin i(l-l'),$$

where i includes all positive integers.

Then neglecting squares and products of the coefficients a_i, g_i, c, k, and also $\frac{dL}{dt}$ or M in places where it will afterwards appear that its retention would not modify the result to the order we are considering, we have

$$\frac{1}{r}\frac{d^2r}{dt^2}=4n^2c\cos 2(l-L)+\Sigma i^2(n-n')^2a_i\cos i(l-l')$$

$$-\left(\frac{d\theta}{dt}\right)^2=-n^2-4n^2k\cos 2(l-L)-2n(n-n')\Sigma ig_i\cos i(l-l')$$

$$\frac{\mu}{r^3}+\frac{\nu}{r^5}=\frac{\mu}{a^3}[1+3c\cos 2(l-L)+3\Sigma a_i\cos i(l-l')]$$
$$+\frac{\nu}{a^5}[1+5c\cos 2(l-L)+5\Sigma a_i\cos i(l-l')]$$

$$=\frac{\mu}{a^3}[1+f+(3+5f)c\cos 2(l-L)+(3+5f)\Sigma a_i\cos i(l-l')]$$

where $$f=\nu/\mu a^2.$$

Also

$$\frac{d^2\theta}{dt^2}=-4n^2k\sin 2(l-L)-\Sigma i^2(n-n')^2g_i\sin i(l-l'),$$

$$2\frac{1}{r}\frac{dr}{dt}\frac{d\theta}{dt}=4n^2c\sin 2(l-L)+\Sigma 2n(n-n')ia_i\sin i(l-l').$$

Substitute in the equations of motion and write

$$\frac{S}{D^3} = M^2,$$

where M is the mean motion of Jupiter about the Sun, and equate coefficients of the various terms.

We obtain the following equations :—

constant term in the first equation :—

$$\frac{\mu}{a^3}(1+f) = n^2 + \frac{1}{2}M^2 + \frac{1}{2}\Sigma\frac{m'}{a}\frac{dA_0}{da};$$

this gives the relation between a and n;

coefficient of $\cos 2(l-L)$ in the first equation :—

$$4n^2c - 4n^2k + \frac{\mu}{a^3}(3+5f) = \frac{3}{2}M^2;$$

coefficient of $\sin 2(l-L)$ in the second equation :—

$$-4n^2k + 4n^2c \qquad\qquad = \frac{3}{2}M^2;$$

hence

$$c = \left(1 - \frac{2f}{3+5f}\right)\frac{M^2}{n^2}, \quad k = \left(\frac{11}{8} - \frac{2f}{3+5f}\right)\frac{M^2}{n^2},$$

these stand for the inequality which is called the Variation in Lunar Theory. We now see that in omitting $\frac{dL}{dt}$ when forming the differential equations we have omitted quantities of the order of M^3 in the values of c and k;

coefficient of $\cos i(l-l')$ in the first equation :—

$$i^2(n-n')^2 a_i - 2n(n-n')\, ig_i + \frac{\mu}{a^3}(3+5f)\, a_i = \frac{m'}{a}\frac{dA_i}{da};$$

coefficient of $\sin i(l-l')$ in the second equation :—

$$-i^2(n-n')^2 g_i + 2n(n-n')\, ia_i \qquad\qquad = -\frac{m'}{a^2}\, iA_i.$$

Multiply the second of these equations by $2n/i(n-n')$, and subtract from the first :—

$$\left[i(n-n')^2 - 4n^2 + \frac{\mu}{a^3}(3+5f)\right] a_i = \frac{m'}{a^2}\left[a\frac{dA_i}{da} + \frac{2n}{n-n'}A_i\right].$$

Making use of the relation

$$\frac{\mu}{a^3}(1+f) = n^2 + \frac{1}{2}M^2 + \frac{1}{2}\Sigma\frac{m'}{a}\frac{dA_0}{da},$$

we may write the coefficient of α_i,

$$i^2(n-n')^2 - N^2,$$

where

$$N^2 = n^2 - \frac{3}{2}M^2 - 2f\frac{\mu}{a^3} - \frac{3}{2}\Sigma\frac{m'}{a}\frac{dA_0}{da}.$$

Where the highest accuracy is not required we may put

$$-2fn^2 \text{ in place of } -2f\mu/a^3,$$

$$\frac{m'n^2}{\mu}a^2\frac{dA_0}{da} \text{ in place of } \frac{m'}{a}\frac{dA_0}{da};$$

but if we wish to be as exact as possible, we must include a further small correction to the value of N^2 for the following reason. The quantity $\frac{m'}{a}\frac{dQ}{da}$ contains the constant term $\frac{1}{2}\frac{m'}{a}\frac{dA_0}{da}$; hence a small correction xa to the value of r will introduce a correction $xa\frac{d}{da}\left(\frac{1}{2}\frac{m'}{a}\frac{dA_0}{da}\right)$ in $\frac{m'}{a}\frac{dQ}{da}$, which contains the term

$$\alpha_i\cos i(l-l')\left[-\frac{1}{2}m'\left(\frac{d^2A_0}{da^2} - \frac{1}{a}\frac{dA_0}{da}\right)\right],$$

to be added to the right-hand member of the first equation. Terms multiplied by α_{i-1}, α_{i+1}, etc., are also introduced, but we shall ignore them; thus we get the more exact value of N^2,

$$N^2 = n^2 - \frac{3}{2}M^2 - 2f\frac{\mu}{a^3} - \frac{1}{2}\Sigma m'\left(\frac{d^2A_0}{da^2} + 2\frac{1}{a}\frac{dA_0}{da}\right),$$

and then α_i, g_i are found from

$$[i^2(n-n')^2 - N^2]\alpha_i = \frac{m'}{a^2}\left[a\frac{dA_i}{da} + \frac{2n}{n-n'}A_i\right],$$

$$g_i = \frac{2n}{i(n-n')}\alpha_i + \frac{1}{i(n-n')^2}\frac{m'}{a^2}A_i.$$

The coefficients α_i, g_i gain in importance if the divisor $i^2(n-n')^2 - n^2$ be small. Now $i^2(n-n')^2 - n^2 = \{i(n-n') - n\}\{i(n-n') + n\}$, and in the case of

the first and second satellites, and also in the case of the second and third, the quantity $n-2n'$ is small. Hence in the case of the first satellite disturbed by the second, and also in that of the second disturbed by the third, α_2, g_2 will be considerable, while α_1, g_1 will be considerable for the cases of the second satellite disturbed by the first, and the third disturbed by the second.

The difference $n-2n'$ is positive, and the term $-2f\mu/a^3$ in the expression for N^2, which depends upon the ellipticity of Jupiter, increases the divisor $i^2(n-n')^2-N^2$ algebraically; hence the ellipticity of Jupiter will diminish α_2, g_2 where these are considerable, and increase α_1, g_1.

Let us now compute the mutual perturbations of the first and second satellites; the foregoing formulae apply without distinction to the perturbations of Satellite I by Satellites II, III, IV, and Satellite II by Satellites I, III, IV; but as we are conducting the computations simultaneously, let us slightly change the notation, so that where heretofore a, α_i, g_i &c. referred always to the body whose motion was under consideration, they shall now refer always to Satellite I, the corresponding quantities for Satellite II being denoted by accented letters.

Then we have
$$fa^2 = f'a'^2,$$
$$A_i = \frac{1}{a}\,\mathfrak{a}b_i = A_i',$$
$$A_1 = \frac{1}{a}\left(\mathfrak{a}b_1 - \mathfrak{a}^2\right), \qquad A_1' = \frac{1}{a}\left(\mathfrak{a}b_1 - \frac{1}{\mathfrak{a}}\right),$$
$$\frac{dA_i}{da} = \frac{1}{a^2}\left(\mathfrak{a}^2\frac{db_i}{d\mathfrak{a}}\right), \qquad \frac{dA_i'}{da'} = -\frac{1}{a'^2}\left(\mathfrak{a}\frac{db_i}{d\mathfrak{a}} + b_i\right),$$
$$\frac{dA_1}{da} = \frac{1}{a^2}\left(\mathfrak{a}^2\frac{db_1}{d\mathfrak{a}} - \mathfrak{a}^2\right), \qquad \frac{dA_1'}{da'} = -\frac{1}{a'^2}\left(\mathfrak{a}\frac{db_1}{d\mathfrak{a}} + b_1 + \frac{1}{\mathfrak{a}^2}\right),$$
$$a^2\left(\frac{d^2A_0}{da^2} + \frac{2}{a}\frac{dA_0}{da}\right) = \frac{1}{a'}\left(\mathfrak{a}^2\frac{d^2b_0}{d\mathfrak{a}^2} + 2\mathfrak{a}\frac{db_0}{d\mathfrak{a}}\right) = a'^2\left(\frac{d^2A_0'}{da'^2} + \frac{2}{a'}\frac{dA_0'}{da'}\right).$$

Now let us employ the following data:—

The mean motions of the several satellites and of Jupiter in 365·25 days, expressed in centesimal seconds, are*,

[* These are derived from the synodic periods of the satellites, the mean tropical motion of Jupiter, and the precession of the equinoxes adopted in Damoiseau's '*Tables*,' Introduction, p. iii.]

$$
\begin{aligned}
n &= 825826010''\!\cdot\!423,\\
n' &= 411412421\cdot 501,\\
n'' &= 204205627\cdot 040,\\
n''' &= 87542597\cdot 800,\\
M &= 337212\cdot 092\,;
\end{aligned}
$$

these satisfy the condition

$$n - 3n' + 2n'' = 0,$$

making
$$n - 2n' = n' - 2n'' = 3001167''\!\cdot\!421.$$

Laplace gives the following values for the masses*:

$$
\begin{aligned}
m &= 0\cdot 0000173281,\\
m' &= 0\cdot 0000232355,\\
m'' &= 0\cdot 0000884972,\\
m''' &= 0\cdot 0000426591,
\end{aligned}
$$

the unit being the mass of Jupiter;

also
$$\rho - \frac{1}{2}\phi = 0\cdot 0219013,$$

$$\frac{a}{A} = 5\cdot 698491, \qquad \frac{a'}{A} = 9\cdot 066548\,;$$

so that

$$f = \left(\rho - \frac{1}{2}\phi\right)\frac{A^2}{a^2} = 0\cdot 0006744505, \quad f' = \left(\rho - \frac{1}{2}\phi\right)\frac{A^2}{a'^2} = 0\cdot 0002664317,$$

$$(1+m)f = 0\cdot 000674462, \qquad (1+m')f' = 0\cdot 000266438.$$

First from the equations

$$
\begin{aligned}
\frac{\mu}{a^3}(1+f) &= n^2 + \frac{1}{2}M^2 + \frac{1}{2}\Sigma\frac{m'}{a}\frac{dA_0}{da},\\
\frac{\mu'}{a'^3}(1+f') &= n'^2 + \frac{1}{2}M^2 + \frac{1}{2}\Sigma\frac{m}{a'}\frac{dA_0'}{da'},
\end{aligned}
$$

let us find the exact values of a and a' that satisfy them.

[* To elicit these values from the observations a more elaborate theory is required than the one here developed.]

Assuming the values for a*

I and II		I and III		I and IV	
a	0·6285152	a	0·3940191	a	0·2240197

we find†

$\log a^2 \frac{db_0}{da}$	9·6379872	$\log a^2 \frac{db_0}{da}$	8·8686957	$\log a^2 \frac{db_0}{da}$	8·0759879
$\log m'$	5·3661520	$\log m''$	5·9469295	$\log m'''$	5·6300117
$\log \frac{1}{2}$	9·6989700	$\log \frac{1}{2}$	9·6989700	$\log \frac{1}{2}$	9·6989700
	4·7031092		4·5145952		3·4049696
	·0000050479		·0000032704		·0000002541

Again

II and I		II and III		II and IV	
a	0·6285152	a	0·6269046	a	0·3564269

we have

b_0	2·258841	$\log a^2 \frac{db_0}{da}$	9·6330469	$\log a^2 \frac{db_0}{da}$	8·7221062
$a \frac{db_0}{da}$	0·691308	$\log m''$	5·9469295	$\log m'''$	5·6300117
	2·950149	$\log \frac{1}{2}$	9·6989700	$\log \frac{1}{2}$	9·6989700
$\log$	0·4698440		5·2789464		4·0510879
$\log m$	5·2387509		·0000190084		·0000011248
$\log \frac{1}{2}$	9·6989700				
	5·4075649				
	·0000255602				

Hence we have the following terms multiplying $\frac{1}{a^3}$ and $\frac{1}{a'^3}$ respectively,

Satellite I		Satellite II	
$1+m$	1·0000173281	$1+m'$	1·0000232355
$(1+m)f$	·000674462	$(1+m')f'$	·000266438
$-\frac{1}{2}m'a^2\frac{dA_0}{da}$	$-$ ·0000050479	$-\frac{1}{2}ma'^2\frac{dA_0}{da}$	·0000255602
$-\frac{1}{2}m''a^2\frac{dA_0}{da}$	$-$ ·0000032704	$-\frac{1}{2}m''a'^2\frac{dA_0}{da'}$	$-$ ·0000190084
$-\frac{1}{2}m'''a^2\frac{dA_0}{da}$	$-$ ·0000002541	$-\frac{1}{2}m'''a'^2\frac{dA_0}{da'}$	$-$ ·0000011248
	1·0006832177		1·0002951005

[* These are taken from the values of a, a', a'', a''' found on p. 192.]

[† These quantities are taken from Runkle's '*Tables*.']

Also $\quad 1+\frac{1}{2}\frac{M^2}{n^2}=1\cdot0000000834, \qquad 1+\frac{1}{2}\frac{M^2}{n'^2}=1\cdot0000003359;$

hence $\quad n^2a^3=1\cdot0006831343, \qquad n'^2a'^3=1\cdot0002947645.$

These results give

$$\frac{a}{a'}=0\cdot6285146,$$

which agrees pretty closely with the value assumed for α for these two satellites.

Now let us compute the values of a_i, g_i, a_i', g_i' for the values $i=1, 2, 3, 4$.

The equations are

$$[i^2(n-n')^2-N^2]\,a_i=\frac{m'}{a^2}\left[a\frac{dA_i}{da}+\frac{2n}{n-n'}A_i\right],$$

$$g_i=\frac{2n}{i(n-n')}a_i+\frac{1}{i(n-n')^2}\frac{m'}{a^2}A_i,$$

where $\quad N^2=n^2-\frac{3}{2}M^2-2f\frac{\mu}{a^3}-\frac{1}{2}\Sigma m'\left(\frac{d^2A_0}{da^2}+2\frac{1}{a}\frac{dA_0}{da}\right).$

First compute N^2/n^2.

I and II		I and III		I and IV	
$\alpha^2\frac{d^2b_0}{d\alpha^2}$	1·686485	$\alpha^2\frac{d^2b_0}{d\alpha^2}$	0·264603	$\alpha^2\frac{d^2b_0}{d\alpha^2}$	0·0594822
$2\alpha\frac{db_0}{d\alpha}$	1·382614	$2\alpha\frac{db_0}{d\alpha}$	0·375153	$2\alpha\frac{db_0}{d\alpha}$	0·1063486
	3·069099		0·639756		0·1658308
log	0·4870109	log	9·8060144	log	9·2196652
log α	9·7983158	log α	9·5955173	log α	9·3502863
log m'	5·3661520	log m''	5·9469295	log m'''	5·6300117
log $1/n^2a^3$	9·9997034	log $1/n^2a^3$	9·9997034	log $1/n^2a^3$	9·9997034
log $\frac{1}{2}$	9·6989700	log $\frac{1}{2}$	9·6989700	log $\frac{1}{2}$	9·6989700
	5·3501521		5·0471346		3·8986366
	0·000022395		0·000011146		0·000000792

Also

$\log \mu$	0·0000075
$\log 1/n^2a^3$	9·9997034
$\log f$	6·8289501
$\log 2$	·3010300
	7·1296910
	0·001348004

Hence collecting the terms we have

	1
$-\dfrac{3}{2}\dfrac{M^2}{n^2}$	$-0·000000250$
$-2\dfrac{\mu}{n^2a^3}f$	$-0·001348004$
term in m'	$-0·000022395$
,, ,, m''	$-0·000011146$
,, ,, m'''	$-0·000000792$
$\dfrac{N^2}{n^2} =$	0·998617413

Hence we have the following divisors, dividing the equation throughout by n^2:—

$$i=1 \quad D_1 = \left(1-\frac{n'}{n}\right)^2 - \frac{N^2}{n^2} = -0·74679706,$$

$$i=2 \quad D_2 = 4\left(1-\frac{n'}{n}\right)^2 - \frac{N^2}{n^2} = 0·008664072,$$

$$i=3 \quad D_3 = 9\left(1-\frac{n'}{n}\right)^2 - \frac{N^2}{n^2} = 1·2677656,$$

$$i=4 \quad D_4 = 16\left(1-\frac{n'}{n}\right)^2 - \frac{N^2}{n^2} = 3·0305086.$$

Thus completing the calculations

$i=1$

$\frac{2n}{n-n'}(b_1-a)$	0·5013457
$a\frac{db_1}{da}-a$	0·4713909
	0·9727366
log	9·9879953
log a	9·7983158
log m'/n^2a^3	5·3658554
$-\log D_1$	$-$9·8732026n
	5·2789639n

$a_1 = -0·0000190092$

$i=2$

$\frac{2n}{n-n'}b_2$	1·4475767
$a\frac{db_2}{da}$	0·9130762
	2·3606529
log	0·3730321
log a	9·7983158
log m'/n^2a^3	5·3658554
$-\log D_2$	$-$7·9377221
	7·5994812

$a_2 = 0·003976319$

$i=3$

$\frac{2n}{n-n'}b_3$	0·7666132
$a\frac{db_3}{da}$	0·6816803
	1·4482935
log	0·1608565
log a	9·7983158
log m'/n^2a^3	5·3658554
$-\log D_3$	$-$0·1030389
	5·2219888

$a_3 = 0·00001667205$

$i=4$

$\frac{2n}{n-n'}b_4$	0·4244872
$a\frac{db_4}{da}$	0·4859961
	0·9104833
log	9·9592720
log a	9·7983158
log m'/n^2a^3	5·3658554
$-\log D_4$	$-$0·4815155
	4·6419277

$a_4 = 0·00000438458$

Next to find g_i.

$i=1$

$\log \alpha_1$	$5{\cdot}2789639n$
$\log \dfrac{2n}{n-n'}$	$0{\cdot}6004846$
	$5{\cdot}8794485n$
	$-{\cdot}0000757615$
$\log (b_1 - a)$	$9{\cdot}0996527$
$\log a$	$9{\cdot}7983158$
$\log m'/n^2a^3$	$5{\cdot}3658554$
$\log n^2/(n-n')^2$	$0{\cdot}5989092$
	$4{\cdot}8627331$
	$0{\cdot}0000072901$

$$g_1 = -0{\cdot}0000684714$$
$$= -43^{\backprime\backprime}{\cdot}5902$$
$$= -14''{\cdot}1232$$

$i=2$

$\log \alpha_2$	$7{\cdot}5994812$
$\log 2n/2(n-n')$	$0{\cdot}2994546$
	$7{\cdot}8989358$
	$0{\cdot}007923842$
$\log b_2$	$9{\cdot}5601570$
$\log a$	$9{\cdot}7983158$
$\log m'/n^2a^3$	$5{\cdot}3658554$
$\log n^2/2(n-n')^2$	$0{\cdot}2978792$
	$5{\cdot}0222074$
	$0{\cdot}000010525$

$$g_2 = 0{\cdot}007934367$$
$$= 5051^{\backprime\backprime}{\cdot}175$$
$$= 1636''{\cdot}581$$

$i=3$

$\log \alpha_3$	$5{\cdot}2219889$
$\log 2n/3(n-n')$	$0{\cdot}1233633$
	$5{\cdot}3453522$
	$0{\cdot}0000221489$
$\log b_3$	$9{\cdot}2840917$
$\log a$	$9{\cdot}7983158$
$\log m'/n^2a^3$	$5{\cdot}3658554$
$\log n^2/3(n-n')^2$	$0{\cdot}1217879$
	$4{\cdot}5700508$
	$0{\cdot}0000037158$

$$g_3 = 0{\cdot}0000258647$$
$$= 16^{\backprime\backprime}{\cdot}4660$$
$$= 5''{\cdot}3350$$

$i=4$

$\log \alpha_4$	$4{\cdot}6419277$
$\log 2n/4(n-n')$	$9{\cdot}9984246$
	$4{\cdot}6403523$
	$0{\cdot}0000043687$
$\log b_4$	$9{\cdot}027380$
$\log a$	$9{\cdot}7983158$
$\log m'/n^2a^3$	$5{\cdot}3658554$
$\log n^2/4(n-n')^2$	$9{\cdot}9968492$
	$4{\cdot}1884004$
	$0{\cdot}0000015431$

$$g_4 = 0{\cdot}0000059118$$
$$= 3^{\backprime\backprime}{\cdot}5942$$
$$= 1''{\cdot}2194$$

Next consider the perturbations of Satellite II under the influence of Satellite I.

First compute N'^2/n'^2.

II and I		II and III		II and IV	
$a^2\dfrac{d^2b_0}{da^2}$	1·686485	$a^2\dfrac{d^2b_0}{da^2}$	1·663501	$a^2\dfrac{d^2b_0}{da^2}$	0·196132
$2a\dfrac{db_0}{da}$	1·382614	$2a\dfrac{db_0}{da}$	1·370488	$2a\dfrac{db_0}{da}$	0·295914
	3·069099		3·033989		0·492046
log	0·4870109	log	0·4820140	log	9·6920057
$\log m$	5·2387509	$\log a$	9·7972015	$\log a$	9·5519705
$\log 1/n'^2a'^3$	9·9998720	$\log m''$	5·9469295	$\log m'''$	5·6300117
$\log\frac{1}{2}$	9·6989700	$\log 1/n'^2a'^3$	9·9998720	$\log 1/n'^2a'^3$	9·9998720
	5·4246038	$\log\frac{1}{2}$	9·6989700	$\log\frac{1}{2}$	9·6989700
	0·000026583		5·9249870		4·5728299
			0·000084137		·000003740

Also

$\log \mu'f'$	6·4255960
$\log 1/n'^2a'^3$	9·9998720
$\log 2$	0·3010300
	6·7264980
	0·000532719

Collecting the terms,

	1
$-\frac{3}{2}M^2/n'^2$	$-0·000001008$
$-2\mu'f'/n'^2a'^3$	$-0·000532719$
term in m	$-0·000026583$
,, ,, m''	$-0·000084137$
,, ,, m'''	$-0·000003740$
$N'^2/n'^2 =$	0·999351813

Next find the divisors in the final equation for a_i', first dividing throughout by n'^2:—

$$i=1 \quad D_1' = \left(1-\frac{n}{n'}\right)^2 - \frac{N'^2}{n'^2} = 0{\cdot}015290982,$$

$$i=2 \quad D_2' = 4\left(1-\frac{n}{n'}\right)^2 - \frac{N'^2}{n'^2} = 3{\cdot}059219364,$$

$$i=3 \quad D_3' = 9\left(1-\frac{n}{n'}\right)^2 - \frac{N'^2}{n'^2} = 8{\cdot}13243333,$$

$$i=4 \quad D_4' = 16\left(1-\frac{n}{n'}\right)^2 - \frac{N'^2}{n'^2} = 15{\cdot}23493289.$$

Hence to determine a_i':—

$i=1$

$-\frac{n+n'}{n-n'}\left(b_1 - \frac{1}{a^2}\right)$	5·305673
$-\left(a\frac{db_1}{da} + \frac{2}{a^2}\right)$	−6·162795
	−0·857122
log	9·9330426n
$\log m/n'^2a'^3$	5·2386229
$-\log D_1$	−8·1844354
	6·9872301n

$a_1' = -0{\cdot}0009710243$

$i=2$

$-\frac{n+n'}{n-n'}b_2$	−1·0843675
$-a\frac{db_2}{da}$	−0·9130762
	−1·9974437
log	0·3004745n
$\log m/n'^2a'^3$	5·2386229
$-\log D_2$	−0·4856106
	5·0534868n

$a_2' = -0{\cdot}0000113106$

$i=3$

$-\frac{n+n'}{n-n'}b_3$	−0·5742635
$-a\frac{db_3}{da}$	−0·6816803
	−1·2559438
log	0·0989702n
$\log m/n'^2a'^3$	5·2386229
$-\log D_3$	−0·9102205
	4·4273726n

$a_3' = -0{\cdot}0000026753$

$i=4$

$-\frac{n+n'}{n-n'}b_4$	−0·3179798
$-a\frac{db_4}{da}$	−0·4859961
	−0·8039759
log	9·9052430n
$\log m/n'^2a'^3$	5·2386229
$-\log D_4$	−1·1828406
	3·9610253n

$a_4' = -0{\cdot}0000009142$

In conclusion we compute g_i':

$i=1$

$\log a_1'$	$6{\cdot}9872301n$
$\log -2n'/(n-n')$	$0{\cdot}2978735n$
	$7{\cdot}2851036$
	$0{\cdot}001927985$
$\log\left(b_1 - \dfrac{1}{\alpha^2}\right)$	$0{\cdot}2497210n$
$\log m/n'^2a'^3$	$5{\cdot}2386229$
$\log n'^2/(n-n')^2$	$9{\cdot}9936870$
	$5{\cdot}4820309n$
	$-0{\cdot}000030341$

$$g_1' = 0{\cdot}001897644$$
$$= 1208^{\text{``}}{\cdot}078$$
$$= 391''{\cdot}4172$$

$i=2$

$\log a_2'$	$5{\cdot}0534868n$
$\log -2n'/2(n-n')$	$9{\cdot}9968435n$
	$5{\cdot}0503303$
	$0{\cdot}0000112287$
$\log b_2$	$9{\cdot}5601570$
$\log m/n'^2a'^3$	$5{\cdot}2386229$
$\log n'^2/2(n-n')^2$	$9{\cdot}6926570$
	$4{\cdot}4914369$
	$0{\cdot}0000031005$

$$g_2' = 0{\cdot}0000143292$$
$$= 9^{\text{``}}{\cdot}1223$$
$$= 2''{\cdot}9556$$

$i=3$

$\log a_3'$	$4{\cdot}4273726n$
$\log -2n'/3(n-n')$	$9{\cdot}8207522n$
	$4{\cdot}2481248$
	$0{\cdot}0000017706$
$\log b_3$	$9{\cdot}2840917$
$\log m/n'^2a'^3$	$5{\cdot}2386229$
$\log n'^2/3(n-n')^2$	$9{\cdot}5165657$
	$4{\cdot}0392803$
	$0{\cdot}0000010947$

$$g_3' = 0{\cdot}0000028653$$
$$= 1^{\text{``}}{\cdot}8241$$
$$= 0''{\cdot}5910$$

$i=4$

$\log a_4'$	$3{\cdot}9610253n$
$\log -2n'/4(n-n')$	$9{\cdot}6958135n$
	$3{\cdot}6568388$
	$0{\cdot}0000004538$
$\log b_4$	$9{\cdot}027380$
$\log m/n'^2a'^3$	$5{\cdot}2386229$
$\log n'^2/4(n-n')^2$	$9{\cdot}3916270$
	$3{\cdot}6576299$
	$0{\cdot}0000004546$

$$g_4' = 0{\cdot}0000009084$$
$$= 0^{\text{``}}{\cdot}5783$$
$$= 0''{\cdot}1874$$

8.

MASSES OF JUPITER'S SATELLITES CONSISTENT WITH DAMOISEAU'S CONSTANTS.

[IN Damoiseau's "Tables Ecliptiques des Satellites de Jupiter" the following masses are attached to the Satellites, that of Jupiter being unity.

I	0·0000168770
II	0·00002322696
III	0·0000884370
IV	0·0000424751

$$\text{Jupiter's Compression } \frac{1}{13{\cdot}492}.$$

These results as well as others whose determination depends upon theory are based upon constants which Delambre derived from the observations. Damoiseau's own reduction of the observations educed constants sensibly different, and, as he remarks, it is to be expected that the values of the masses, &c., which are consistent with them will also differ sensibly from the above. However he does not consider the degree of exactness to be expected from such tables warrants a new computation. It appears from the following note that his constants imply very considerable departures from the above values.]

The process followed is that of the *Mécanique Céleste*, l. VIII. ch. IX., with certain modifications.

The first equation is derived from the coefficient of the great inequality of I which depends upon the action of II. In his *Introduction*, p. iv, Damoiseau gives this the value $3^{m}\ 13^{s}{\cdot}079 = 0^{d}{\cdot}00223471$. This is the

value employed by Laplace; hence the mass of II which is found thence is the same as that which Laplace gives, or

$$m' = 0{\cdot}232355.$$

We next take the great inequality of II which the actions of I and III jointly produce. Damoiseau's coefficient (p. vi) is

$$15^{\mathrm{m}}\ 6^{\mathrm{s}}{\cdot}331 = 0^{\mathrm{d}}{\cdot}01048995,$$

or in angle $11806''{\cdot}03$. Hence on the model of Laplace's equation (1), we find

$$m + m''1{\cdot}7419336 = 1{\cdot}6983516,$$

and if we take Laplace's value of m'' as an approximation, this equation is satisfied by

$$m = 0{\cdot}1567892, \qquad m'' = 0{\cdot}884972,$$

with corrections to be presently determined

$$\frac{\delta m}{m} = -\frac{\delta m''}{m''} \,.\, 9{\cdot}832069.$$

By p. ii, the annual sidereal motion of the perijove of IV is $41'\ 51''{\cdot}57$ or $7751''{\cdot}759$. We substitute this in the third and fourth equations for g of Ch. VII., and thus form equations analogous to Laplace's (5) and (2) of Ch. IX. The values of the ratios of the eccentricities which refer to the perijove of IV are then evaluated thus:—Damoiseau gives the terms

$$\begin{array}{lrl}
\mathrm{I} & - & 0^{\mathrm{s}}{\cdot}709 \sin(u_1 - \varpi_4), \\
\mathrm{II} & - & 12^{\mathrm{s}}{\cdot}022 \sin(u_2 - \varpi_4), \\
\mathrm{III} & - & 1^{\mathrm{m}}\ 5^{\mathrm{s}}{\cdot}073 \sin(u_3 - \varpi_4), \\
\mathrm{IV} & - & 55^{\mathrm{m}}\ 37^{\mathrm{s}}{\cdot}390 \sin(u_4 - \varpi_4).
\end{array}$$

These are values at conjunction, and represent the sum of two terms

$$-2h \sin(u - \varpi_4) - \frac{15}{4}\frac{M}{n} h \sin(u - 2u_0 + \varpi_4).$$

Hence we find

$$h = 9''{\cdot}28, \quad h' = 78''{\cdot}42, \quad h'' = 210''{\cdot}85, \quad h''' = 4644''{\cdot}74,$$

and $$\frac{h}{h'''} = [7{\cdot}30059], \quad \frac{h'}{h'''} = [8{\cdot}22747], \quad \frac{h''}{h'''} = [8{\cdot}65700].$$

We also take Laplace's m''' as an approximation. We then find the equations between the residuals :—

$$\text{Laplace's (5)},\quad 0 = \quad 804{\cdot}05 - \quad 970{\cdot}68\,\frac{\delta\mu}{\mu} - 31{\cdot}45\,\frac{\delta m}{m} + 1{\cdot}05\,\frac{\delta m''}{m''} + 1590{\cdot}24\,\frac{\delta m'''}{m'''},$$

$$\text{Laplace's (2)},\quad 0 = -298{\cdot}48 - 2963{\cdot}45\,\frac{\delta\mu}{\mu} - 56{\cdot}90\,\frac{\delta m}{m} - 3807{\cdot}27\,\frac{\delta m''}{m''}.$$

Finally the annual sidereal motion of the node of II as derived from Damoiseau's p. iii, is $134207''{\cdot}06$. Another value is given on p. i, namely $12°\ 4'\ 40''{\cdot}4 = 134198''{\cdot}76$, but the former appears to be that employed in constructing the tables. Substitute this in the second equation for p of Ch. VII., and we get an equation corresponding to Laplace's (6) of Ch. IX.; and the quantities l must be evaluated from Damoiseau's coefficients. l is the coefficient of a term in s which refers to the node of II. And disregarding inequalities the semi duration of an eclipse is, by Ch. VIII. $T\sqrt{1-(1+\rho')^2\frac{s^2}{\beta^2}}$, where we may also take $\beta = T \times$ synodic motion. The value of ρ' is given in the same chapter. Hence by comparison with Damoiseau we find $(1+\rho')\frac{s}{\beta} =$ Damoiseau's M; also

	I	II	III	IV
Value of T	$1^h\ 7^m\ 52^s$	$1^h\ 25^m\ 49^s$	$1^h\ 46^m\ 50^s$	$2^h\ 22^m{\cdot}42^s$
Coeff. of Term in M	$-0{\cdot}001097$	$-0{\cdot}083957$	$0{\cdot}004656$	$0{\cdot}000222$

Concerning ρ' there is some doubt as to the value used by Damoiseau as his coefficients are not consistent with each other. We take $1+\rho' = 1{\cdot}0796$, which is consistent with several of his coefficients.

We then find

$$l = -108''{\cdot}235, \qquad l' = -5216''{\cdot}0, \qquad l'' = 178''{\cdot}59, \qquad l''' = 4''{\cdot}865,$$

whence $$\frac{l}{l'} = {\cdot}020750, \quad \frac{l''}{l'} = -{\cdot}034239, \quad \frac{l'''}{l'} = -{\cdot}000933,$$

differing very little from the values in the *Mécanique Céleste.* With these constants the equation for p yields

$$0 = 859{\cdot}8 - 109613{\cdot}2\,\frac{\delta\mu}{\mu} - 4847{\cdot}7\,\frac{\delta m}{m} - 17908{\cdot}8\,\frac{\delta m''}{m''} - 770{\cdot}3\,\frac{\delta m'''}{m'''}.$$

Hence by solving the four equations found

$$\frac{\delta\mu}{\mu} = -0{\cdot}0109117,$$

$$\frac{\delta m}{m} = 0{\cdot}805692,$$

$$\frac{\delta m''}{m''} = -0{\cdot}0819453,$$

$$\frac{\delta m'''}{m'''} = -0{\cdot}496287,$$

and the results will be, restoring Jupiter's mass as the unit;

$$\rho - \frac{1}{2}\phi = 0{\cdot}0216623,$$

$$m = 0{\cdot}0000283113,$$

$$m' = 0{\cdot}0000232355,$$

$$m'' = 0{\cdot}0000812453,$$

$$m''' = 0{\cdot}0000214880.$$

(August 31, 1875.)

9.

REPETITION OF SOME NUMERICAL CALCULATIONS IN THE *MÉCANIQUE CÉLESTE* RELATING TO THE THEORY OF JUPITER'S SATELLITES.

THE calculations in question are those of Livre VIII., Nos. 20, 22, 23. In No. 20 Laplace calculates the ratios of the mean distances of the Satellites by help of a formula of No. 3, but neglects the corrections to those ratios that are produced by the mutual perturbations of the Satellites. Employing the exact formula, with Damoiseau's values of the masses, and $\rho - \frac{1}{2}\phi = 0{\cdot}0220021$, which results from substituting Damoiseau's value of p for Satellite II, together with his values of the masses and the values of $l : l'$, &c. found by Adams from Damoiseau's constants (see p. 188) in the second equation for p of Livre VIII., No. 23, we find

$$a = 5{\cdot}698464,$$

$$a' = 9{\cdot}066548,$$

$$a'' = 14{\cdot}462403,$$

$$a''' = 25{\cdot}437328,$$

where the value found by Laplace is that ascribed to a', for the following reason: that the compression of Jupiter is found from the motion of the node of Satellite II, and the form under which this compression enters the perturbations of that Satellite is $(\rho - \frac{1}{2}\phi)/a'^2$, and this is the only considerable term in which the absolute value of the distance appears (see the formal equation for p in No. 9); hence as one of the four distances is indeterminate by our method, we may save a correction to the equation for p that we employ, by choosing the unit of length so that a' has the value Laplace ascribes to it.

Conducting the remaining calculations of No. 20 with the values of a or $a : a'$ derived from these results, there appears no difference to mention except for Satellites III and IV, where we should have

$$\frac{db_{\frac{1}{2}}^{\circ}}{da} = 0{\cdot}879085, \qquad \frac{db_{\frac{1}{2}}'}{da} = 1{\cdot}546154,$$

in place of $\quad 0{\cdot}878931, \qquad 1{\cdot}545882.$

(12—23 April, 1862.)

In No. 22, Laplace takes $n - 2n' = 3001300''$; employing the mean motions derived from Damoiseau which are given on p. 179, its value is $3001167''{\cdot}4$; further Q'' is erroneous; each term is too great by the factor $1{\cdot}0001686$;

in the second equation for g,	for	143201	read	143401,
	for	-196037	read	-193744,
	for	34596	read	34590,
	for	22518	read	22005,
	for	-94603	read	-94587;
in the third equation for g,	for	28176	read	28171,
	for	-10279	read	-10302;
in the third equation for p,	for	$-15494{\cdot}62$	read	$-15492{\cdot}62$.

If these corrections are made it is verified that the value of g associated with the perijove of Satellite IV, and that of p associated with the node of Satellite II, together with the other constants adopted or derived by Laplace, still satisfy closely the equations for g and p. This amounts to a verification of Laplace's calculation of the masses.

(20—24 August, 1875.)

10.

PERTURBATION OF THE ORBIT OF THE NOVEMBER METEORS.

[In a paper "On the Orbit of the November Meteors" (*Mon. Not.* 1867 April) Adams considers the problem of discriminating between a number of possible periods deduced by Professor H. A. Newton as consistent with the observed recurrences of the display of meteors, by computing the perturbation of the node of the orbit corresponding to each period, and comparing it with the observed perturbation. After excluding a number of orbits by this test, the only remaining and, as it appears, the true orbit is highly elliptical, with eccentricity 0·9047. It remains to compute the perturbation of the node of this orbit, and the method and result of his investigation Adams states as follows:—"In order to determine the secular "motion of the node in this orbit, I employed the method given by Gauss "in his beautiful investigation 'Determinatio attractionis &c.'

"It may be proved that if two planets revolve about the Sun in "periodic times that are incommensurable with each other, the secular "variations which either of these bodies produces in the elements of the "orbit of the other would be the same as if the whole mass of the "disturbing body had been distributed over its orbit in such a manner "that the portion of the mass distributed over any given arc should be "always proportional to the time which the body takes to describe that "arc. In the memoir just referred to, Gauss shews how to determine the "attraction of such an elliptic ring on a point in any given position. "When this attraction has been calculated for any point in the orbit of "the meteors, we can at once deduce the changes which it would produce "in the elements of the orbit, while the meteors are describing any small "arc contiguous to the given point. Hence, by dividing the orbit of the "meteors into a number of small portions, and summing up the changes

"corresponding to these portions, we may find the total secular changes "of the elements produced in a complete period of the meteors.

"In this manner I have found that during a period of 33·25 years, "the longitude of the node is increased 20′ by the action of *Jupiter*, nearly "7′ by the action of *Saturn*, and about 1′ by that of *Uranus*. The other "planets produce scarcely any sensible effects, so that the entire calculated "increase of the longitude of the node in the above-mentioned period is "about 28′.

"As already stated, the observed increase of longitude in the same "time is 29′. This remarkable accordance between the results of theory "and observation appears to me to leave no doubt as to the correctness "of the period of 33·25 years.

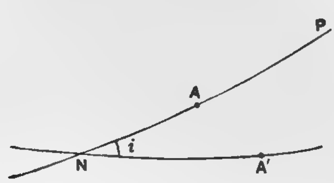
"In order to attain a sufficient degree of approximation it is requisite "to break up the orbit of the meteors into a considerable number of "portions, for each of which the attractions of the elliptic rings corre-"sponding to the several disturbing planets have to be determined; hence "the calculations are necessarily very long, although I have devised a "modification of Gauss's formulae which greatly facilitates their application "to the present problem."

The following extracts from MSS. give a sketch of the mathematical details of this remarkable calculation.]

Let NAP be the orbit of the disturbed body, and NA' that of the disturber;

A, A' are the perihelia;

$NP=\theta$, $NA=\omega$, $NA'=\omega'$, $ANA'=i$;



S the force at P perpendicular to the orbit;

N the longitude of the node;

then as in *Lectures on the Lunar Theory*, xv.,

$$\frac{dN}{dt}=\frac{Sr\sin\theta}{H\sin i};$$

but if u be the eccentric anomaly of the disturbed body, n its mean motion, and a its semi-axis major, and the square of the disturbing force is ignored, we may put

$$\frac{dt}{du} = \frac{1}{n}(1 - e\cos u) = \frac{r}{an},$$

$$H = na^2(1-e^2)^{\frac{1}{2}};$$

or taking the Sun's mass unity, so that

$$n^2a^3 = 1,$$

we get
$$\frac{dN}{du} = \frac{Sr^2}{(1-e^2)^{\frac{1}{2}}}\frac{\sin\theta}{\sin i}.$$

Now the quantities which Gauss has shewn how to compute are the components of the whole attraction at P of the disturbing ring of matter NA', resolved parallel to the major and minor axes of this ring and perpendicular to its plane. Call these components X, Y, Z respectively; then in the above formula

$$S = -X\sin\omega'\sin i - Y\cos\omega'\sin i + Z\cos i.$$

[2 March, 1867.]

Take the origin at the centre of the orbit of the disturbing body and the axes parallel to the above-mentioned directions;

let A, B, C be the coordinates of P;

a', b', e', u' refer with the usual meanings to the disturbing body;

then by Gauss's formulae the elements of X, Y, Z, due to the attraction of that portion of the ring into which the disturbing body is distributed which lies between u' and $u' + du'$, are given by

$$dX = \frac{(A - a'\cos u')(1 - e'\cos u')\,du'}{2\pi D^3},$$

$$dY = \frac{(B - b'\sin u')(1 - e'\cos u')\,du'}{2\pi D^3},$$

$$dZ = \frac{(C \qquad\qquad)(1 - e'\cos u')\,du'}{2\pi D^3},$$

where

$$D^2 = (A - a' \cos u')^2 + (B - b' \sin u')^2 + C^2.$$

If we ignore the square of e', we have

$$a' = b',$$

and

$$D^2 = A^2 + B^2 + C^2 - 2a'(A \cos u' + B \sin u') + a'^2$$
$$= \kappa^2 [1 - 2\alpha \cos(u' - \beta) + \alpha^2],$$

if

$$\kappa^2 (1 + \alpha^2) = A^2 + B^2 + C^2 + a'^2,$$
$$\kappa^2 \alpha \cos \beta = Aa', \qquad \kappa^2 \alpha \sin \beta = Ba'.$$

Hence

$$dX = \frac{A - (Ae' + a') \cos u' + \frac{1}{2} a'e' (1 + \cos 2u')}{\kappa^3 [1 - 2\alpha \cos(u' - \beta) + \alpha^2]^{\frac{3}{2}}} \frac{du'}{2\pi},$$

$$dY = \frac{B - Be' \cos u - b' \sin u' + \frac{1}{2} b'e' \sin 2u'}{\kappa^3 [1 - 2\alpha \cos(u' - \beta) + \alpha^2]^{\frac{3}{2}}} \frac{du'}{2\pi},$$

$$dZ = \frac{C - Ce' \cos u'}{\kappa^3 [1 - 2\alpha \cos(u' - \beta) + \alpha^2]^{\frac{3}{2}}} \frac{du'}{2\pi};$$

and to obtain the complete values of X, Y, Z, these expressions must be integrated between the limits $u' = 0$ and $u' = 2\pi$. Change the variable from u' to ϕ when

$$\phi = u' - \beta;$$

then the limits of ϕ are also 0 and 2π; further write

$$\frac{1}{(1 - 2\alpha \cos \phi + \alpha^2)^{\frac{3}{2}}} = \frac{1}{2} b_0 + b_1 \cos \phi + b_2 \cos 2\phi + \dots,$$

so that

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos n\phi d\phi}{(1 - 2\alpha \cos \phi + \alpha^2)^{\frac{3}{2}}};$$

then noticing that

$$0 = \int_0^{2\pi} \frac{\sin n\phi d\phi}{(1 - 2\alpha \cos \phi + \alpha^2)^{\frac{3}{2}}},$$

we have

$$X=\frac{1}{2\kappa^3}\left[\left(A+\frac{1}{2}a'e'\right)b_0-(Ae'+a')\cos\beta\, b_1+\frac{1}{2}a'e'\cos 2\beta\, b_2\right],$$

$$Y=\frac{1}{2\kappa^3}\left[Bb_0-(Be'\cos\beta+a'\sin\beta)\, b_1+\frac{1}{2}a'e'\sin 2\beta\, b_2\right],$$

$$Z=\frac{1}{2\kappa^3}\left[Cb_0-Ce'\cos\beta\, b_1\right].$$

Now writing, after Gauss,

$$P=\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^2\theta\, d\theta}{(m^2\cos^2\theta+n^2\sin^2\theta)^{\frac{3}{2}}},\qquad Q=\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\theta\, d\theta}{(m^2\cos^2\theta+n^2\sin^2\theta)^{\frac{3}{2}}},$$

put

$$2\theta=\pi+\phi,$$

$$m^2=\kappa^2(1+\alpha)^2,\qquad n^2=\kappa^2(1-\alpha)^2,$$

and we have

$$P=\frac{1}{4\pi\kappa^3}\int_0^{2\pi}\frac{(1-\cos\phi)\, d\phi}{(1-2\alpha\cos\phi+\alpha^2)^{\frac{3}{2}}}=\frac{1}{4\kappa^3}(b_0-b_1),$$

$$Q=\frac{1}{4\pi\kappa^3}\int_0^{2\pi}\frac{(1+\cos\phi)\, d\phi}{(1-2\alpha\cos\phi+\alpha^2)^{\frac{3}{2}}}=\frac{1}{4\kappa^3}(b_0+b_1);$$

moreover we know that

$$b_2-2\left(\alpha+\frac{1}{\alpha}\right)b_1+3b_0=0.$$

Hence

$$\frac{b_0}{\kappa^3}=2(P+Q),$$

$$\frac{b_1}{\kappa^3}=2(Q-P),$$

$$\frac{b_2}{\kappa^3}=4\left(\alpha+\frac{1}{\alpha}\right)(Q-P)-6(P+Q).$$

Now returning to the formula for S,

$$S=-X\sin\omega'\sin i-Y\cos\omega'\sin i+Z\cos i,$$

and remarking that the focus of the orbit of the disturber lies in the

plane of the disturbed orbit, that is in the plane of which the direction cosines of the normal are

$$-\sin\omega'\sin i, \quad -\cos\omega'\sin i, \quad \cos i,$$

we have

$$0 = -(A - a'e')\sin\omega'\sin i - B\cos\omega'\sin i + C\cos i,$$

we get

$$S = a'\sin(\omega'+\beta)\sin i\,\frac{b_1}{2\kappa^3} - \frac{3}{2}a'e'\sin\omega'\sin i\,\frac{b_0}{2\kappa^3} - \frac{1}{2}a'e'\sin(\omega'+2\beta)\sin i\,\frac{b_2}{2\kappa^3}$$

$$= a'\sin(\omega'+\beta)\sin i\,(Q-P) + 3a'e'\sin\beta\cos(\omega'+\beta)(P+Q)$$

$$- a'e'\sin(\omega'+2\beta)\sin i\left(a+\frac{1}{a}\right)(Q-P).$$

[13 March, 1867.]

We substitute this value of S in the right-hand member of the equation for dN/du. The right-hand member is thus expressed as a function of known quantities and the co-ordinates of the disturbed body. Taking n the eccentric anomaly in the disturbed orbit to define what is variable in these co-ordinates, we compute the value of dN/du for a sufficient number of different values of u. Thus for instance we find

Perturbations by Jupiter.

u	$\frac{dN}{du}$	u	$\frac{dN}{du}$	u	$\frac{dN}{du}$
	$''$		$''$		$''$
0°	0·04	90°	11·90	270°	31·54
$22\frac{1}{2}$	9·97	$112\frac{1}{2}$	1·62	$281\frac{1}{4}$	79·5
$33\frac{3}{4}$	34·3	135	0·76	$292\frac{1}{2}$	263·59
45	126·39	$157\frac{1}{2}$	0·00	$298\frac{1}{8}$	495·32
$50\frac{5}{8}$	286·83	180	0·20	$303\frac{3}{4}$	642·1
$56\frac{1}{4}$	698·4	$202\frac{1}{2}$	0·65	$309\frac{3}{8}$	414·04
$61\frac{7}{8}$	634·50	225	2·43	315	194·14
$67\frac{1}{2}$	223·60	$247\frac{1}{2}$	7·68	$326\frac{1}{4}$	45·1
$78\frac{3}{4}$	40·5	270	31·54	$337\frac{1}{2}$	9·93
90	11·90			360	0·04

To deduce from these figures the change in N while u ranges from

0° to 360° is a problem of Mechanical Quadrature; the following are the results.

Range of u	Change in N	Range of u	Change in N
	$''$		$''$
0° to $33\frac{3}{4}$°	11·23	0° to $202\frac{1}{2}$°	566·00
$33\frac{3}{4}$ ... $50\frac{5}{8}$	82·62	$202\frac{1}{2}$ ... $247\frac{1}{2}$	6·02
$50\frac{5}{8}$ ... $61\frac{7}{8}$	306·34	$247\frac{1}{2}$ ... $281\frac{1}{4}$	38·36
$61\frac{7}{8}$ ... $78\frac{3}{4}$	149·78	$281\frac{1}{4}$ ... $298\frac{1}{8}$	169·78
$78\frac{3}{4}$ ... $112\frac{1}{2}$	14·00	$298\frac{1}{8}$ ... $309\frac{3}{8}$	286·23
$112\frac{1}{2}$ ... $157\frac{1}{2}$	1·55	$309\frac{3}{8}$ ... $326\frac{1}{4}$	123·15
$157\frac{1}{2}$ ... $202\frac{1}{2}$	0·48	$326\frac{1}{4}$ ... 360	11·19
0° to $202\frac{1}{2}$°	566·00	0° to 360°	1200·73

Thus the change in the longitude of the node produced by the action of Jupiter during one complete revolution of the meteors is 20′.

11.

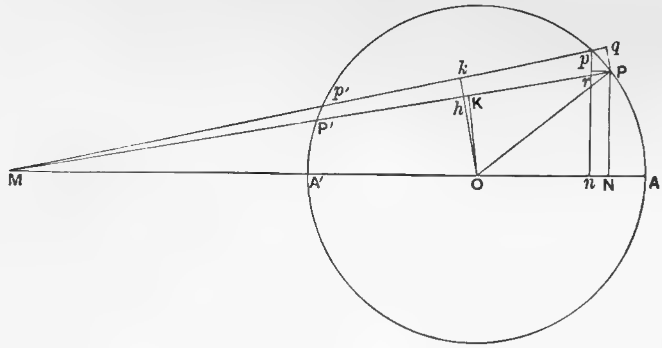
THEORY OF THE FIGURE OF THE EARTH.

[LECTURES on the Theory of the Figure of the Earth were delivered by Adams in the Lent Term of 1871 and were repeated once or twice with little alteration. They were prefaced by a historical sketch of the problem, translated from Laplace (*Mécanique Céleste,* livre XI. ch. 1), continued by Chasles (*Recueil des Savans Etrangers,* t. IX.). It does not seem proper to reprint this matter here, and it has been omitted. The concluding lectures have also been omitted; these dealt with attractions of ellipsoids and ellipsoidal shells; the greater part has since become familiar in current text-books, and what was most interesting and original in it was published by Adams himself (*Camb. Phil. Soc. Proc.,* II. p. 213), and has been reprinted in the first volume of his papers, No. 53, p. 414. The form of lectures, never very marked, has not been preserved in the remainder. It divides roughly into three portions. §§ 1—5 make an informal commentary upon certain propositions in Newton's *Principia,* lib. I. sect. xii. xiii. It should be remarked that these propositions by no means represent all that Newton contributed to the theory of the Figure of the Earth. §§ 6—10 give a characteristic discussion of the potential and attraction of a spheroid of small ellipticity on any point. §§ 11—15 demonstrate Clairaut's theorem, deducing it from hypotheses as to the internal strata of equal density, and these hypotheses are further considered in a theory of the internal state of a fluid earth of heterogeneous structure, supposed to rotate as if solid.]

ATTRACTION OF SPHERES AND SPHEROIDS BY METHODS ANALOGOUS TO NEWTON'S.

§ 1. *Attraction of a uniform spherical shell.*

Let O be the centre of a spherical shell, a its radius; M an attracted point, $MO = r$.



Join MO meeting the surface in A, A'. Through M draw any line MP cutting the surface in P, P', and let Mp be drawn in the plane OMP making an indefinitely small angle with MP and cutting the surface in p, p'.

Also draw OK, Ok perpendicular to MP, Mp respectively; PN, pn perpendicular to MO, Pq perpendicular to Mp and Pr perpendicular to pn; let Ok meet MP in h.

The surface of the elementary zone of the shell generated by revolution of the arc Pp about MO is

$$2\pi \,.\, PN \,.\, Pp;$$

but by similar triangles $\quad Pp : Pr = OP : PN.$

Therefore $$PN \,.\, Pp = OP \,.\, Pr = OP \,.\, Nn,$$

and the surface of the elementary zone is

$$2\pi \,.\, OP \,.\, Nn = 2\pi a \,.\, Nn,$$

and the surface of the whole sphere, obtained by summing all such elements,

$$2\pi a \,.\, 2a = 4\pi a^2.$$

The zone in question evidently attracts M in the direction MO, and since each element of the zone attracts M in a direction making the same angle with MO, we have, resolving the attractions in this direction and measuring the mass of the zone by its surface,

$$\text{Attraction of zone} = 2\pi \,.\, PN \,.\, Pp \,.\, \frac{MN}{MP^3}.$$

But by similar triangles

$$PN : MP = OK : MO, \quad MN : MP = MK : MO.$$

Therefore $$\text{Attraction} = 2\pi \frac{OK \,.\, MK}{MO^2} \,.\, \frac{Pp}{MP}.$$

Also by similar triangles

$$Pp : Pq = OP : PK \text{ and } Pq : hk = MP : MK \text{ ultimately.}$$

Hence $$\frac{Pp}{hk} = \frac{OP \,.\, MP}{PK \,.\, MK} \text{ and } \frac{Pp}{MP} = \frac{OP \,.\, hk}{PK \,.\, MK};$$

and $$\text{Attraction of zone} = 2\pi \frac{OP \,.\, OK}{MO^2} \,.\, \frac{hk}{PK}.$$

But $$Ok^2 - OK^2 = PK^2 - pk^2;$$

therefore ultimately $$hk \,.\, OK = (PK - pk)\, PK.$$

Hence the attraction of the zone is

$$2\pi \frac{OP}{MO^2}(PK - pk) = -\frac{2\pi a}{r} \,.\, \delta PK.$$

Similarly the attraction of the zone generated by revolution of $P'p'$ may be shewn equal to the same quantity, and

$$\text{Attraction of the two zones} = -\frac{2\pi a}{r^2} . \delta PP'.$$

Now if we sum the attractions of all the elements that make up the entire shell, since PP' varies from $2a$ when MP passes through O to zero when MP becomes a tangent, we have

$$\text{Attraction of spherical shell} = \frac{2\pi a}{r^2} . 2a = \frac{4\pi a^2}{r^2} = \frac{M}{r^2},$$

since the mass is supposed to be measured by its surface.

Again, the potential at M of the zone generated by Pp

$$= 2\pi \frac{PN}{MP} Pp = 2\pi \frac{OK . OP . MP}{MO . PK . MK} . hk;$$

and the potential of the zone generated by $P'p'$

$$= 2\pi \frac{OK . OP . MP'}{MO . PK . MK} . hk.$$

Together, the potential of the two zones is

$$2\pi \frac{OK . OP (MP + MP')}{MO . PK . MK} . hk = 4\pi \frac{OP . OK}{MO . PK} . hk,$$

or, as above, $$= -\frac{2\pi a}{r} . \delta PP',$$

and the potential of the whole shell $= \dfrac{2\pi a}{r} . 2a = \dfrac{M}{r}$.

In the same way we may prove that the attraction at an internal point is zero and the potential constant*.

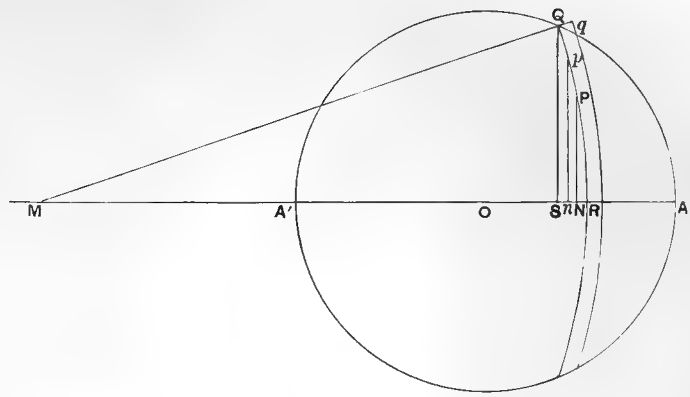
§ 2. *Attraction of a sphere, supposing the law of attraction to be any whatever.*

Let O be the centre of the sphere, a its radius, M an attracted point, $OM = r$, and let the attraction at distance ρ be denoted by $f(\rho)$.

Let a sphere described with centre M and any radius $MQ = \rho$ cut the attracting sphere in the spherical segment which is generated by revolution of the arc QR about MO. Let Pp be an element of the arc RQ and imagine an indefinitely thin shell of the attracting matter included between

* Compare Newton's *Principia*, Book I. Prop. LXXI.

two spherical surfaces with the common centre M, their radii being ρ and $\rho+\delta\rho$.



Then the elementary surface generated by Pp

$$=2\pi PN\,.\,Pp=2\pi\rho\,.\,Nn,$$

and the volume corresponding to this surface contained between the surfaces of the shell $=2\pi\rho\delta\rho\,.\,Nn.$

The attraction of this matter upon M

$$=2\pi\rho\delta\rho\,.\,Nn\,.f(\rho)\,MN/\rho=\pi f(\rho)\,\delta\rho\,(MN^2-Mn^2)=\pi f(\rho)\,\delta\rho\,.\,\delta\,(MN^2).$$

Hence if QS be drawn perpendicular to MO, the attraction of the whole shell whose internal and external radii are ρ and $\rho+\delta\rho$ is

$$\pi f(\rho)\,\delta\rho\,(MR^2-MS^2)=\pi f(\rho)\,\delta\rho\,.\,QS^2\,*.$$

Now in the triangle MQO, since QS is drawn perpendicular to MO, we have

$$MS=\frac{\rho^2+r^2-a^2}{2r};$$

hence

$$QS^2=\rho^2-\left(\frac{\rho^2+r^2-a^2}{2r}\right)^2$$

$$=\frac{[(\rho+r)^2-a^2]\,[a^2-(\rho-r)^2]}{4r^2}$$

$$=\frac{(\rho+r+a)\,(\rho+r-a)\,(a+\rho-r)\,(a+r-\rho)}{4r^2},$$

and the attraction of the whole sphere

$$=\frac{\pi}{4r^2}\int_{r-a}^{r+a}f(\rho)\,(\rho+r+a)\,(\rho+r-a)\,(\rho+a-r)\,(a+r-\rho)\,d\rho$$

$$=\frac{\pi}{4r^2}\int_{r-a}^{r+a}f(\rho)\,[(r+a)^2-\rho^2]\,[\rho^2-(r-a)^2]\,d\rho.$$

* This is equivalent to Newton's Prop. LXXIX.

The value of the integral in this expression may be written down in several cases.

Ex. 1. $f(\rho)=1/\rho$. Integral $=2ar(r^2+a^2)-(r^2-a^2)^2\log_e\dfrac{r+a}{r-a}$.

2. $f(\rho)=1/\rho^2$. Integral $=\dfrac{16}{3}a^3$.

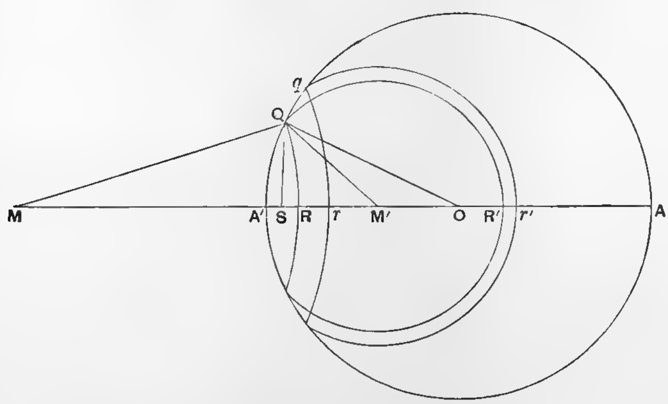
3. $f(\rho)=1/\rho^3$. Integral $=-4ra+2(r^2+a^2)\log_e\dfrac{r+a}{r-a}$.

4. $f(\rho)=1/\rho^4$. Integral $=\dfrac{16}{3}\dfrac{a^3}{r^2-a^2}$.

§ 3. *Comparison of the attractions of a solid sphere upon any external point and a related internal point.*

Let M be the external point and O the centre of the sphere. In OM take M' so that OM, OA, OM' are in continued proportion; thus if

$$OM'=r',\quad r'=a^2/r.$$



Let Q be any point of the circle with centre O and radius OA, and let Qq be an indefinitely small arc of this circle. Join MQ, $M'Q$.

Then since
$$M'O : OQ = OQ : OM,$$
the triangles $M'OQ$, QOM are similar, and we have
$$M'Q : MQ = OQ : OM;$$
so that $M'Q$ is to MQ in the constant ratio $OQ : OM$ or $a : r$.

Let shells be described in the sphere with centres M, M' and internal and external radii respectively MQ, Mq, $M'Q$, $M'q$. Call MQ, Mq, ρ, $\rho+\delta\rho$ respectively, and $M'Q$, $M'q$, ρ', $\rho'+\delta\rho'$. Then if we take the attraction of a particle to vary inversely as the nth power of the distance, we have

$$\text{Attraction on } M \text{ of shell with centre } M=\frac{\pi}{\rho^n}\delta\rho\,.\,QS^2,$$

and again, Attraction on M' of shell with centre $M' = \frac{\pi}{\rho'^n}\,\delta\rho' \,.\, QS^2$.

But we have seen $\quad \frac{\rho'}{\rho} = \frac{\rho' + \delta\rho'}{\rho + \delta\rho} = \frac{\delta\rho'}{\delta\rho} = \frac{a}{r} = \frac{r'}{a} = \sqrt{\frac{r'}{r}}$.

Hence

Attraction of former shell on M : attraction of latter shell on M'

$$= \left(\frac{a}{r}\right)^n \,.\, \frac{r}{a} : 1$$

$$= (r'/r)^{\frac{n-1}{2}} : 1,$$

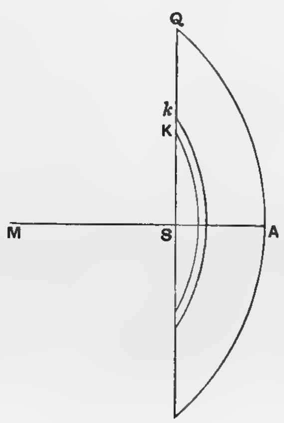
and it is immediately clear that the attractions of the whole sphere are in the same ratio*.

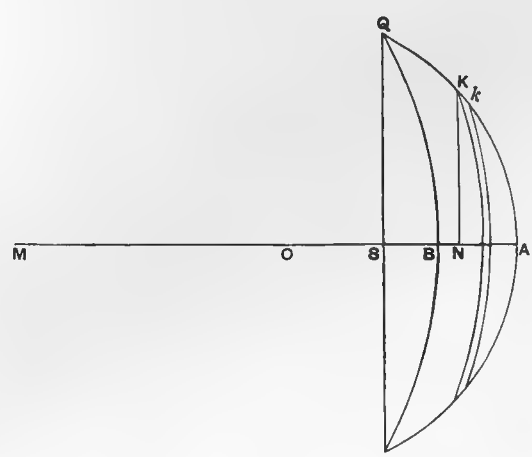
§ 4. *Attraction of a segment of a sphere upon a particle in the axis of the segment.*

First†, let the particle M lie at the centre of the sphere, and let a shell be described in the segment, M being the common centre of its internal and external surfaces, and MK, Mk the radii of these surfaces; call MK, Mk respectively ρ, $\rho + \delta\rho$. Then the attraction of this shell upon M

$$= \pi f(\rho)\,\delta\rho \,.\, KS^2 = \pi f(\rho)\,\delta\rho\,(\rho^2 - c^2),$$

where c stands for MS; and the value of this integrated from $\rho = c$ to $\rho = r$ gives the attraction of the whole segment.





Secondly‡, let the point M lie anywhere on the axis.

With M as centre describe a spherical segment QB. The attraction of such a segment we have just found. To find the attraction of the

* Compare *Principia*, Prop. LXXXII.

† Compare *Principia*, Prop. LXXXIII.

‡ Compare *Principia*, Prop. LXXXIV.

remainder, divide it into shells with centre M. The attraction of a shell with inner and outer radii MK, Mk, say ρ, $\rho+\delta\rho$, is

$$\pi f(\rho)\,\delta\rho\,.\,KN^2,$$

where KN is perpendicular to MO; that is

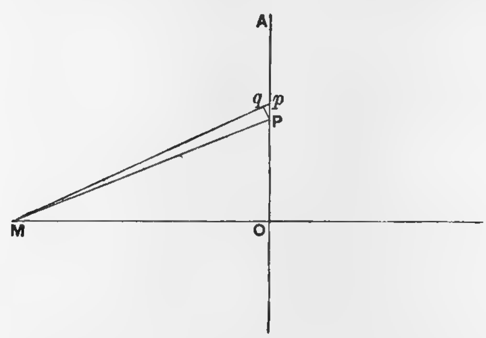
$$\pi f(\rho)\,\delta\rho\,.\,\frac{1}{4r^2}[(r+a)^2-\rho^2][\rho^2-(r-a)^2],$$

in which r stands for MO.

This expression summed from $\rho=(r^2+a^2+2rc)^{\frac{1}{2}}$ to $\rho=r+a$.

§ 5. *Attraction of a solid of revolution upon any point in its axis.*

Let us find the attraction of a circular lamina on a point situated anywhere on a straight line through its centre perpendicular to its plane.



Let M be the point, O the centre of the circle, $MO=a$, $AO=r$. Let Pp be an element of AO, $MP=\rho$, and let the attraction at distance ρ be $f(\rho)$.

Join MP, Mp; draw Pq perpendicular to Mp.

The area generated by revolution of Pp about O is

$$2\pi\,.\,OP\,.\,Pp.$$

But by similar triangles $\quad Pp : pq = MP : PO$;

therefore $\quad OP\,.\,Pp = MP\,.\,pq$,

and the area in question is $\quad 2\pi MP\,.\,pq$.

The attraction of this elementary area upon M is in the direction MO and is equal to

$$2\pi\,.\,MP\,.\,pq\,.f(MP)\,.\,\frac{MO}{MP}=2\pi af(\rho)\,\delta\rho\,;$$

and the attraction of the whole circular lamina is the integral of this taken from $\rho=a$ to $\rho=(r^2+a^2)^{\frac{1}{2}}$.

If $f(\rho)=\frac{1}{\rho^2}$, the expression for the attraction of the circular lamina is

$$2\pi a\left[1-\frac{a}{(a^2+r^2)^{\frac{1}{2}}}\right].$$

To deduce the attraction of a solid of revolution divide it into plates by planes perpendicular to its axis. The attraction of any plate is found as above, and the sum of the attractions for all the plates gives the attraction of the whole solid.

1. Let the body be a cylinder of radius a; let b, c be the distances of the attracted point from its two ends. Then if $f(\rho)=1/\rho^2$, the attraction of the whole cylinder is

$$2\pi\left[c-b-(c^2+a^2)^{\frac{1}{2}}+(b^2+a^2)^{\frac{1}{2}}\right].$$

2. Let the body be a spheroid, its equatorial and polar axes being respectively a and c, and the distance of the attracted point from the centre being r; the attraction of a slice at distance x from the centre is

$$2\pi\,.\left\{1-\frac{r+x}{\sqrt{(r+x)^2+\frac{c^2}{a^2}(a^2-x^2)}}\right\}dx.$$

The integral is to be taken from $x=-c$ to $x=c$; its expression requires circular or logarithmic functions according as c is less or greater than a; in the former case, the oblate spheroid, we find the attraction

$$2\pi\left\{\frac{2a^2c}{a^2-c^2}-\frac{ra^2c}{(a^2-c^2)^{\frac{3}{2}}}(\phi-\phi')\right\},$$

where $$\sin\phi=\frac{rc+a^2-c^2}{a\sqrt{r^2+a^2-c^2}},\quad \sin\phi'=\frac{rc-a^2+c^2}{a\sqrt{r^2+a^2-c^2}}.$$

It may be verified that this agrees with the result of Newton's Prop. XCI. Cor. 2, when his conic KRM is an ellipse.

§ 6. *Attraction and potential of a spheroid differing little from a sphere, upon a point situated upon the axis of revolution.*

Let M be the point; draw MP intersecting the inscribed sphere in P; let $MP=\rho$, and take ρ as independent variable; draw Mp a consecutive, $P'PN$ perpendicular to the axis, OPL passing through O the centre, OK, pq perpendicular to MP. Let $OP=c$, $OM=r$.

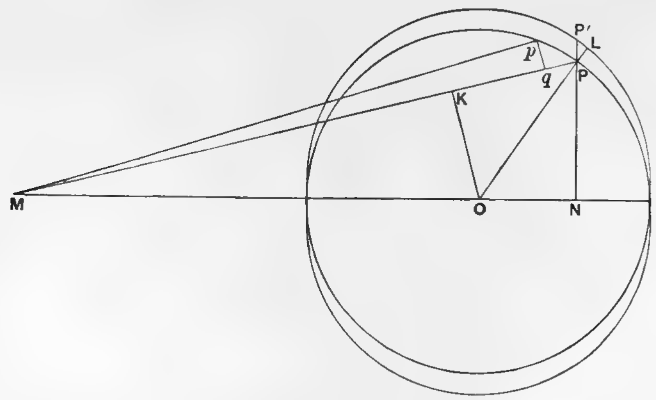
The surface of the elementary zone generated by revolution of Pp about the axis is $$2\pi\,.\,PN\,.\,Pp;$$

but $$Pp : Pq :: OP : OK,$$

and $$PN : OK :: MP : MO;$$

thus the elementary surface is

$$2\pi\, PN \cdot \frac{OP}{OK} \cdot Pq = 2\pi \frac{OP}{MO} \cdot MP \cdot Pq = 2\pi \frac{c}{r} \rho\, d\rho.$$



Now the thickness of the shell at P is PL.

Thus if a be the equatorial axis of the spheroid and $a = c(1+\epsilon)$, then $NP' = NP(1+\epsilon)$, or $PP' = \epsilon \,.\, NP$; so that

$$PL = \frac{PN}{c} \,.\, PP' = \frac{\epsilon}{c} PN^2,$$

but
$$PN^2 = PO^2 - ON^2$$
$$= c^2 - \left(\frac{\rho^2 - r^2 - c^2}{2r}\right)^2 = \frac{1}{4r^2} \{(r+c)^2 - \rho^2\}\{\rho^2 - (r-c)^2\}.$$

Hence the volume of the elementary zone of the shell is

$$\frac{\pi\epsilon}{2r^3} \rho \{(r+c)^2 - \rho^2\}\{\rho^2 - (r-c)^2\}\, d\rho.$$

To find the potential, divide by ρ, and integrate from $\rho = r - c$ to $\rho = r + c$. This gives

$$V = \frac{8}{15} \pi\epsilon c^3 \left(\frac{5}{r} - \frac{c^2}{r^3}\right),$$

and the attraction,
$$-\frac{dV}{dr} = \frac{8}{15} \pi\epsilon c^3 \left(\frac{5}{r^2} - \frac{3c^2}{r^4}\right),$$

consisting of two parts, one varying inversely as the square of the distance, and the other inversely as the fourth power.

The method of § 1 may also be adapted to find this quantity.

To find the potential of the whole spheroid add for the inscribed sphere the quantity

$$\frac{4}{3} \pi c^3 \frac{1}{r}.$$

Now the volume of the spheroid $=\frac{4}{3}\pi a^2 c$; call this M. Hence the potential of the whole spheroid at any point upon its axis is

$$V=\frac{M}{r}-\frac{8}{15}\pi\epsilon\frac{c^5}{r^3}.$$

§ 7. *Attraction and potential of the same at any point in the equatorial plane.*

We can find this without further integration. Let the equation of the spheroid be

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1,$$

where the axis of revolution is that of z, and the point considered lies in the axis of x. Put as before

$$a=c(1+\epsilon),$$

where ϵ is supposed small; then

$$r=a-\frac{\epsilon}{a}z^2,$$

so that the thickness at any point, measured from the circumscribed sphere, is $\frac{\epsilon}{a}z^2$. Let V be the potential of this shell at a point on the axis of x; then it is clear that V is also the value of the potential at the same point of the shell

$$\frac{x^2+z^2}{a^2}+\frac{y^2}{c^2}=1,$$

which is the same shell turned through a right angle about Ox. Hence the potential of these two shells together is $2V$.

But together these shells make up a shell of thickness $\frac{\epsilon}{a}(y^2+z^2)$, that is to say a shell having yz for its equatorial plane; this is the case we have already discussed, the polar radius being a, and the equatorial radius $a(1-\epsilon)$. Hence

$$2V=\frac{8}{3}\pi\epsilon\frac{a^3}{r}-\frac{8}{15}\pi\epsilon\frac{a^5}{r^3}, \text{ or } V=\frac{4}{3}\pi\epsilon\frac{a^3}{r}-\frac{4}{15}\pi\epsilon\frac{a^5}{r^3},$$

which, subtracted from the potential of the circumscribed sphere $\left(\frac{4}{3}\pi\frac{a^3}{r}\right)$, gives for the potential of the spheroid

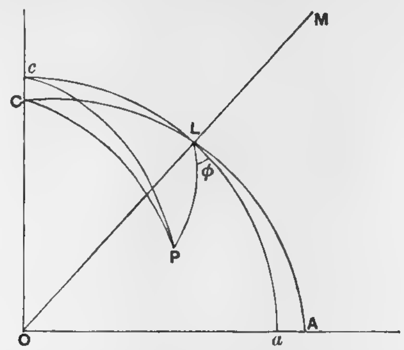
$$\frac{4}{3}\pi a^3(1-\epsilon)\frac{1}{r}+\frac{4}{15}\pi\epsilon\frac{a^5}{r^3};$$

the volume of the spheroid, $M=\frac{4}{3}\pi a^3(1-\epsilon)$; hence

$$V=\frac{M}{r}+\frac{4}{15}\pi\epsilon\frac{a^5}{r^3}.$$

§ 8. We will now proceed to the more general case of a point situated anywhere with respect to the spheroid.

Let the line OM intersect the surface of the spheroid in L, and describe a sphere with centre O and radius OL. This sphere will in general



lie partly within and partly without the spheroid; we shall take the thickness of the shell between them as negative in one case and positive in the other. Let CLA be the meridian through L, and P any point on the surface of the sphere. Join PL with an arc of a great circle, and call the angle PLa between PL and the meridian of the sphere, ϕ.

Also let $OA=a=c(1+\epsilon)$, $OM=r$, $\cos CL=\lambda$, $\cos LP=\mu$.

Then $$\cos CP=\cos CL\cos LP-\sin CL\sin LP\cos\phi$$
$$=\lambda\mu-(1-\lambda^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\cos\phi.$$

At the point L the radius vector of the spheroid exceeds c by

$$\epsilon c\sin^2 CL=\epsilon c(1-\lambda^2).$$

Similarly at P the radius vector of the spheroid exceeds c by $\epsilon c\sin^2 CP=\epsilon c(1-\cos^2 CP)$. Hence the thickness between the sphere ca and the spheroid is, at P,

$$\epsilon c(\lambda^2-\cos^2 CP)$$
$$=\epsilon c\{\lambda^2(1-\mu^2)+2\lambda\mu(1-\lambda^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\cos\phi-(1-\lambda^2)(1-\mu^2)\cos^2\phi\}.$$

It is to be remarked that all the elements of the shell for which LP or μ remain the same may be considered to be at equal distances from M, and the directions of the attractions make equal angles with MO. Consequently the potential of the shell at M and its attraction in the direction MO will not be affected if we replace any two elements

corresponding to equal elements of the spherical surface by two other elements the thickness of each of which is equal to the mean of the thickness of the two former.

If ϕ be increased by π all the terms are the same as before except the term involving $\cos\phi$; therefore the potential at M and the attraction in OM will not be altered if we suppose the thickness of the shell at any point P to be the mean of the two, or

$$\epsilon c\{\lambda^2(1-\mu^2)-(1-\lambda^2)(1-\mu^2)\cos^2\phi\}.$$

But if ϕ be increased by $\frac{\pi}{2}$, the thickness of the shell at each of these points will now be represented by

$$\epsilon c\{\lambda^2(1-\mu^2)-(1-\lambda^2)(1-\mu^2)\sin^2\phi\};$$

we may suppose the thickness at the four points referred to, to be equal to the mean of these two quantities, that is to

$$\epsilon c\left\{\lambda^2(1-\mu^2)-\frac{1}{2}(1-\lambda^2)(1-\mu^2)\right\}$$
$$=\epsilon c(1-\mu^2)\left\{\frac{3}{2}\lambda^2-\frac{1}{2}\right\}.$$

We have thus brought down the shell to one of uniform thickness, and the same law applies as in the case of a shell included between a spheroid of revolution and a sphere touching it at the poles, the quantity ϵ being replaced by

$$\epsilon\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right).$$

In the term multiplied by ϵ it matters not whether we take c or OL. Thus we have

$$V=\frac{M}{r}-\frac{8}{15}\pi\epsilon\frac{c^5}{r^3}\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right),$$

where M is the volume either of the original spheroid or of its substitute.

The attraction in the direction MO is

$$-\frac{dV}{dr}=\frac{M}{r^2}-\frac{8}{5}\pi\epsilon\frac{c^5}{r^4}\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right),$$

and perpendicular to MO it is $-dV/ds$, where it is easily seen

$$d\lambda=-\sin CM\,ds/r,$$

so that it is
$$-\frac{dV}{ds}=\frac{8}{5}\pi\epsilon\frac{c^5}{r^4}\lambda(1-\lambda^2)^{\frac{1}{2}},$$

or it varies as the sine of twice the latitude and is positive towards the equator.

§ 9. From these expressions we can deduce the potential or attraction exerted at any point within the spheroid.

First consider the solid sphere of radius $OL = r'$. The potential of this at any point M $(OM = r)$ may be divided into two parts, first, the consecutive shell outside M, and second, the sphere of radius r within M. The potential of the latter is the same as if the whole were concentrated in O, that of the former is the same at M as it is at O; hence the potential of the sphere at M is

$$\frac{4}{3}\pi r^2 + 2\pi(r'^2 - r^2) = 2\pi\left(r'^2 - \frac{1}{3}r^2\right).$$

Next consider the spheroidal shell surrounding the inscribed sphere.

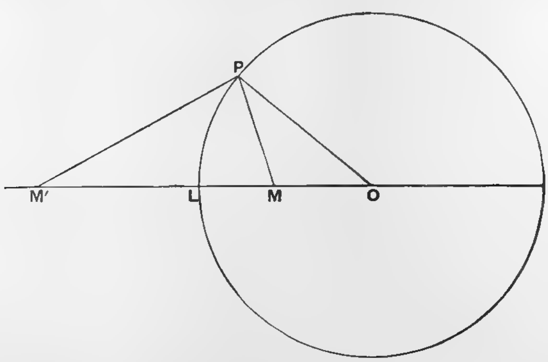
Employ the construction of § 3, and take a point M' in OM produced such that

$$OM : OL :: OL : OM';$$

then $$MP : M'P :: r : r',$$

and for a particle at P,

$$\text{Potential at } M : \text{Potential at } M' :: r' : r,$$



therefore it is independent of the position of P, and the potential of the whole shell at M may be found by multiplying the potential at M' by r'/r. But if L be the point the sine of whose latitude is λ, the potential of the spheroidal shell at M' is

$$\left\{\frac{8}{3}\pi r'^3 \frac{r}{r'^2} - \frac{8}{15}\pi r'^5 \left(\frac{r}{r'^2}\right)^3\right\} \epsilon \left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right) = \left\{\frac{8}{3}\pi r r' - \frac{8}{15}\pi \frac{r^3}{r'}\right\} \epsilon \left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right).$$

Multiply this by r'/r, and we have the potential of the shell at M

$$\left\{\frac{8}{3}\pi r'^2 - \frac{8}{15}\pi r^2\right\} \epsilon \left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right);$$

add the potential of the sphere of radius r', and we have the potential at M of the solid spheroid

$$V = 2\pi\left(r'^2 - \frac{1}{3}r^2\right) + \left\{\frac{8}{3}\pi r'^2 - \frac{8}{15}\pi r^2\right\} \epsilon \left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right).$$

The attraction in the direction MO is

$$-\frac{dV}{dr}=\frac{4}{3}\pi r+\frac{16}{15}\pi r\epsilon\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right),$$

from which r' has disappeared, so that the attraction is independent of the dimensions of the spheroid, and varies directly as the distance MO.

It will be convenient to eliminate r', which varies with λ, from the expression for V. We have seen in § 6,

$$r'=c+\frac{\epsilon}{c}PN^2 \text{ where } PN=c\left(1-\lambda^2\right)^{\frac{1}{2}},$$

or if we introduce a quantity κ, the radius of a sphere of volume equal to the spheroid, so that $\kappa=c\left(1-\frac{2}{3}\epsilon\right)$, then

$$r'=c\left\{1+\epsilon\left(1-\lambda^2\right)\right\}=\kappa\left(1-\frac{2}{3}\epsilon\right)\left\{1+\epsilon\left(1-\lambda^2\right)\right\}=\kappa\left\{1+\epsilon\left(\frac{1}{3}-\lambda^2\right)\right\}.$$

Substitute above for r', and collect the terms in ϵ; we find

$$V=2\pi\left(\kappa^2-\frac{1}{3}r^2\right)-\frac{8}{15}\pi r^2\epsilon\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right).$$

The potential at an external point expressed in a parallel manner is

$$V=\frac{4}{3}\pi\frac{\kappa^3}{r}-\frac{8}{15}\pi\frac{\kappa^5}{r^3}\epsilon\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right).$$

We have supposed the density uniform and have taken it as unit. If it be called ρ, these expressions must be multiplied throughout by ρ.

§ 10. Let us now consider the case of a heterogeneous spheroid, the surfaces of equal density being also spheroidal, concentric and coaxial with the external surface, but not necessarily similar to it.

Take a shell of homogeneous matter of density ρ contained between two spheroidal surfaces; let the semiaxes of the interior surface be c, $c(1+\epsilon)$, and the semiaxes of the exterior surface c', $c'(1+\epsilon')$; also let κ, κ' be the radii of the spherical surfaces enclosing volumes equal to the spheroidal. Then for an attracted particle inside this shell

$$V=2\pi\rho\left(\kappa'^2-\kappa^2\right)-\frac{8}{15}\pi\rho r^2\left(\epsilon'-\epsilon\right)\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right),$$

and for an attracted particle outside

$$V=\frac{4}{3}\pi\rho\left(\kappa'^3-\kappa^3\right)\frac{1}{r}-\frac{8}{15}\pi\rho\left(\kappa'^5\epsilon'-\kappa^5\epsilon\right)\frac{1}{r^3}\left(\frac{3}{2}\lambda^2-\frac{1}{2}\right).$$

These expressions hold whether the shell is indefinitely thin or of finite thickness.

In the former case, let $\kappa' = \kappa + \delta\kappa$; then for a point within the shell

$$\delta V = 4\pi\rho\kappa\delta\kappa - \frac{8}{15}\pi\rho r^2 \frac{d\epsilon}{d\kappa}\delta\kappa\left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right);$$

for a point outside

$$\delta V = 4\pi\rho\kappa^2\delta\kappa\frac{1}{r} - \frac{8}{15}\pi\rho\frac{d(\kappa^5\epsilon)}{d\kappa}\delta\kappa\frac{1}{r^3}\left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right).$$

If κ_0, ϵ_0 refer to the stratum passing through the attracted particle we may express r thus

$$r = \kappa_0\left\{1 + \epsilon_0\left(\frac{1}{3} - \lambda^2\right)\right\},$$

and then to obtain the potential of a solid heterogeneous spheroid at any internal point we must take the sum of the latter expression for δV integrated from $\kappa = 0$ to $\kappa = \kappa_0$, and of the former expression integrated from $\kappa = \kappa_0$ to its value at the bounding surface; but for an external point the former expression must be used exclusively. Thus at any point of the bounding surface or external to it

$$V = \frac{4\pi}{r}\int\rho\kappa^2 d\kappa - \frac{8}{15}\frac{\pi}{r^3}\left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right)\int\rho\frac{d(\kappa^5\epsilon)}{d\kappa}d\kappa.$$

But the whole mass of the spheroid is

$$M = 4\pi\int\rho\kappa^2 d\kappa;$$

and we may write $E = \frac{4}{5}\pi\int\rho\frac{d(\kappa^5\epsilon)}{d\kappa}d\kappa$, a constant depending only upon the structure of the spheroid; thus

$$V = \frac{M}{r} - \frac{E}{r^3}\left(\lambda^2 - \frac{1}{3}\right).$$

§ 11. Let us apply this result to demonstrate Clairaut's theorem on the variation of gravity at the surface of the earth.

Consider the equilibrium of the fluid portions at the surface of the earth. The equilibrium of this requires that at the surface

$$V + \frac{\omega^2}{2}r^2(1 - \lambda^2) = C,$$

where

$$V = \frac{M}{r} - \frac{E}{r^3}\left(\lambda^2 - \frac{1}{3}\right),$$

if we suppose the analysis of § 10 to be applicable to the earth; that is to say, if we suppose the surfaces of equal density within the earth to be concentric coaxial spheroids of small ellipticity.

Substitute for V; then

$$\frac{M}{r} - \frac{E}{r^3}\left(\lambda^2 - \frac{1}{3}\right) + \frac{\omega^2}{2} r^2 (1 - \lambda^2) = C.$$

Now let κ_0, ϵ_0, refer to the surface, so that

$$r = \kappa_0 \left\{1 + \epsilon_0 \left(\frac{1}{3} - \lambda^2\right)\right\};$$

substitute for r, neglecting ϵ_0 when multiplied by the small quantities E or ω^2; then

$$\frac{M}{\kappa_0}\left\{1 + \epsilon_0 \left(\lambda^2 - \frac{1}{3}\right)\right\} - \frac{E}{\kappa_0^3}\left(\lambda^2 - \frac{1}{3}\right) + \frac{\omega^2}{2} \kappa_0^2 (1 - \lambda^2) = C;$$

this must be true for all values of λ; we get the two equations

$$\frac{M}{\kappa_0} + \frac{\omega^2}{3} \kappa_0^2 = C, \qquad \frac{M}{\kappa_0} \epsilon_0 - \frac{E}{\kappa_0^3} - \frac{\omega^2}{2} \kappa_0^2 = 0;$$

the latter equation gives ϵ_0 in terms of E and ω.

Now gravity at the Earth's surface is the resultant of the attraction of the body of the earth and the centrifugal force; that is to say, at any latitude, writing g for gravity,

$$\begin{aligned}
g &= -\frac{d}{dr}\left\{V + \frac{\omega^2}{2} r^2 (1 - \lambda^2)\right\} \\
&= \frac{M}{r^2} - \frac{3E}{r^4}\left(\lambda^2 - \frac{1}{3}\right) - \omega^2 r (1 - \lambda^2) \\
&= \frac{M}{\kappa_0^2} - \frac{2}{3} \omega^2 \kappa_0 + \left\{2 \frac{M}{\kappa_0^2} \epsilon_0 - \frac{3E}{\kappa_0^4} + \omega^2 \kappa_0\right\}\left(\lambda^2 - \frac{1}{3}\right).
\end{aligned}$$

Eliminate E by means of the relation found above; we find

$$\begin{aligned}
g &= \frac{M}{\kappa_0^2} - \frac{2}{3} \omega^2 \kappa_0 + \left(\frac{5}{2} \omega^2 \kappa_0 - \frac{M}{\kappa_0^2} \epsilon_0\right)\left(\lambda^2 - \frac{1}{3}\right) \\
&= \frac{M}{\kappa_0^2}\left(1 + \frac{1}{3} \epsilon_0\right) - \frac{3}{2} \omega^2 \kappa_0 + \left(\frac{5}{2} \omega^2 \kappa_0 - \frac{M}{\kappa_0^2} \epsilon_0\right) \lambda^2.
\end{aligned}$$

Let G_0 denote gravity at the equator, where $\lambda = 0$; then

$$G_0 = \frac{M}{\kappa_0^2}\left(1 + \frac{1}{3} \epsilon_0\right) - \frac{3}{2} \omega^2 \kappa_0.$$

Also the centrifugal force at the equator is $\omega^2 \kappa_0 \left(1 + \frac{1}{3} \epsilon_0\right)$; let ϕ be the ratio this bears to G_0; then

$$\phi = \frac{\omega^2 \kappa_0 \left(1 + \frac{1}{3} \epsilon_0\right)}{\frac{M}{\kappa_0^2}\left(1 + \frac{1}{3} \epsilon_0\right) - \frac{3}{2} \omega^2 \kappa_0} = \frac{\omega^2 \kappa_0^3}{M}$$

to the range of approximation we have used.

Hence
$$g = G_0\left\{1 + \left(\frac{5}{2}\phi - \epsilon_0\right)\lambda^2\right\},$$
or the coefficient of λ^2 in this expression + the compression of the earth $= \frac{5}{2}\phi$. This is Clairaut's Theorem.

§ 12. In this discussion we have not made any supposition as to the internal fluidity or otherwise of the spheroid. Let us now suppose the interior fluid, and let us consider what forms the surfaces of equal density would assume.

In the position of relative equilibrium, when the whole rotates as if solid, the surfaces of equal density will coincide with the surfaces of equal pressure.

Let us consider the following problem:—

Suppose a spheroid which is rotating as if solid to be made up of a number of different fluids that do not mix, to find the ellipticity of the bounding surfaces, given the volume and density of each different mass of fluid.

Thus we are given the quantities κ_1, ρ_1, κ_2, ρ_2, ... for each stratum, and it is required to determine ϵ_1, ϵ_2, ...

The advantage of employing κ rather than c will be obvious in this case.

Since the common surfaces are surfaces of equal pressure
$$V + \frac{\omega^2}{2} r^2 (1 - \lambda^2)$$
must be constant over each such surface.

Let us first solve the case of three fluids, and afterwards proceed to the general case of any number.

It is to be observed that the portions of V which are due to the different layers of fluid assume different forms according as the point lies within or without that layer.

Thus if V_1, V_2, ... be the parts contributed by the first, second, ... layer (beginning at the inmost layer), the first surface of separation is external to the first fluid and internal to all the rest; hence at the first surface of division V is the sum of V_1, V_2, V_3, where
$$V_1 = \frac{4}{3}\pi\rho_1\kappa_1^3\frac{1}{r} - \frac{8}{15}\pi\rho_1\frac{1}{r^3}\kappa_1^5\epsilon_1\left(\frac{3}{2}\lambda^2 - \frac{1}{2}\right)$$
$$= \frac{4}{3}\pi\rho_1\kappa_1^3\frac{1}{\kappa_1} + \frac{4}{3}\pi\rho_1\kappa_1^3\frac{1}{\kappa_1}\epsilon_1\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\pi\rho_1\frac{1}{\kappa_1^3}\kappa_1^5\epsilon_1\left(\lambda^2 - \frac{1}{3}\right),$$

$$V_2 = 2\pi\rho_2(\kappa_2^2 - \kappa_1^2) - \frac{4}{5}\pi\rho_2\kappa_1^2(\epsilon_2 - \epsilon_1)\left(\lambda^2 - \frac{1}{3}\right),$$

$$V_3 = 2\pi\rho_3(\kappa_3^2 - \kappa_2^2) - \frac{4}{5}\rho_3\kappa_1^2(\epsilon_3 - \epsilon_2)\left(\lambda^2 - \frac{1}{3}\right).$$

In addition to V there is the term

$$\frac{1}{2}\omega^2 r^2(1 - \lambda^2) = \frac{1}{3}\omega^2\kappa_1^2 - \frac{1}{2}\omega^2\kappa_1^2\left(\lambda^2 - \frac{1}{3}\right).$$

Now $V + \frac{1}{2}\omega^2 r^2(1 - \lambda^2)$ is constant; equate to zero the coefficient of $\lambda^2 - \frac{1}{3}$ in it; we get the equation

$$\text{(I)}\quad \frac{M_1}{\kappa_1}\epsilon_1 - \frac{4}{5}\pi\frac{\rho_1}{\kappa_1^3}\kappa_1^5\epsilon_1 - \frac{4}{5}\pi\rho_2\kappa_1^2(\epsilon_2 - \epsilon_1) - \frac{4}{5}\pi\rho_3\kappa_1^2(\epsilon_3 - \epsilon_2) - \frac{1}{2}\omega^2\kappa_1^2 = 0,$$

where $M_1 = \frac{4}{3}\pi\rho_1\kappa_1^3$, the mass of the inmost fluid. This is the condition of equilibrium of the first stratum.

For the second stratum, any point of it is external to two fluids and internal to the third;

$$V_1 = \frac{4}{3}\pi\kappa_1^3\rho_1\frac{1}{\kappa_2} + \frac{4}{3}\pi\kappa_1^3\rho_1\frac{\epsilon_2}{\kappa_2}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\pi\rho_1\kappa_1^5\epsilon_1\frac{1}{\kappa_2^3}\left(\lambda^2 - \frac{1}{3}\right),$$

$$V_2 = \frac{4}{3}\pi\rho_2(\kappa_2^3 - \kappa_1^3)\frac{1}{\kappa_2} + \frac{4}{3}\pi\rho_2(\kappa_2^3 - \kappa_1^3)\frac{\epsilon_2}{\kappa_2}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\frac{1}{\kappa_2^3}\left(\lambda^2 - \frac{1}{3}\right),$$

$$V_3 = 2\pi\rho_3(\kappa_3^2 - \kappa_2^2) - \frac{4}{5}\pi\rho_3\kappa_2^2(\epsilon_3 - \epsilon_2)\left(\lambda^2 - \frac{1}{3}\right);$$

in addition there is the term

$$\frac{1}{2}\omega^2 r^2(1 - \lambda^2) = \frac{1}{3}\omega^2\kappa_2^2 - \frac{1}{2}\omega^2\kappa_2^2\left(\lambda^2 - \frac{1}{3}\right).$$

Then, as before, these give the condition of equilibrium of the second stratum

$$\text{(II)}\quad (M_1 + M_2)\frac{\epsilon_2}{\kappa_2} - \frac{4}{5}\pi\rho_1\kappa_1^5\epsilon_1\frac{1}{\kappa_2^3} - \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\frac{1}{\kappa_2^3} - \frac{4}{5}\pi\rho_3(\epsilon_3 - \epsilon_2)\kappa_2^2 - \frac{1}{2}\omega^2\kappa_2^2 = 0,$$

where $M_2 = \frac{4}{3}\pi\rho_2(\kappa_2^3 - \kappa_1^3)$, the mass of the second layer.

Lastly, for the exterior surface

$$V_1 = \frac{4}{3}\pi\rho_1\kappa_1^3\frac{1}{\kappa_3} + \frac{4}{3}\pi\rho_1\kappa_1^3\frac{\epsilon_3}{\kappa_3}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\pi\rho_1\kappa_1^5\epsilon_1\frac{1}{\kappa_1^3}\left(\lambda^2 - \frac{1}{3}\right),$$

$$V_2 = \frac{4}{3}\pi\rho_2(\kappa_2^3 - \kappa_1^3)\frac{1}{\kappa_3} + \frac{4}{3}\pi\rho_2(\kappa_2^3 - \kappa_1^2)\frac{\epsilon_3}{\kappa_3}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\frac{1}{\kappa_3^3}\left(\lambda^2 - \frac{1}{3}\right),$$

$$V_3 = \frac{4}{3}\pi\rho_3(\kappa_3^3 - \kappa_2^3)\frac{1}{\kappa_3} + \frac{4}{3}\pi\rho_3(\kappa_3^3 - \kappa_2^3)\frac{\epsilon_3}{\kappa_3}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\pi\rho_3(\kappa_3^5\epsilon_3 - \kappa_2^5\epsilon_2)\frac{1}{\kappa_3^3}\left(\lambda^2 - \frac{1}{3}\right),$$

and the term $$\frac{1}{3}\omega^2\kappa_3^2 - \frac{1}{2}\omega^2\kappa_3^2\left(\lambda^2 - \frac{1}{3}\right)$$

must be added. We obtain the condition

$$\text{(III)}\quad (M_1+M_2+M_3)\frac{\epsilon_3}{\kappa_3} - \frac{4}{5}\pi\rho_1\kappa_1^5\epsilon_1\frac{1}{\kappa_3^2} - \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\frac{1}{\kappa_3^3}$$
$$-\frac{4}{5}\pi\rho_3(\kappa_3^5\epsilon_3 - \kappa_2^5\epsilon_2)\frac{1}{\kappa_3^3} - \frac{1}{2}\omega^2\kappa_3^2 = 0.$$

The three equations (I), (II), (III) serve to determine the quantities ϵ_1, ϵ_2, ϵ_3. Write

$$S_1 = M_1,\quad S_2 = M_1 + M_2,\quad S_3 = M_1 + M_2 + M_3,$$

and divide the equations by κ_1^2, κ_2^2, κ_3^2 respectively. Then

$$\frac{S_1}{\kappa_1^3}\epsilon_1 - \frac{4}{5}\pi\rho_1\epsilon_1 - \frac{4}{5}\pi\rho_2(\epsilon_2-\epsilon_1) - \frac{4}{5}\pi\rho_3(\epsilon_3-\epsilon_2) - \frac{1}{2}\omega^2 = 0,$$
$$\frac{S_2}{\kappa_2^3}\epsilon_2 - \frac{4}{5}\pi\rho_1\epsilon_1\frac{\kappa_1^5}{\kappa_2^5} - \frac{4}{5}\pi\rho_2\left(\epsilon_2 - \frac{\kappa_1^5}{\kappa_2^5}\epsilon_1\right) - \frac{4}{5}\pi\rho_3(\epsilon_3-\epsilon_2) - \frac{1}{2}\omega^2 = 0,$$
$$\frac{S_3}{\kappa_3^3}\epsilon_3 - \frac{4}{5}\pi\rho_1\epsilon_1\frac{\kappa_1^5}{\kappa_3^5} - \frac{4}{5}\pi\rho_2\left(\frac{\kappa_2^5}{\kappa_3^5}\epsilon_2 - \frac{\kappa_1^5}{\kappa_3^5}\epsilon_1\right) - \frac{4}{5}\pi\rho_3\left(\epsilon_3 - \frac{\kappa_2^5}{\kappa_3^5}\epsilon_2\right) - \frac{1}{2}\omega^2 = 0.$$

Take the differences (II) − (I), (III) − (II):

$$\frac{S_2}{\kappa_2^3}\epsilon_2 - \frac{S_1}{\kappa_1^3}\epsilon_1 - \frac{4}{5}\pi\rho_1\epsilon_1\kappa_1^5\left(\frac{1}{\kappa_2^5} - \frac{1}{\kappa_1^5}\right) + \frac{4}{5}\pi\rho_2\epsilon_1\kappa_1^5\left(\frac{1}{\kappa_2^5} - \frac{1}{\kappa_1^5}\right) = 0,$$
$$\frac{S_3}{\kappa_3^3}\epsilon_3 - \frac{S_2}{\kappa_2^3}\epsilon_2 - \frac{4}{5}\pi\rho_1\epsilon_1\kappa_1^5\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) - \frac{4}{5}\pi\rho_2\epsilon_2\kappa_2^5\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right)$$
$$+\frac{4}{5}\pi\rho_2\epsilon_1\kappa_1^5\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) + \frac{4}{5}\pi\rho_3\epsilon_2\kappa_2^5\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) = 0,$$

or, as they may be written,

$$\frac{S_2}{\kappa_2^3}\epsilon_2 - \frac{S_1}{\kappa_1^3}\epsilon_1 - \frac{4}{5}\pi\epsilon_1\kappa_1^5(\rho_1-\rho_2)\left(\frac{1}{\kappa_2^5} - \frac{1}{\kappa_1^5}\right) = 0,$$
$$\frac{S_3}{\kappa_3^3}\epsilon_3 - \frac{S_2}{\kappa_2^3}\epsilon_2 - \frac{4}{5}\pi\epsilon_1\kappa_1^5(\rho_1-\rho_2)\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) - \frac{4}{5}\pi\epsilon_2\kappa_2^5(\rho_2-\rho_3)\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) = 0.$$

From the former we find ϵ_2 in terms of ϵ_1 and known quantities; substitute in the latter and ϵ_3 is found in terms of ϵ_1, and ϵ_1 is then found from (I). The case when there is a solid nucleus may be dealt with in the same way. If ϵ_0 be the ellipticity of its outer surface, ϵ_0 is known and there is no condition of uniformity of pressure at its surface. Proceeding with the series, ϵ_1 may be found in terms of ω^2 and ϵ_0, ϵ_2 in terms of ϵ_1, ϵ_0, ω^2, and so on.

§ 13. Now take the more general case of any number of fluids whose

densities $\rho_1, \rho_2, \ldots \rho_n$ are known, and also their volumes, and let them rest upon a solid nucleus of known ellipticity ϵ_0. Let M_0 be the mass of the nucleus, $\frac{4}{3}\pi\kappa_0^3$ its volume, and let $E_0 = \frac{4}{5}\pi\int_0^{\kappa_0}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa$. Further, take

$$M_1 = \frac{4}{3}\pi\rho_1(\kappa_1^3 - \kappa_0^3), \quad M_2 = \frac{4}{3}\pi\rho_2(\kappa_2^3 - \kappa_1^3), \ldots M_n = \frac{4}{3}\pi\rho_n(\kappa_n^3 - \kappa_{n-1}^3).$$

Then exactly as before we have the equations

$$(M_0 + M_1)\frac{\epsilon_1}{\kappa_1} - \frac{E_0}{\kappa_1^3} - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\frac{1}{\kappa_1^3} - \frac{4}{5}\pi\rho_2(\epsilon_2 - \epsilon_1)\kappa_1^2 - \ldots$$
$$- \frac{4}{5}\pi\rho_n(\epsilon_n - \epsilon_{n-1})\kappa_1^2 - \frac{1}{2}\omega^2\kappa_1^2 = 0,$$

$$(M_0 + M_1 + M_2)\frac{\epsilon_2}{\kappa_2} - \frac{E_0}{\kappa_2^3} - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\frac{1}{\kappa_2^3} - \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\frac{1}{\kappa_2^3} - \ldots$$
$$- \frac{4}{5}\pi\rho_n(\epsilon_n - \epsilon_{n-1})\kappa_2^2 - \frac{1}{2}\omega^2\kappa_2^2 = 0,$$

$$(M_0 + M_1 + M_2 + M_3)\frac{\epsilon_3}{\kappa_3} - \frac{E_0}{\kappa_3^3} - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\frac{1}{\kappa_3^3} - \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\frac{1}{\kappa_3^3}$$
$$- \frac{4}{5}\pi\rho_3(\kappa_3^5\epsilon_3 - \kappa_2^5\epsilon_2)\frac{1}{\kappa_3^3} - \frac{4}{5}\pi\rho_4(\epsilon_4 - \epsilon_3)\kappa_3^2 - \ldots - \frac{4}{5}\pi\rho_n(\epsilon_n - \epsilon_{n-1})\kappa_3^2 - \frac{1}{2}\omega^2\kappa_3^2 = 0,$$

....

$$(M_0 + M_1 + \ldots + M_{n-1})\frac{\epsilon_{n-1}}{\kappa_{n-1}} - \frac{E_0}{\kappa_{n-1}^3} - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\frac{1}{\kappa_{n-1}^3} - \ldots$$
$$- \frac{4}{5}\pi\rho_{n-1}(\kappa_{n-1}^5\epsilon_{n-1} - \kappa_{n-2}^5\epsilon_{n-2})\frac{1}{\kappa_{n-1}^3} - \frac{4}{5}\pi\rho_n(\epsilon_n - \epsilon_{n-1})\kappa_{n-1}^2 - \frac{1}{2}\omega^2\kappa_{n-1}^2 = 0,$$

$$(M_0 + M_1 + \ldots + M_n)\frac{\epsilon_n}{\kappa_n} - \frac{E_0}{\kappa_n^3} - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0) - \ldots$$
$$- \frac{4}{5}\pi\rho_n(\kappa_n^5\epsilon_n - \kappa_{n-1}^5\epsilon_{n-1})\frac{1}{\kappa_n^3} - \frac{1}{2}\omega^2\kappa_n^2 = 0.$$

By means of these equations the n quantities $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ may be determined. The process of solution may be simplified if we divide in succession by $\kappa_1^2, \kappa_2^2, \ldots \kappa_n^2$ respectively, and write for brevity $M_0 + M_1 + \ldots + M_i = S_i$. Then we shall have

$$\frac{S_2}{\kappa_2^3}\epsilon_2 - \frac{S_1}{\kappa_1^3}\epsilon_1 - E_0\left(\frac{1}{\kappa_2^5} - \frac{1}{\kappa_1^5}\right) - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\left(\frac{1}{\kappa_2^5} - \frac{1}{\kappa_1^5}\right) + \frac{4}{5}\pi\epsilon_1\kappa_1^5\rho_2\left(\frac{1}{\kappa_2^5} - \frac{1}{\kappa_1^5}\right) = 0,$$

$$\frac{S_3}{\kappa_3^3}\epsilon_3 - \frac{S_2}{\kappa_2^3}\epsilon_2 - E_0\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right)$$
$$- \frac{4}{5}\pi\rho_2(\kappa_2^5\epsilon_2 - \kappa_1^5\epsilon_1)\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) + \frac{4}{5}\pi\epsilon_2\rho_3\kappa_2^5\left(\frac{1}{\kappa_3^5} - \frac{1}{\kappa_2^5}\right) = 0,$$

....

$$\frac{S_n}{\kappa_n^3}\epsilon_n - \frac{S_{n-1}}{\kappa_{n-1}^3}\epsilon_{n-1} - E_0\left(\frac{1}{\kappa_n^5} - \frac{1}{\kappa_{n-1}^5}\right) - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\left(\frac{1}{\kappa_n^5} - \frac{1}{\kappa_{n-1}^5}\right) - \dots$$
$$- \frac{4}{5}\pi\rho_{n-1}(\kappa_{n-1}^5\epsilon_{n-1} - \kappa_{n-2}^5\epsilon_{n-2})\left(\frac{1}{\kappa_n^5} - \frac{1}{\kappa_{n-1}^5}\right) + \frac{4}{5}\pi\rho_n\epsilon_{n-1}\kappa_{n-1}^5\left(\frac{1}{\kappa_n^5} - \frac{1}{\kappa_{n-1}^5}\right) = 0.$$

The first of these gives ϵ_2 in terms of ϵ_1 and ϵ_0, the second gives ϵ_3 in terms of ϵ_2, ϵ_1, ϵ_0, and so on; by this means we may express all in terms of ϵ_1, which may then be determined from the equation

$$\frac{S_1}{\kappa_1^3}\epsilon_1 - \frac{E_0}{\kappa_1^5} - \frac{4}{5}\pi\rho_1(\kappa_1^5\epsilon_1 - \kappa_0^5\epsilon_0)\frac{1}{\kappa_1^5} - \frac{4}{5}\pi\rho_2(\epsilon_2 - \epsilon_1) - \dots - \frac{4}{5}\pi\rho_n(\epsilon_n - \epsilon_{n-1}) - \frac{1}{2}\omega^2 = 0.$$

§ 14. Let us now pass to the case where each layer of fluid outside the nucleus is indefinitely thin. We shall then pass from sums to integrals and from difference equations to differential equations.

Let κ be taken as independent variable, and let ρ be known in terms of κ; then if M be the mass interior to the stratum defined by κ,

$$M = M_0 + \int_{\kappa_0}^{\kappa} 4\pi\kappa^2\rho d\kappa;$$

and it is required to determine ϵ in terms of κ. Then we have at any point

$$V = \frac{M}{r} - \frac{E_0}{r^3}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\frac{\pi}{r^3}\left(\lambda^2 - \frac{1}{3}\right)\int_{\kappa_0}^{\kappa}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa$$
$$+ 4\pi\int_{\kappa}^{K}\rho\kappa d\kappa - \frac{4}{5}\pi r^2\left(\lambda^2 - \frac{1}{3}\right)\int_{\kappa}^{K}\rho\frac{d\epsilon}{d\kappa}\,d\kappa,$$

where K is the value κ assumes at the external surface; the second term on the right being obtained from the nucleus, the third from the mass internal to the point, and the fourth and fifth from the mass external to the point. Substitute for $1/r$ its value

$$\left\{1 + \epsilon\left(\lambda^2 - \frac{1}{3}\right)\right\}\Big/\kappa,$$

and we get

$$\frac{M}{\kappa}\left\{1 + \epsilon\left(\lambda^2 - \frac{1}{3}\right)\right\} - \frac{E_0}{\kappa^3}\left(\lambda^2 - \frac{1}{3}\right) - \frac{4}{5}\frac{\pi}{\kappa^3}\left(\lambda^2 - \frac{1}{3}\right)\int_{\kappa_0}^{\kappa}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa$$
$$+ 4\pi\int_{\kappa}^{K}\rho\kappa d\kappa - \frac{4}{5}\pi\kappa^2\left(\lambda^2 - \frac{1}{3}\right)\int_{\kappa}^{K}\rho\frac{d\epsilon}{d\kappa}\,d\kappa + \frac{1}{2}\omega^2\kappa^2(1 - \lambda^2) = C,$$

where C is a quantity independent of λ, but varying with κ. Hence we have the two equations

$$\frac{M}{\kappa} + 4\pi\int_{\kappa}^{K}\rho\kappa d\kappa + \frac{1}{3}\omega^2\kappa^2 = C,$$

$$M\frac{\epsilon}{\kappa} - \frac{E_0}{\kappa^3} - \frac{4}{5}\frac{\pi}{\kappa^3}\int_{\kappa_0}^{\kappa}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa - \frac{4}{5}\pi\kappa^2\int_{\kappa}^{K}\rho\frac{d\epsilon}{d\kappa}\,d\kappa - \frac{1}{2}\omega^2\kappa^2 = 0.$$

Just as we formerly divided by $\kappa_1^2, \kappa_2^2, \ldots$ and took the difference of two successive equations, so here divide the second equation by κ^2, and then differentiate it with respect to κ:

$$\frac{d}{d\kappa}\left(\frac{M}{\kappa^3}\epsilon\right)+5\frac{E_0}{\kappa^6}+4\frac{\pi}{\kappa^6}\int_{\kappa_0}^{\kappa}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa-\frac{4}{5}\frac{\pi}{\kappa^5}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)+\frac{4}{5}\pi\rho\frac{d\epsilon}{d\kappa}=0;$$

or since
$$\frac{d}{d\kappa}(\kappa^5\epsilon)=\kappa^5\frac{d\epsilon}{d\kappa}+5\kappa^4\epsilon,$$

the equation becomes

$$\frac{d}{d\kappa}\left(\frac{M}{\kappa^3}\epsilon\right)+5\frac{E_0}{\kappa^6}+4\frac{\pi}{\kappa^6}\int_{\kappa_0}^{\kappa}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa-4\pi\rho\frac{\epsilon}{\kappa}=0,$$

or
$$\frac{M}{\kappa^3}\frac{d\epsilon}{d\kappa}-3\frac{M}{\kappa^4}\epsilon+5\frac{E_0}{\kappa^6}+4\frac{\pi}{\kappa^6}\int_{\kappa_0}^{\kappa}\rho\frac{d}{d\kappa}(\kappa^5\epsilon)\,d\kappa=0,$$

remembering that
$$\frac{dM}{d\kappa}=4\pi\rho\kappa^2.$$

Multiply by κ^6 and differentiate again; we shall thus remove the remaining definite integral and obtain

$$\frac{d^2\epsilon}{d\kappa^2}+\frac{8\pi\rho\kappa^2}{M}\frac{d\epsilon}{d\kappa}+\left(\frac{8\pi\rho\kappa}{M}-\frac{6}{\kappa^2}\right)\epsilon=0.$$

This will determine ϵ, when ρ (and therefore M) is known in terms of κ. Its solution will introduce two arbitrary constants, but we must remember that this equation has been derived by two differentiations from the actual equation which we have to satisfy, and the two constants must be adapted so as to satisfy this; they will evidently depend upon E_0 and ω^2. The equation may be put under a simpler form; for

$$\frac{4\pi\rho\kappa^2}{M}=\frac{1}{M}\frac{dM}{d\kappa};$$

therefore the equation may be written

$$M\frac{d^2\epsilon}{d\kappa^2}+2\frac{dM}{d\kappa}\frac{d\epsilon}{d\kappa}+\left(8\pi\rho\kappa-\frac{6M}{\kappa^2}\right)\epsilon=0,$$

or since
$$\frac{d^2M}{d\kappa^2}=8\pi\rho\kappa+4\pi\kappa^2\frac{d\rho}{d\kappa},$$

$$\frac{d^2}{d\kappa^2}(M\epsilon)-\left(\frac{4\pi\kappa^2}{M}\frac{d\rho}{d\kappa}+\frac{6}{\kappa^2}\right)M\epsilon=0.$$

We cannot treat this equation further unless we know ρ in terms of κ; but it is worth while to shew how it will solve the inverse problem, viz.: *Given the law of ellipticity, to find what the law of density must be.*

Return to the earlier form of the equation and replace $8\pi\rho\kappa$ by

$$2\frac{dM}{d\kappa}\Big/\kappa;$$

it becomes
$$\frac{d^2\epsilon}{d\kappa^2}+\frac{2}{M}\frac{dM}{d\kappa}\frac{d\epsilon}{d\kappa}+2\frac{dM}{d\kappa}\frac{\epsilon}{M\kappa}-\frac{6\epsilon}{\kappa^2}=0,$$

or
$$\frac{2}{M}\frac{dM}{d\kappa}=\frac{\dfrac{6\epsilon}{\kappa^2}-\dfrac{d^2\epsilon}{d\kappa^2}}{\dfrac{d\epsilon}{d\kappa}+\dfrac{\epsilon}{\kappa}};$$

the right-hand member is supposed known in terms of κ. Integrating we get

$$2\log M=\int\frac{\dfrac{6\epsilon}{\kappa^2}-\dfrac{d^2\epsilon}{d\kappa^2}}{\dfrac{d\epsilon}{d\kappa}+\dfrac{\epsilon}{\kappa}}\,d\kappa;$$

this will give M, and ρ then follows from the equation

$$\rho=\frac{1}{4\pi\kappa^2}\frac{dM}{d\kappa}.$$

§ 15. Let us assume a particular law of density and find the associated law of ellipticity; let us take

$$\rho=A\,\frac{\sin q\kappa}{\kappa};$$

at the centre where $\kappa=0$, this gives $\rho=Aq$, and a gradual diminution as we ascend. Then

$$\frac{d\rho}{d\kappa}=\frac{A}{\kappa^2}(q\kappa\cos q\kappa-\sin q\kappa);$$

and
$$\frac{dM}{d\kappa}=4\pi\rho\kappa^2=4\pi A\kappa\sin q\kappa,$$

$$M=\frac{4\pi A}{q^2}\{\sin q\kappa-q\kappa\cos q\kappa\},$$

and
$$\frac{4\pi\kappa^2}{M}\frac{d\rho}{d\kappa}=-q^2.$$

Therefore the equation becomes

$$\frac{d^2(M\epsilon)}{d\kappa^2}+\left(q^2-\frac{6}{\kappa^2}\right)M\epsilon=0,$$

a case of Riccati's equation of which the solution is

$$M\epsilon=\frac{A}{q^4\kappa^2}\{(3-\kappa^2q^2)\sin(q\kappa+a)-3q\kappa\cos(q\kappa+a)\}.$$

The two arbitrary constants must be adapted to the values of E_0 and ω^2.

12.

EFFECT OF THE LONG INEQUALITY OF JUPITER AND SATURN UPON THE MOTIONS OF JUPITER'S SATELLITES.

[In his "Continuation of Damoiseau's Tables" (*Supplement to Nautical Almanac*, 1881; *Works*, vol. I. p. 113), Adams remarks:—

"The terms which involve $\sin(5u - 2u_0 - 34^\circ{\cdot}542)$ in Damoiseau's formulae for Table III. of each Satellite are sufficiently accurate as they stand."

These are the terms that express the effect of the long inequality of Jupiter and Saturn upon the Satellites, and Damoiseau's values do not differ much from those of the *Mécanique Céleste*, l. XVI. ch. VII.

In Oct. 1878, after the publication of the above, M. Souillart wrote to Adams to remark that he believed Laplace's determinations to be vitiated by an error of theory, and Adams re-examined them and detected the curious sequence of numerical errors detailed below, whose removal produced a close accordance with the results of M. Souillart; subsequently, at the request of M. Puiseux, the corrections were communicated to him and are given in a note at the end of t. V. of the new edition of the *Mécanique Céleste*, 1882.

When it became necessary to continue the Tables from 1890 to 1900, Adams derived values of these inequalities from Le Verrier's Tables of Jupiter, and Souillart's theory, and used them for calculating the continuation of Table III.; but no communication of the adopted expressions was made to the *Nautical Almanac* office, and the approximate values given in vol. I. p. 124 are not Adams's own, but were derived *à posteriori*.]

In the *Mécanique Céleste*, l. XVI. ch. VI. Laplace finds the effect of the great inequality of Saturn and Jupiter upon the satellites of the latter, but his numbers are in fault at several points and require substantial corrections.

In t. V. p. 462, line 6 from bottom, quoting from the new edition, the number $-44'',334$, being the coefficient of the inequality of the fourth satellite, is not consistent with the formula preceding it, which gives $-42'',863$, and the coefficients of the inequalities of the first three satellites should be diminished in the same ratio; but in the formula itself a term is

wrong, viz.: $-0{,}000264 \cos(x+48^\circ{,}27)$, which is taken from t. IV. p. 341, line 8 from bottom, and represents the term of t. III. p. 138, last line but one,

$$-(1+\mu^{\text{v}}) \,.\, 0{,}0003042733 \cos(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\epsilon^{\text{v}} - 2\epsilon^{\text{iv}} - 13^\circ{,}4966),$$

modified by the change in the mass of Saturn introduced in t. IV. p. 337; the correctly modified coefficient is $-0{,}000290$, and it is curious to notice that the same mistake occurs again in t. IV. p. 341, line 16, where the coefficient of $\cos(q^{\text{iv}} - 2q^{\text{v}} - 24^\circ{,}88 + t \,.\, 58''{,}0)$ should be $-0{,}000290$ in place of $-0{,}000264$, the two terms from which this is derived (t. III. p. 128, lines 4, 5) uniting to give a coefficient $-0{,}00030503$ which modifies into $-0{,}000290$; finally the term above upon which all depends

$$-(1+\mu^{\text{v}}) \,.\, 0{,}0003042733 \cos(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\epsilon^{\text{v}} - 2\epsilon^{\text{iv}} - 13^\circ{,}4966)$$

is got by taking with wrong sign the term (t. III. p. 138, line 7 from bottom)

$$-a^{\text{iv}}e^{\text{iv}}H \cos(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\epsilon^{\text{v}} - 2\epsilon^{\text{iv}} - \varpi^{\text{iv}} + A).$$

To correct these errors, read in t. III. p. 138, last line but one,

$$-(1+\mu^{\text{v}})\, 0{,}0002465 \cos(5n^{\text{v}}t - 2n^{\text{iv}}t + 5\epsilon^{\text{v}} - 2\epsilon^{\text{iv}} + 32^\circ{,}41),$$

using centesimal degrees;

in t. IV. p. 341, line	16,	$-0{,}000290$ for $-0{,}000264$;
,, ,,	25,	$-0{,}0002343 \cos(5q^{\text{v}} - 2q^{\text{iv}} + 32^\circ{,}41)$;
t. V. p. 462, line	6,	$-0{,}0002343 \cos(x + 94^\circ{,}18)$;

and for the coefficients of the inequalities of the satellites

I	II	III	IV
$-0''{,}618$	$-8''{,}788$	$-12''{,}872$	$-30''{,}404$

each being multiplied into $\sin(x+48^\circ{,}09)$ or $\sin(5q^{\text{v}} - 2q^{\text{iv}} - 13^\circ{,}68)$; or if we employ Laplace's values of the ratios which deduce I, II, III from IV (values which are only approximate), we get

I	II	III	IV
$-0''{,}682$	$-8''{,}810$	$-12''{,}874$	$-30''{,}404$

in place of the values of the *Mécanique Céleste*:—

$-0''{,}994$	$-12''{,}847$	$-18''{,}774$	$-44''{,}334$.

The degrees and seconds are centesimal. [Oct. 1881.

If we take the formula of M. Souillart*

$$\frac{d\Delta l}{dt} = -\frac{3N^2}{n}\left(-\frac{\delta A}{A} + E\delta E\right),$$

and use Le Verrier's formulae (*Mém. Ob. Paris*, t. XII. pp. 14, 23), we have

$$-\frac{\delta A}{A} + E\delta E = (12''{\cdot}119 - 0''{\cdot}112v + 0''{\cdot}022v^2)\cos(5l' - 2l$$
$$- 16^\circ{\cdot}63 - 11^\circ{\cdot}57v + 0^\circ{\cdot}27v^2),$$

* *Théorie...des Satellites de Jupiter, Mem. R. A. S.* XLV. p. 89.

where $v = \frac{t-1850}{500}$ and the degrees, &c. are sexagesimal; and (*ibid.* tome XI. pp. 106, 273),

$$\left.\begin{aligned} l &= 160^\circ\ \ 1' 20''\cdot 3 + 109256''\cdot 719 t \\ l' &= \ \ 14^\circ\, 50' 40''\cdot 6 + \ \ 43996''\cdot 127 t \end{aligned}\right\} \text{ Epoch 1850.}$$

If we integrate we obtain for Satellite IV the inequality

$$(11''\cdot 0525 - 0''\cdot 1332 v + 0''\cdot 0201 v^2) \sin (5l' - 2l - 16^\circ\cdot 835 - 11^\circ\cdot 508 v + 0^\circ\cdot 2696 v^2),$$

or in seconds of synodic time, the coefficients are for epoch 1850 :—

		coefficient	change in 500 years
	IV	$12^s\cdot 344$	$-0^s\cdot 1488$,
whence we derive			
	I	$0^s\cdot 0265$	$-0^s\cdot 0003$,
	II	$0^s\cdot 757$	$-0^s\cdot 0091$,
	III	$2^s\cdot 235$	$-0^s\cdot 0269$.

[June, 1885.

13.

STUDIES ON NEWTON'S LUNAR THEORY.

[It appears from an undated fragment among his papers that Adams at one time proposed to give an outline of Newton's methods and results in the Lunar Theory, of which he says "there is no part of Newton's great work which displays more conspicuously the genius of the author, or better illustrates his manner of working." His attention was directed to the subject in relation to the Portsmouth papers*, and the second investigation below is an analytical parallel to a method of obtaining the motion of the apse, found among those papers and described and, in part, published in the *Catalogue* (pp. xii, xxvi).]

ANALYTICAL INTERPRETATION OF NEWTON'S INVESTIGATION OF THE LUNAR INEQUALITY OF THE VARIATION.

Let a be the actual mean distance of the Moon from the Earth; $a(1-x)$ and $a(1+x)$ the least and greatest distances; a', n, n', m, μ, r, θ, θ' have the meanings attached to these symbols in the Lectures, *passim*; e, e', i are supposed to vanish;

then as in Lecture VI. p. 24, we have

$$\frac{1}{H}\frac{dH}{d\theta} = -\frac{3}{2}n'^2\left(\frac{dt}{d\theta}\right)^2 \sin 2(\theta-\theta').$$

Put for $\frac{dt}{d\theta}$ its approximate value $\frac{1}{n}$; then

$$\frac{1}{H}\frac{dH}{d\theta} = -\frac{3}{2}m^2 \sin 2(\theta-\theta').$$

Integrate, considering $\frac{d\theta'}{d\theta}=m$, as it is, very nearly;

$$\log\frac{H}{h} = \frac{3}{2}\frac{m^2}{2-2m}\cos 2(\theta-\theta'),$$

or putting $h=na^2$,

$$H = na^2\left\{1+\frac{3}{4}\frac{m^2}{1-m}\cos 2(\theta-\theta')\right\}.$$

This agrees exactly with the result of Prop. XXVI. Lib. III.

* See "A Catalogue of the Portsmouth Collection of Books and Papers written by or belonging to Sir Isaac Newton," Cambridge, 1888.

Now assume, as in Newton's Proposition XXVIII., that the orbit is an ellipse, with shorter axis in the line of syzygy, that is to say, that its equation is of the form

$$\frac{1}{r^2} = \frac{\cos^2(\theta-\theta')}{a^2(1-x)^2} + \frac{\sin^2(\theta-\theta')}{a^2(1+x)^2} = \frac{1}{a^2(1-x^2)^2}\left[1+x^2+2x\cos 2(\theta-\theta')\right].$$

If the line of syzygy were fixed in direction the curvature at the extremity of the minor axis (A) would be

$$\left(\frac{1-x}{1+x}\right)^2 \frac{1}{a(1-x)} = \frac{1}{a(1-x)}\left[1-\frac{4x}{(1+x)^2}\right];$$

but if we take

$$\frac{d\theta'}{d\theta} = m,$$

the curvature at the same apse of the (now revolving) ellipse appears by differentiation of the above value of r,

at A, the nearer apse $\quad \dfrac{1}{a(1-x)}\left[1-\dfrac{4x}{(1+x)^2}(1-m)^2\right],$

at C, the further apse $\quad \dfrac{1}{a(1+x)}\left[1+\dfrac{4x}{(1-x)^2}(1-m)^2\right]$*,

and the former curvature is to the latter as

$$(1-x)\left[(1-x)^2(1-m)^2+(1+x)^2(2m-m^2)\right]$$
$$: (1+x)\left[(1+x)^2(1-m)^2+(1-x)^2(2m-m^2)\right],$$

which agrees with Newton's result, p. 401 (Second Ed.), line 4.

Now the forces upon the Moon at the points A and C, according to Lecture II. p. 8, are

$$\text{at } A,\ \frac{\mu}{r^2} - 2\frac{m'r}{r'^3},$$

$$\text{at } C,\ \frac{\mu}{r^2} + \frac{m'r}{r'^3},$$

or if

$$\frac{\mu}{a^3} = n^2\left(1+\frac{1}{2}m^2\right), \quad \frac{m'}{r'^3} = n^2m^2,$$

these are as

$$\frac{1+\frac{1}{2}m^2}{(1-x)^2} - 2m^2(1-x) : \frac{1+\frac{1}{2}m^2}{(1+x)^2} + m^2(1+x),$$

a ratio which Newton gives, not quite correctly (p. 399, line 9) as

$$\frac{1}{a^2(1-x)^2} - \frac{2m^2}{a^2(1+x)} : \frac{1}{a^2(1+x)^2} + \frac{m^2}{a^2(1-x)}.$$

* [This is the analytical verification of Newton's "Rationes autem ineundo invenio quod differentia inter curvaturam &c." Prop. XXVIII.]

Again, the velocities at A and C are the values of H/r; that is to say, they are in the proportion

$$\frac{1}{1-x}\left\{1+\frac{3}{4}\frac{m^2}{1-m}\right\} : \frac{1}{1+x}\left\{1-\frac{3}{4}\frac{m^2}{1-m}\right\}.$$

Hence the curvatures of the orbit at A and C are as

$$[(1+x)^2-2m^2(1-x)^2(1+x)](1-x)^2\Big/\left(1+\frac{3}{4}\frac{m^2}{1-m}\right)^2$$
$$: [(1-x)^2+m^2(1+x)^2(1-x)](1+x)^2\Big/\left(1-\frac{3}{4}\frac{m^2}{1-m}\right)^2.$$

This ratio must be equal to the ratio of the curvatures already found, viz.

$$(1-x)[(1-x)^2(1-m)^2+(1+x)^2(2m-m^2)]$$
$$: (1+x)[(1+x)^2(1-m)^2+(1-x)^2(2m-m^2)].$$

Hence we have a proportion, from which, multiplying extremes and means, we obtain the equation

$$\left(1-\frac{3}{4}\frac{m^2}{1-m}\right)^2[(1+x)^3(1-m)^2+(1-x)^2(1+x)(2m-m^2)$$
$$-2m^2(1-m)^2(1-x)^2(1+x)^2-2m^2(2m-m^2)(1-x)^4]$$
$$=\left(1+\frac{3}{4}\frac{m^2}{1+m}\right)^2[(1-x)^3(1-m)^2+(1-x)(1+x)^2(2m-m^2)$$
$$+m^2(1-m)^2(1-x)^2(1+x)^2+m^2(2m-m^2)(1+x)^4],$$

which is Newton's equation of p. 401, expressed symbolically.

This equation may be developed into the form

$$\left\{1+\frac{9}{16}\frac{m^4}{(1-m)^2}\right\}[-3m^2+x\{6-4(2-m^2)(2m-m^2)\}$$
$$+x^2\{6m^2-24m^2(2m-m^2)\}+x^3\{2+4m^2(2m-m^2)\}-x^4 3m^2]$$
$$=\frac{3}{2}\frac{m^2}{1-m}[2-m^2+x\{12m^2(2m-m^2)\}$$
$$+x^2\{6-8(2m-m^2)+2m^2-8m^2(2m-m^2)\}+x^3\{12m^2(2m-m^2)\}-x^4m^2].$$

Neglecting at first all terms of the fourth order or of the order $m^2\alpha_1$, we get

$$x=\frac{3}{2}m^2\frac{2-m}{(1-m)(3-2m)(1-2m)}.$$

Taking as Newton does, $m^2=\dfrac{1}{178\cdot725}$, *i.e.* $m=\cdot0748011$, this gives

$$x=\cdot00720475.$$

In the first edition Newton gives $\cdot0072036$, thus apparently taking account of the first power of x only. The complete equation for x is

$$0=\cdot03487783-4\cdot851179x+\cdot02978676x^2-2\cdot003176x^3+\cdot016735x^4,$$

which will be found to agree closely with that given by Cotes. (See Edleston, *Correspondence of Newton and Cotes*, p. 98.) Taking ·00719, the value given in the second edition of the *Principia* as an approximation, we find $x = \cdot 00718973.$

Cotes's coefficients give $x = \cdot 00719000.$ [June, 1873.

A METHOD OF NEWTON'S FOR FINDING THE MOTION OF THE APSE.

["Two lemmas are first established* which give the motion of the apogee in an elliptic orbit of very small eccentricity due to given small disturbing forces acting (1) in the direction of the radius vector, and (2) in the direction perpendicular to it....... Newton assumes that the form of the orbit in which the moon really moves will be related to the form of the oval orbit [which the Variation produces] nearly as an elliptic orbit of small eccentricity with the earth in its focus is related to a circular orbit about the earth in the centre." *Catalogue*, p. xii.]

If $\frac{\mu}{r^2} + P$, Q be the forces in the direction of the radius vector and perpendicular to it, we have the equations for the changes in the elliptic elements, e, ϖ, of an orbit,

$$\frac{de}{dt}\cos(\theta - \varpi) + e\sin(\theta - \varpi)\frac{d\varpi}{dt} = 2\frac{h}{\mu}Q,$$

$$-\frac{de}{dt}\sin(\theta - \varpi) + e\cos(\theta - \varpi)\frac{d\varpi}{dt} = \frac{h}{\mu}P - \frac{h}{\mu}Q\frac{e\sin(\theta - \varpi)}{1 + e\cos(\theta - \varpi)};$$

whence

$$e\frac{d\varpi}{dt} = \frac{h}{\mu}P\cos(\theta - \varpi) + \frac{h}{\mu}Q\sin(\theta - \varpi)\frac{2 + e\cos(\theta - \varpi)}{1 + e\cos(\theta - \varpi)},$$

a result which becomes identical with Newton's two lemmas, provided that we neglect e in the second member on the right.

Now we will attempt from this result to find the motion of the Moon's apogee in a way similar to that which was in Newton's mind.

Let θ_0, r_0 be coordinates of the moon moving in an elliptic orbit; let θ, r be the actual coordinates; and let us assume that θ, r are related to θ_0, r_0 by the equations

$$\theta = \theta_0 + \frac{11}{8}m^2\sin 2\theta_0, \quad r = r_0\{1 - m^2\cos 2\theta_0\},$$

an assumption which reduces to a known result when the elliptic orbit (r_0, θ_0) degenerates into a circle.

For simplicity we will suppose the Sun stationary.

* [Published in the *Catalogue*, p. xxvi.]

Also let h_0, h be the double areal velocity in the two orbits; then

$$h_0 = na^2,$$

and since

$$\frac{d\theta}{dt} = \left\{1 + \frac{11}{4} m^2 \cos 2\theta_0\right\} \frac{d\theta_0}{dt} = \frac{h_0}{r_0^2}\left\{1 + \frac{11}{4} m^2 \cos 2\theta_0\right\},$$

$$r^2 = r_0^2 \{1 - 2m^2 \cos 2\theta_0\};$$

$$\therefore\ h = h_0 \left\{1 + \frac{3}{4} m^2 \cos 2\theta_0\right\}.$$

The force on the Moon perpendicular to the Radius Vector would be, if h_0 were constant,

$$\frac{1}{r}\frac{dh}{dt} = \frac{h_0^2}{r_0^3}\left\{-\frac{3}{2} m^2 \sin 2\theta_0\right\}.$$

But the actual force in this direction is

$$-\frac{3}{2} m^2 n^2 r \sin 2\theta = \frac{h_0^2}{a^4} r_0 \left\{-\frac{3}{2} m^2 \sin 2\theta_0\right\}, \text{ approximately.}$$

The latter minus the former quantity

$$= n^2 a \{6m^2 e \sin 2\theta_0 \cos(\theta_0 - \varpi)\}$$

if we write $r_0 = a/\{1 + e\cos(\theta - \varpi)\}$, and neglect e^2. If this difference were zero, the elliptic elements, e, ϖ, would not be liable to change in respect to forces perpendicular to the radius vector; hence it measures Q, the force perpendicular to the radius vector which disturbs the elliptic orbit (r_0, θ_0).

Again, in the direction of the radius vector the force is

$$\frac{h^2}{r^3} - \frac{d^2 r}{dt^2} = n^2 a \left\{1 + 2e\cos(\theta_0 - \varpi) + \frac{1}{2} m^2 \cos 2\theta_0 + \frac{5}{2} m^2 e \cos 2\theta_0 \cos(\theta_0 - \varpi)\right\},$$

if the elements of the elliptic orbit be constant; but the actual force is

$$\frac{\mu}{r^2} - \frac{1}{2} n'^2 r - \frac{3}{2} n'^2 r \cos 2\theta = \frac{\mu}{a^2}\{1 + 2m^2 \cos 2\theta_0\}\{1 + 2e\cos(\theta_0 - \varpi)\}$$

$$- \frac{1}{2} m^2 n^2 a \{1 - e\cos(\theta_0 - \varpi)\} - \frac{3}{2} m^2 n^2 a \cos 2\theta_0 \{1 - e\cos(\theta_0 - \varpi)\},$$

putting r_0 for r, θ_0 for θ in terms multiplied by m^2. If

$$\frac{\mu}{a^2} = n^2 a \left(1 + \frac{1}{2} m^2\right),$$

the terms independent of e agree with the like terms in the expression above, and the latter force minus the former

$$= n^2 a \left\{\frac{3}{2} m^2 e \cos(\theta_0 - \varpi) + 3m^2 e \cos 2\theta_0 \cos(\theta_0 - \varpi)\right\}.$$

This is the quantity P.

Hence we have the equation

$$\frac{d\varpi}{d\theta}=\cos(\theta_0-\varpi)\left\{\frac{3}{2}m^2\cos(\theta_0-\varpi)\right\}\{1+2\cos 2\theta_0\}$$
$$+2\sin(\theta_0-\varpi)\{6m^2\sin 2\theta_0\cos(\theta_0-\varpi)\}$$
$$=\frac{3}{4}m^2+\frac{3}{4}m^2\cos 2(\theta_0-\varpi)+\frac{3}{2}m^2\cos 2\theta_0+\frac{15}{4}m^2\cos 2\varpi-\frac{9}{4}m^2\cos(4\theta_0-2\varpi).$$

If we compare this with the result of p. 62 it will be seen that while the periodic terms of short period are in error, the coefficients of the terms which produce most effect are correct to the order to which we have carried them, remembering that at the beginning we neglected the Sun's motion. If we simply take

$$\frac{d\varpi}{d\theta}=\frac{3}{4}m^2+\frac{15}{4}m^2\cos 2\varpi,$$

we find the mean rate of change of ϖ with respect to θ, by the methods of Lecture XIII.,

$$\frac{3}{4}m^2+\frac{225}{32}m^3.$$

14.

VARIOUS ERRATA.

[The errata in Plana and Pontécoulant seem to have been noticed by Adams in course of his scrutiny of those authors when constructing his tables of the Moon's parallax, and later in the controversy upon the Secular Acceleration.]

Damoiseau, *Théorie de la Lune.*

p. 321, to the value of $\left(\frac{dQ}{du}\right)$ add the terms $-\frac{3m'u'^4}{u^4}\left[3(1-4s^2)\cos(\nu-\nu')+5\cos(3\nu-3\nu')\right]$, but the equations for u, s, and t just below are correct.

p. 329, last line, for $\int \frac{u'^3 d\nu}{u}\,\delta u \sin(2\nu-2\nu')$ read $\int \frac{u'^3 d\nu}{u^5}\,\delta u \sin(2\nu-2\nu')$

p. 332, the terms in the equations for s which depend on the Sun's parallax, should be

$$\frac{33}{8}\frac{m'u'^4 s}{h^2u^5}\cos(\nu-\nu')-\frac{3}{8}\frac{m'u'^4}{h^2u^5}\frac{ds}{d\nu}\sin(\nu-\nu')+\frac{15}{8}\frac{m'u'^4 s}{h^2u^5}\cos(3\nu-3\nu')-\frac{15}{8}\frac{m'u'^4}{h^2u^5}\frac{ds}{d\nu}\sin(3\nu-3\nu')$$

p. 338, last line but one, for $1-\frac{15}{2}\gamma^2$ read $1-\frac{15}{2}s^2$

" last line, for u_0+&c. read $\frac{a}{a_1}\{u_0+\text{\&c.}\}$

p. 452, line 1, the argument corresponding to C_{92} should be $(\nu-m\nu+c\nu+c'm\nu-\varpi-\varpi')$

p. 559, in the value of $b^{(122)}$ the term $-\frac{1}{8}(4-4m+c)^2 C_1 C_{30}^2$ has been omitted.

p. 589, in the value of g^2-1, for $-\frac{3}{2}e'^2$ read $+\frac{3}{2}e'^2$, but the work is correct.

Damoiseau, *Tables de la Lune.*

Introduction, p. iii, second edition, for $1''\cdot3\sin(\bar{x}+\bar{y})$ read $-1''\cdot3\sin(\bar{x}+\bar{y})$, but the Table in p. 68 is correct.

" p. vi, line 6, first edition, for $t-x$ read $t+x$

Plana, *Théorie du Mouvement de la Lune.*

Tome I., p. 488, in coefficient of $\sin(c\nu+c'm\nu)\,ee'm^4$ for $\frac{2061269}{2048}$ read $\frac{1605365}{2048}$

" p. 545, " " $\sin(cnt+c'mnt)\,ee'm^4$ " " " "

" pp. 545, 576, " " for $-\frac{4173895}{6144}$ read $-\frac{2806183}{6144}$

Tome I., p. 490, in coefficient of $\sin 2E\nu\, m^5\epsilon'^2$ for $\frac{19729}{256}$ read $\frac{19729}{216}$
" p. 548, " " $\sin 2Ent\, m^5\epsilon'^2$ " " " "
" " " " " " $\frac{65051}{768}$ " $\frac{85513}{864}$
" p. 577, " " " " $\frac{70043}{768}$ " $\frac{91129}{864}$
" p. 511, " " $\cos 2E\nu\, m^3\epsilon'^2$ for $\frac{67}{4}$ read $\frac{33}{2}$
" p. 548, " " $\sin 2Ent\, m^3\epsilon'^2$ " " " "
" pp. 548, 577, " " for $\frac{697}{24}$ read $\frac{691}{24}$
" p. 548, " " $\sin 2Ent\, m^4\epsilon'^2$ add $-(\frac{1}{4})$
" " " " " for $\frac{15949}{144}$ read $\frac{15985}{144}$
" p. 577, " " " " $\frac{16543}{144}$ " $\frac{16579}{144}$
" p. 523, " " $\sin c\nu\, em^4\epsilon'^2$ for $\frac{1005}{32}$ read $\frac{990}{32}$
" " " " " " $\frac{33621}{64}$ " $\frac{33591}{64}$
" p. 542, " " $\sin cnt\, em^4\epsilon'^2$ for $-\frac{11207}{128}$ read $-\frac{11197}{128}$
" " " " " " $\frac{303981}{512}$ " $\frac{304021}{512}$
" p. 574, " " " " $\frac{281095}{512}$ " $\frac{281135}{512}$
" p. 529, " " $\sin(2E\nu - c\nu)\, em^3\epsilon'^2$ for $\frac{67}{4}$ read $\frac{33}{2}$
" " " " " " " 16 " $\frac{63}{4}$
" p. 530, " " $\sin(2E\nu + c\nu)\, em^3\epsilon'^2$ " $\frac{67}{4}$ " $\frac{33}{2}$
" " " " " " " $\frac{199}{4}$ " $\frac{99}{2}$
" p. 550, " " $\sin(2Ent - cnt)\, em^3\epsilon'^2$ add $-\frac{1}{24}$
" " " " " " for $\frac{331001}{1536}$ read $\frac{330937}{1536}$
" p. 578, " " " " for $-\frac{326681}{1536}$ read $-\frac{326617}{1536}$
" p. 550, " " $\sin(2Ent - cnt)\, em^4\epsilon'^2$ add $+\frac{1}{6}$
" " " " " " for $\frac{36136889}{9216}$ read $\frac{36138425}{9216}$
" p. 578, " " " " for $-\frac{36126953}{9216}$ read $-\frac{36128489}{9216}$
" p. 551, " " $\sin(2Ent + cnt)\, em^3\epsilon'^2$ add $+\frac{3}{8}$
" pp. 551, 578, " " " for $-\frac{1579}{24}$ read $-\frac{785}{12}$
" p. 551, " " $\sin(2Ent + cnt)\, em^4\epsilon'^2$ add $-\frac{1}{2}$
" " " " " " for $-\frac{934399}{2304}$ read $-\frac{935551}{2304}$
" p. 578, " " " " " $-\frac{967447}{2304}$ " $-\frac{968599}{2304}$
" p. 524, " " $\sin 5c\nu\, e^5$ for $\frac{91}{16}$ read $\frac{91}{96}$
" p. 530, " " $\sin(2E\nu + c\nu)\, em^5$ read $-(\frac{1815}{512} + \frac{1815}{1024} = \frac{5445}{1024})$
" p. 550, " " $\sin(2Ent + cnt)\, em^5$ for $-\frac{5445}{512}$ read $-\frac{16335}{2048}$
" pp. 550, 578, " " " " $\frac{1869001}{55296}$ " $\frac{126001}{3456}$
" p. 531, " " $\sin(2E\nu + 3c\nu)\, e^3m^2$ for $-\frac{191}{32}$ read $-\frac{291}{32}$
" p. 557, " " $\sin(2Ent + 3cnt)\, e^3m^2$ for $-\frac{4775}{192}$ read $-\frac{7275}{192}$
" pp. 557, 580, " " " " $\frac{1093}{64}$ " $\frac{779}{192}$
" p. 622 (note) " $\sin(2E\nu + 3c\nu)\, e^3m^3$ for $\frac{955}{48}$ read $\frac{1455}{48}$
" " " " " " " $\frac{2317}{576}$ " $\frac{8317}{576}$
" " (text) " $\sin(2Ent + 3cnt)$ for $3''{\cdot}252 + 0''{\cdot}057 = 3''{\cdot}309$ read $0''{\cdot}773 + 0''{\cdot}206 = 0''{\cdot}979$
" p. 531, " " $\sin(2E\nu + 2g\nu)\, \gamma^2me^2$ add $-\frac{15}{16}$
" " " " " " for $-\frac{15}{8}$ read $-\frac{45}{16}$
" p. 554, " " $\sin(2Ent + 2gnt)\, \gamma^2me^2$ for 5 read $\frac{15}{2}$
" pp. 554, 579, " " " " $\frac{35}{64}$ " $\frac{195}{64}$
" p. 554, " " $\sin(2Ent + 2gnt)\, m^3\gamma^2$ for $-\frac{59}{48}$ read $-\frac{59}{24}$
" pp. 554, 579, " " " " 0 " $-\frac{59}{48}$
" p. 621, " " $\sin(2Ent + 2gnt)$ read $\{-3''{\cdot}218(4) - 1''{\cdot}960(5) - 0''{\cdot}6 \text{ (ind.)}\} = -5''{\cdot}778$
" p. 544, " " $\sin c'mnt\, \epsilon'm^7$ for $\frac{964444099}{55296}$ read $\frac{964445099}{55296}$
" p. 575, " " " " $\frac{964470235}{55296}$ " $\frac{964471235}{55296}$
" p. 559, " " $\sin(2Ent - 4cnt)\, me^4$ omit $\frac{15}{8}$
" pp. 559, 582, " " " " for $\frac{25}{4}$ read $\frac{35}{8}$
" p. 624, " " $\sin(2Ent - 4cnt)$ for $0''{\cdot}873$ read $0''{\cdot}611$
" pp. 581, 624, for $2Ent + 2c'mnt + 2cnt$ read $2Ent + 2c'mnt - 2cnt$
" p. 624, for $-0''{\cdot}190$ read $+0''{\cdot}190$

Tome I., p. 615, in coefficient of $\sin(g\nu - c\nu + c'm\nu)$ for $0''{\cdot}144$ read $0''{\cdot}431$

„ „ „ „ $\sin(2E\nu + g\nu)$ for $-1''{\cdot}059$ read $+1''{\cdot}059$

„ „ „ „ „ „ $-1''{\cdot}513$ „ $+0''{\cdot}605$

„ p. 637, „ „ $\cos c'm\nu\, \epsilon' m^4$ for $+\frac{1}{4}$ read $-\frac{1}{4}$

„ p. 638, „ „ $\cos 2E\nu\, mg^4$ add $-(-\frac{3}{128})$

„ „ „ „ „ for $\frac{3}{128}$ read 0

„ p. 667, „ „ $\cos 2Ent\, mg^4$ for $-\frac{3}{128}$ read 0

„ p. 640, „ „ $\cos(2E\nu - c'm\nu - 2g\nu)\, m^2\epsilon'\gamma^2$ for $\frac{7}{4}$ read $\frac{7}{2}$

„ p. 643, „ „ $\cos(4E\nu - c\nu)$ for $\frac{45}{256}m^2\gamma^2$ read $\frac{45}{256}m^3\gamma^2$

„ p. 645, „ „ $\cos(2E\nu - 2g\nu)$ read $-0''{\cdot}1552 - 0''{\cdot}0275 = -0''{\cdot}1827$

„ „ „ „ $\cos(2E\nu + 2g\nu)$ „ $0''{\cdot}0388 + 0''{\cdot}0063 = 0''{\cdot}0451$

„ p. 646, „ „ $\cos(2E\nu + 2g\nu - c\nu)$ read $0''{\cdot}0267 + 0''{\cdot}0084 = 0''{\cdot}0351$

„ „ „ „ $\cos(2E\nu - 2g\nu + c\nu)$ „ $-0''{\cdot}0907 + 0''{\cdot}0005 = -0''{\cdot}0902$

„ „ „ „ $\cos(2E\nu - c'm\nu - 2g\nu)$ read $+0''{\cdot}0305 - 0''{\cdot}0091 = 0''{\cdot}0214$

„ p. 647, „ „ $\cos(E\nu + c'm\nu)$ read $+0''{\cdot}1812 - 0''{\cdot}0610 + 0''{\cdot}0236 = +0''{\cdot}1438$

„ p. 648, „ „ $\cos(4E\nu - c\nu)$ read $-0''{\cdot}0920 - 0''{\cdot}0279 = -0''{\cdot}1199$

„ „ „ „ $\cos(4E\nu + c\nu)$ for $+0''{\cdot}0546$ read $+0''{\cdot}00546$

„ p. 651, „ „ $\cos(2g\nu + 2c\nu)\, e^2\gamma^2$ for $\frac{5}{8}$ read $\frac{1}{2}$

„ p. 665, „ „ $\cos(2gnt + 2cnt)\, e^2\gamma^2$ read $\frac{1}{2} - \frac{1}{2} = 0$

„ p. 675, „ „ $\cos(2gnt + 2cnt)$ for $+0''{\cdot}0104$ read 0

„ p. 656, „ „ $\cos(E\nu - c'm\nu)\, \epsilon' b^2 m^2$ for $-\frac{305}{32}$ read $-\frac{305}{64}$

„ p. 672, „ „ $\cos(Ent - c'mnt)\, \epsilon' b^2 m^2$ for $\frac{305}{32}$ read $\frac{305}{64}$

„ „ „ „ „ „ „ $\frac{641}{32}$ „ $\frac{977}{64}$

„ p. 662, „ „ $\sin(2E\nu + c'm\nu - 3c\nu)\, e^3\epsilon' m$ for $-\frac{45}{8}$ read $-\frac{45}{16}$

„ p. 671, „ „ $\cos(2Ent + c'mnt - 3cnt)\, e^3\epsilon' m$ read $-\frac{15}{8} + \frac{45}{64} + \frac{45}{16} = \frac{105}{64}$

„ p. 670, „ „ $\cos(2Ent + c'mnt + 2cnt)\, e^2\epsilon'$ for $-\frac{7}{2}$ read $-\frac{7}{4}$

„ p. 677, „ „ $\cos(2Ent + c'mnt + 2cnt)$ for $-0''{\cdot}0034$ read $-0''{\cdot}0017$

„ p. 671, „ „ $\cos(2Ent - 4cnt)$ for $\frac{15}{16}$ read $\frac{25}{16}$

„ p. 735, „ „ $2g - 2f$ for $-10''{\cdot}608$ read $-0''{\cdot}0608$

„ p. 748, „ „ „ „ „ „ „ in order to agree with p. 727

„ pp. 746, 756, in all terms with coefficient x^2, for + read −

Tome II., p. 76, in coefficient of $\cos(2E\nu - 2c\nu)\, me^4$ for $-\frac{45}{8}$ read $+\frac{45}{8}$, which is given correctly in Tome III., p. 847.

Tome III., p. 286, line 3, in coefficient of m^6 for $\frac{230401}{192}$ read $\frac{230401}{384}$

„ p. 289, line 2, „ „ „ for $-\frac{230401}{192}$ read $-\frac{230401}{384}$

„ pp. 381, 845, „ „ $\cos(c\nu + c'm\nu)\, e\epsilon' m^4$ for $-\frac{1351233}{4096}$ read $-\frac{1351333}{4096}$

„ p. 567, in coefficient of $\sin(c\nu + c'm\nu)\, e\epsilon' m^4$ for $\frac{2061269}{2048}$ read $\frac{1605365}{2048}$

„ p. 840, „ „ $\cos 2E\nu\gamma^2 m^4$ for $\frac{2047}{2048}$ read $\frac{2947}{2048}$

„ p. 841, „ „ $\cos(2E\nu - c'm\nu - 2g\nu)\, \epsilon'\gamma^2$ for $-\frac{7}{8}m$ read $+\frac{7}{8}m$

„ „ „ „ $\cos(2E\nu - c'm\nu - 2g\nu)\, \epsilon'^2\gamma^2$ for $-\frac{51}{32}m$ read $+\frac{51}{32}m$

„ p. 842, add the term $\cos(E\nu - c'm\nu + c\nu)\, e\epsilon'\gamma^2 b^2 (\frac{195}{64}m)$

„ p. 844, in coefficient of $\cos(2g\nu + c\nu)\, e\gamma^2 m^4$ for $-\frac{9711}{2048}$ read $-\frac{9903}{2048}$

„ p. 851, „ „ $\cos(2E\nu + 3c'm\nu - 2g\nu)\, \epsilon'^3\gamma^2$ for $-\frac{1}{128}m^2$ read $-\frac{1}{128}m$

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Tome III., p. 418, line 3 from bottom, for $+0{\cdot}0000263449$ read $+0{\cdot}000016349$

Tome IV., p. 248, line 7 from bottom, for $+\frac{33}{128}m^2$ read $+\frac{33}{128}m^3$

„ p. 263, in value of $\left(\delta\frac{1}{r}\right)^3$ omit $-\frac{215}{16}m^5e'^2$

„ p. 303, „ „ $\frac{1}{r^3} + \frac{m^2a'^3}{r'^3}$ for $+\frac{1055}{32}m^5e'^2$ read $+\frac{1485}{32}m^5e'^2$

„ „ „ „ „ for $\frac{153}{64}m^3e'^2$ read $\frac{153}{64}m^4e'^2$

„ p. 304, in coefficient of $\sin\eta\, m^5e'^2\gamma$ for $-\frac{10135}{64}$ read $-\frac{9275}{64}$

Tome IV., p. 306, in value of g^2 for $-\frac{80869}{512}m^5e'^2$ read $-\frac{73989}{512}m^5e'^2$

" p. 307, " " g " $-\frac{79753}{1024}m^5e'^2$ " $-\frac{72873}{1025}m^5e'^2$

" p. 281, " " a_8 and a_9 for m^2 read m

" p. 282, " " a_{34} for $\frac{15}{52}me^2e'$ read $\frac{15}{52}me^2e'^2$

" p. 303, " " $\frac{1}{r^3}+\frac{m^2a'^2}{r'^3}$ for $+\frac{153}{64}m^3e'^2$ read $+\frac{153}{64}m^4e'^2$

" p. 317, in coefficient of $\sin\phi' m^5e'\gamma^2$ for $-\frac{23209}{512}$ read $-\frac{10897}{512}$

" pp. 319, 344, " $\cos\phi' m^4e'\gamma^2$ " $+\frac{23209}{512}$ " $\frac{10897}{512}$

" p. 345, " " " " $+\frac{6837}{512}$ " $-\frac{5475}{512}$

" p. 352, in value of mb_6 for $-\frac{6785}{512}m^4$ read $-\frac{19097}{512}m^4$

" p. 354, " " b_6 " $-\frac{6785}{512}m^3$ " $-\frac{19097}{512}m^3$

" p. 573, in coefficient of $\sin\phi' m^3e'\gamma^2$ for $-\frac{6785}{512}$ read $-\frac{19097}{512}$

" p. 336, in value of a_{18} for $-\frac{m^2}{2}$ read $+\frac{m^2}{2}$

" " " " " for $+\frac{135}{32}m^2e^2$ read $+\frac{135}{32}me^2$

" p. 437, " " c for $\frac{2475}{128}m^3e^2e'^2$ read $\frac{2475}{64}m^3e^2e'^2$

" p. 439, " " q' for $-\frac{84517}{1024}m^5$ read $-\frac{79753}{1024}m^5$

" p. 566, in the Secular Equation of the nodes for $-\frac{84517}{1024}m^5$ read $-\frac{79753}{1024}m^5$ but this is not the true value.

" p. 509, in value of b_{44} for $\frac{779}{182}m^2$ read $\frac{779}{192}m^2$

" p. 534, " " A for $\frac{319}{3024}$ read $-\frac{5}{32}$

" p. 572, in coefficient of $\sin\phi\, em^4$ for $-\frac{26659}{256}$ read $-\frac{6659}{256}$

" p. 573, " " $\sin(\phi+\phi')\, ee'm^5$ for $-\frac{19962409}{2216}$ read $-\frac{19962409}{9216}$

" p. 588, " " $\sin(3\xi-\eta)$ for $-\frac{95}{192}$ read $-\frac{95}{128}$

" p. 603, " " $\sin(2\xi+3\phi)$ read $0''\cdot768+0''\cdot181=0''\cdot949$

" p. 620, " " $\sin(\xi+\phi')$ for $-13''\cdot7$ read $+13''\cdot7$

" p. 625, line 6 from bottom, for $2\xi-\phi-2\phi'$ read $2\xi-\phi+2\phi'$

15.

ADDENDUM TO PAPER ON THE SECULAR ACCELERATION OF THE MOON'S MEAN MOTION, No. 4, p. 120.

[SINCE the sheets of the paper No. 4 were printed off, I have found details of the calculations referred to on p. 127. Start with the equations

$$-\frac{d^2l}{dt^2}+\left(\frac{dl}{dt}\right)^2-\left(\frac{d\theta}{dt}\right)^2+\frac{1}{r^3}=\frac{1}{2}m^2\left\{1+\frac{3}{2}e'^2+3e'\cos mt\right\}$$

$$+\frac{3}{2}m^2\left\{\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2mt)+\frac{7}{2}e'\cos(2\theta-3mt)-\frac{1}{2}e'\cos(2\theta-mt)\right\},$$

$$-\frac{d^2\theta}{dt^2}+2\frac{dl}{dt}\frac{d\theta}{dt}=\frac{3}{2}m^2\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2\theta-2mt)\right.$$

$$\left.+\frac{7}{2}e'\sin(2\theta-3mt)-\frac{1}{2}e'\sin(2\theta-mt)\right\},$$

where $l=\log(1/r)$, and for simplicity n has been taken equal to unity, and $m=0{\cdot}0748013\,(1+f)$, where f is an undetermined small quantity equal to $-(1-m)\,\delta n'/n'$ of p. 111. We then proceed in the manner of Lecture VII. from an approximate numerical solution. In this way Adams obtained the numbers of p. 127 on Dec. 6, 1859. The work seems to have suggested to him the general method which he outlined in *Mon. Not.* XXXVIII.; see p. 104; for MSS. of various dates in 1860, 1861, 1863, exist which are preliminary to the developments of p. 108 *et seqq.* Among them are the numbers below, not included in those developments:—]

$$l = \log\frac{1}{r} = 0{\cdot}00089{,}40885, + 0{\cdot}00169{,}77375f + 0{\cdot}00101{,}3438e'^2$$
$$+ \{-0{\cdot}00692{,}91739{,}6e'\}\cos\phi'$$
$$+ \{0{\cdot}00717{,}98881, + 0{\cdot}01624{,}5654f - 0{\cdot}02411{,}1402e'^2\}\cos 2\xi$$
$$+ \{0{\cdot}03036{,}07348{,}e'\}\cos(2\xi - \phi')$$
$$+ \{-0{\cdot}00445{,}13839{,}5e'\}\cos(2\xi + \phi')$$
$$+ \{0{\cdot}00003{,}29131, + 0{\cdot}00014{,}9717f - 0{\cdot}00039{,}3997e'^2\}\cos 4\xi$$
$$+ \{0{\cdot}00027{,}99970{,}e'\}\cos(4\xi - \phi')$$
$$+ \{-0{\cdot}00004{,}06265{,}e'\}\cos(4\xi + \phi').$$

$$\theta - nt = \{-0{\cdot}19057{,}02812{,}e'\}\sin\phi'$$
$$+ \{0{\cdot}01021{,}13626{,}0 + 0{\cdot}02348{,}38286{,}f - 0{\cdot}03435{,}26286e'^2\}\sin 2\xi$$
$$+ \{0{\cdot}04396{,}36669{,}5e'\}\sin(2\xi - \phi')$$
$$+ \{-0{\cdot}00624{,}23804{,}1e'\}\sin(2\xi + \phi')$$
$$+ \{0{\cdot}00004{,}23730{,}4 + 0{\cdot}00019{,}38852{,}5f - 0{\cdot}00050{,}76714{,}e'^2\}\sin 4\xi$$
$$+ \{0{\cdot}00036{,}27788{,}6e'\}\sin(4\xi - \phi')$$
$$+ \{-0{\cdot}00005{,}20361{,}2e'\}\sin(4\xi + \phi').$$

[19 Sep. 1863.

16.

LAPLACE'S THEOREMS ON THE DEVELOPMENT OF FUNCTIONS IN SERIES.

LET
$$x = \xi + aP,$$
$$y = \eta + bQ,$$
$$z = \zeta + cR,$$
where P, Q, R are functions of x, y, z. It is required to develop a function of x, y, z, in powers of a, b, c.

We may prove that if D be the Jacobian

$$\frac{d(\xi,\ \eta,\ \zeta)}{d(x,\ y,\ z)} = 1 \Big/ \frac{d(x,\ y,\ z)}{d(\xi,\ \eta,\ \zeta)}$$
$$= 1 - a\frac{dP}{dx} - b\frac{dQ}{dy} - c\frac{dR}{dz} + bc\frac{d(Q,\ R)}{d(y,\ z)}$$
$$+ ca\frac{d(R,\ P)}{d(z,\ x)} + ab\frac{d(P,\ Q)}{d(x,\ y)} - abc\frac{d(P,\ Q,\ R)}{d(a,\ b,\ c)},$$

then for any function $F(x,\ y,\ z)$

$$\frac{d}{da}\left(\frac{F}{D}\right) = \frac{d}{d\xi}\left(\frac{PF}{D}\right), \quad \frac{d}{db}\left(\frac{F}{D}\right) = \frac{d}{d\eta}\left(\frac{QF}{D}\right), \quad \frac{d}{dc}\left(\frac{F}{D}\right) = \frac{d}{d\zeta}\left(\frac{RF}{D}\right),$$

and generally

$$\frac{d^{l+m+n}}{da^l db^m dc^n}\left(\frac{F}{D}\right) = \frac{d^{l+m+n}}{d\xi^l d\eta^m d\zeta^n}\left(P^l Q^m R^n \frac{F}{D}\right).$$

Hence by Maclaurin's Theorem

$$\frac{F}{D}=\Sigma\frac{a^l b^m c^n}{l!\,m!\,n!}\left[\frac{d^{l+m+n}}{da^l db^m dc^n}\left(\frac{F}{D}\right)\right]_{a=b=c=0}$$

$$=\Sigma\frac{a^l b^m c^n}{l!\,m!\,n!}\left[\frac{d^{l+m+n}}{d\xi^l d\eta^m d\zeta^n}\left(P^l Q^m R^n\frac{F}{D}\right)\right]_{a=b=c=0}$$

$$=\Sigma\frac{a^l b^m c^n}{l!\,m!\,n!}\,\frac{d^{l+m+n}}{d\xi^l d\eta^m d\zeta^n}(P^l Q^m R^n F),$$

when ξ, η, ζ replace x, y, z, in P, Q, R, F on the right. Now

$$F=\frac{F}{D}\left\{1-a\frac{dP}{dx}-\&\text{c.}\right\}.$$

Collect the developments of

$$F/D,\quad -aF\frac{dP}{dx}\bigg/D,\ \&\text{c.};$$

we have

$$F=\Sigma\frac{a^l b^m c^n}{l!\,m!\,n!}\,\frac{d^{l+m+n}}{d\xi^l d\eta^m d\zeta^n}\left[P^l Q^m R^n F\left\{1-a\frac{dP}{d\xi}-\ldots-abc\frac{d(P,\,Q,\,R)}{d(\xi,\,\eta,\,\zeta)}\right\}\right].$$

If we collect the coefficients of $a^l b^m c^n$ from the different terms, the result agrees with that given by Laplace, *Mémoires de l'Académie*, 1777.

[March, 1869.

PART II.

TERRESTRIAL MAGNETISM.

SECTION I.

USEFUL FORMULAE, CONNECTING LEGENDRE'S COEFFICIENTS, WHICH ARE EMPLOYED IN THE THEORY OF TERRESTRIAL MAGNETISM.

1. IF r be less than unity, and if

$$V=(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$$

be expanded in a series of ascending powers of r, then

$$V=P_0+P_1r+P_2r^2+\ldots+P_nr^n+\ldots=\Sigma(P_nr^n),$$

where P_0, P_1, &c. are functions of θ only, and are known as Legendre's Coefficients or as Zonal Surface Harmonics.

When r is greater than unity,

$$V=\frac{1}{r}\left(1-\frac{2}{r}\cos\theta+\frac{1}{r^2}\right)^{-\frac{1}{2}}=\frac{1}{r}\left[P_0+P_1\cdot\frac{1}{r}+\text{\&c.}+P_n\cdot\frac{1}{r^n}+\ldots\right]$$

$$=P_0\cdot\frac{1}{r}+P_1\cdot\frac{1}{r^2}+\ldots+P_n\cdot\frac{1}{r^{n+1}}+\ldots=\Sigma\left(P_n\cdot\frac{1}{r^{n+1}}\right).$$

Putting $\cos\theta=\mu$, we get from Laplace's equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\}+n(n+1)P_n=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

2. Let μ be a variable which changes between the limits -1 and 1, and let $f(\mu)$ be a function of μ of n dimensions, such that when multiplied by any function of lower dimensions the integral of the product taken between the limits -1 and 1 shall always vanish, so that

$$\int f(\mu)\,d\mu=0,\quad \int \mu f(\mu)\,d\mu=0,\quad \text{\&c.},\quad \int \mu^{n-1}f(\mu)\,d\mu=0,$$

the limits being -1 and 1 in each case.

31—2

Let the indefinite integral of $f(\mu)=f_1(\mu)$, the integral being supposed to vanish when $\mu=-1$; also let the indefinite integral of $f_1(\mu)$ on the same supposition be $f_2(\mu)$ and so on, till we come to

$$\int f_{n-1}(\mu)\,d\mu = f_n(\mu).$$

Then integrating by parts we have

$$\int f(\mu)\,d\mu = f_1(\mu),\quad \int \mu f(\mu)\,d\mu = \mu f_1(\mu) - f_2(\mu),$$

$$\int \mu^2 f(\mu)\,d\mu = \mu^2 f_1(\mu) - 2\mu f_2(\mu) + 2f_3(\mu),\quad \text{\&c.} = \text{\&c.},$$

$$\int \mu^{n-1} f(\mu)\,d\mu = \mu^{n-1} f_1(\mu) - (n-1)\,\mu^{n-2} f_2(\mu) + (n-1)(n-2)\,\mu^{n-3} f_3(\mu) + \dots$$
$$+ (-1)^{n-1}(n-1)(n-2)\dots 2\,.\,1 f_n(\mu),$$

all the integrals on the left-hand sides of equations being supposed to vanish when $\mu=-1$.

Now put $\mu=1$ in these integrals, and we have

$$f_1(\mu)=0,\quad f_2(\mu)=0,\ \dots f_n(\mu)=0,$$

so that $f_n(\mu)$ and its first $n-1$ differential coefficients vanish when $\mu=1$, as well as when $\mu=-1$.

Hence $f_n(\mu)$ is divisible both by $(1-\mu)^n$ and by $(1+\mu)^n$, and since it is of $2n$ dimensions in μ it must be of the form

$$c\,(1-\mu)^n(1+\mu)^n = c\,(1-\mu^2)^n.$$

Hence
$$f(\mu) = c\,\frac{d^n}{d\mu^n}(1-\mu^2)^n.$$

If c be chosen so that $f(\mu)=1$ when $\mu=1$, we have

$$c\,\frac{d^n}{d\mu^n}\left[(1-\mu)^n.\{2-(1-\mu)\}^n\right] = 2^n(-1)^n n\,.\,(n-1)\dots 2\,.\,1\,.\,c = 1\,;$$

therefore
$$c = \frac{(-1)^n}{2^n}\cdot\frac{1}{n!}\,;$$

or
$$f(\mu) = \frac{(-1)^n}{2^n}\cdot\frac{1}{n!}\cdot\frac{d^n}{d\mu^n}(1-\mu^2)^n$$

$$= \frac{1}{2^n.\,n!}\cdot\frac{d^n}{d\mu^n}(\mu^2-1)^n = \frac{1}{n!}\cdot\frac{d^n}{d\mu^n}\left(\frac{\mu^2-1}{2}\right)^n;$$

$f(\mu)$ is the coefficient of r^n in the development of

$$V=(1-2\mu r+r^2)^{-\frac{1}{2}}$$

in ascending powers of r.

3. Let $$(1-2\mu r+r^2)^{\frac{1}{2}}=1-rx,$$

then $$1-2\mu r+r^2=1-2rx+r^2x^2;$$

$$\therefore \quad 2\mu r=2rx+r^2(1-x^2),$$

or $$\mu=x+\frac{r}{2}(1-x^2);$$

$$\frac{d\mu}{dx}=1-rx,$$

so that $$\frac{dx}{d\mu}=\frac{1}{1-rx}=(1-2\mu r+r^2)^{-\frac{1}{2}}=V.$$

Since $$x=\mu-\frac{r}{2}(1-x^2),$$

we may develop x in terms of μ by Lagrange's theorem and get

$$x=\mu-\frac{r}{2}(1-\mu^2)+\frac{1}{1\,.\,2}\left(\frac{r}{2}\right)^2\frac{d}{d\mu}(1-\mu^2)^2-\&c.$$

Differentiating with respect to μ we get

$$(1-2\mu r+r^2)^{-\frac{1}{2}}=\frac{dx}{d\mu}=1-\frac{r}{2}\frac{d}{d\mu}(1-\mu^2)+\frac{1}{1\,.\,2}\left(\frac{r}{2}\right)^2\frac{d^2}{d\mu^2}(1-\mu^2)^2-\&c.$$

Hence the equation

$$f(\mu)=\frac{1}{2^n\,.\,n!}\cdot\frac{d^n}{d\mu^n}(\mu^2-1)^n=P_n \qquad \text{.......................(2)}$$

gives the value of the Legendre's coefficient.

4. Several convenient relations may be found between successive Legendre's coefficients, which are useful for determining or checking the values of these coefficients.

Thus $$V=(1-2\mu r+r^2)^{-\frac{1}{2}},$$

$$\log V=-\frac{1}{2}\log(1-2\mu r+r^2).$$

Differentiating we get

$$\frac{1}{V}\cdot\frac{dV}{dr}=\frac{\mu-r}{1-2\mu r+r^2}.$$

For V put $\Sigma(r^n . P_n)$, then

$$(1-2\mu r+r^2)\frac{d}{dr}\Sigma(r^n.P_n)+(r-\mu)\Sigma(r^n.P_n)=0.$$

The coefficient of r^n in this equation becomes

$$(n+1)P_{n+1}-2\mu nP_n+(n-1)P_{n-1}-\mu P_n+P_{n-1}.$$

Equating this to zero we get

$$(n+1)P_{n+1}-(2n+1)\mu P_n+nP_{n-1}=0 \quad\text{.........................(3).}$$

5. If we express the values of the Legendre's coefficients by means of equation (2) we get

$$2^n . n!\left\{P_{n+1}-P_{n-1}-(2n+1)\int P_n d\mu\right\}$$

$$=\frac{1}{2(n+1)}\cdot\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^{n+1}-2n\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^{n-1}-(2n+1)\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n,$$

$$=\frac{d^n}{d\mu^n}\{\mu(\mu^2-1)^n\}-2n\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^{n-1}-(2n+1)\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n,$$

$$=\frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^n+2n\mu^2(\mu^2-1)^{n-1}\}$$
$$-2n\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^{n-1}-(2n+1)\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n,$$

$$=\frac{d^{n-1}}{d\mu^{n-1}}\{(2n+1)(\mu^2-1)^n+2n(\mu^2-1)^{n-1}\}$$
$$-2n\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^{n-1}-(2n+1)\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n,$$

$=0$ identically.

Hence
$$P_{n+1}-P_{n-1}-(2n+1)\int P_n d\mu=0 \quad\text{.........................(4).}$$

Differentiating the left-hand side of (4) with respect to μ we get

$$\frac{dP_{n+1}}{d\mu}-\frac{dP_{n-1}}{d\mu}=(2n+1)P_n.$$

6. A very short and simple proof connects the relations expressed by equations (1) and (2)—

$$\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^{n+1}=\frac{d^n}{d\mu^n}\{2(n+1)\mu(\mu^2-1)^n\}=2(n+1)\frac{d^n}{d\mu^n}\{\mu(\mu^2-1)^n\}$$

$$=2(n+1)\left\{\mu\frac{d^n(\mu^2-1)^n}{d\mu^n}+n\frac{d^{n-1}(\mu^2-1)^n}{d\mu^{n-1}}\right\}.$$

Dividing by $2^{n+1}.(n+1)!$ we get from equation (2)

$$P_{n+1}=\mu P_n+n\int P_n d\mu,$$

where $\int P_n d\mu$ vanishes when $\mu=1$.

But we have also by Leibnitz' theorem

$$\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^{n+1}=\frac{d^{n+1}}{d\mu^{n+1}}\{(\mu^2-1)(\mu^2-1)^n\}$$

$$=(\mu^2-1)\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^n+(n+1)\,2\mu\frac{d^n}{d\mu^n}(\mu^2-1)^n+(n+1)\,n\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n.$$

Equating the above expressions for

$$\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^{n+1},$$

and omitting the term common to both sides, we get

$$(\mu^2-1)\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^n+(n+1)n.\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n=2(n+1)n.\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n,$$

or
$$(\mu^2-1)\frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^n=n(n+1)\frac{d^{n-1}}{d\mu^{n-1}}(\mu^2-1)^n.$$

Hence
$$(\mu^2-1)\frac{dP_n}{d\mu}=n(n+1)\int P_n d\mu,$$

or differentiating

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\}+n(n+1)P_n=0.$$

From the above equations we also obtain

$$P_{n+1}=\mu P_n+\frac{\mu^2-1}{n+1}\cdot\frac{dP_n}{d\mu},$$

which also is a simple and elegant formula.

7. Equation (3) gives

$$(n+1)P_{n+1}-(2n+1)\mu P_n+nP_{n-1}=0.$$

Differentiating this m times with respect to μ we have

$$(n+1)\frac{d^mP_{n+1}}{d\mu^m}=(2n+1)\left\{\mu\frac{d^mP_n}{d\mu^m}+m\frac{d^{m-1}P_n}{d\mu^{m-1}}\right\}-n\frac{d^mP_{n-1}}{d\mu^m}\text{..........}(\alpha),$$

but from equation (4),

$$(2n+1)\int P_n d\mu=P_{n+1}-P_{n-1};$$

therefore $$(2n+1)\frac{d^{m-1}P_n}{d\mu^{m-1}}=\frac{d^mP_{n+1}}{d\mu^m}-\frac{d^mP_{n-1}}{d\mu^m}\text{..........................}(5),$$

or $$m\frac{d^mP_{n+1}}{d\mu^m}=(2n+1)\,m\frac{d^{m-1}P_n}{d\mu^{m-1}}+m\frac{d^mP_{n-1}}{d\mu^m}\text{..........}(\beta).$$

Subtracting (β) from (α) we get

$$(n-m+1)\frac{d^mP_{n+1}}{d\mu^m}=(2n+1)\,\mu\frac{d^mP_n}{d\mu^m}-(n+m)\frac{d^mP_{n-1}}{d\mu^m}\text{..............}(6).$$

8. Let $$Q_n^m=(1-\mu^2)^{\frac{m}{2}}\frac{d^mP_n}{d\mu^m}.$$

Then multiplying by $(1-\mu^2)^{\frac{m}{2}}$ in equation (5), we get

$$Q_{n+1}^m-Q_{n-1}^m=(2n+1)\,(1-\mu^2)^{\frac{1}{2}}Q_n^{m-1}\text{ ....}(7),$$

or putting n for $\overline{n+1}$ we get

$$Q_n^m=Q_{n-2}^m+(2n-1)\,(1-\mu^2)^{\frac{1}{2}}Q_{n-1}^{m-1}\text{ ....}(7').$$

Also let $$H_n^m=\frac{1\,.\,2\,.\,3\,\ldots\,(n-m)}{1\,.\,3\,.\,5\,\ldots\,(2n-1)}\cdot Q_n^m.$$

Multiplying in equation (6) by $(1-\mu^2)^{\frac{m}{2}}$, we get

$$(n-m+1)\,Q_{n+1}^m=(2n+1)\,\mu Q_n^m-(n+m)\,Q_{n-1}^m\text{...............}(8).$$

Multiplying this equation by $\dfrac{1\,.\,2\,.\,3\,\ldots\,(n-m)}{1\,.\,3\,.\,5\,\ldots\,(2n+1)}$, we get

$$H_{n+1}^m=\mu H_n^m-\frac{(n-m)\,(n+m)}{(2n-1)\,(2n+1)}H_{n-1}^m\text{..........................}(9).$$

In the case when $m=0$,

$$H_{n+1}^0 = \mu H_n^0 - \frac{n^2}{(2n-1)(2n+1)} H_{n-1}^0.$$

Differentiating equation (8) with respect to μ, we have

$$(n-m+1)\frac{dQ_{n+1}^m}{d\mu} = (2n+1)\,\mu\frac{dQ_n^m}{d\mu} - (n+m)\frac{dQ_{n-1}^m}{d\mu} + (2n+1)\,Q_n^m.$$

9. From equation (1) we get

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\} + n(n+1)P_n = 0.$$

Differentiating $m-1$ times with regard to μ, we have

$$(1-\mu^2)\frac{d^{m+1}P_n}{d\mu^{m+1}} - 2\mu m\frac{d^mP_n}{d\mu^m} + (n-m+1)(n+m)\frac{d^{m-1}P_n}{d\mu^{m-1}} = 0.$$

Multiplying by $(1-\mu^2)^{\frac{m-1}{2}}$ we get

$$Q_n^{m+1} - 2\mu m(1-\mu^2)^{-\frac{1}{2}}Q_n^m + (n-m+1)(n+m)Q_n^{m-1} = 0 \quad \text{.............(10).}$$

Also we get from the same equation by multiplying by $(1-\mu^2)^{m-1}$,

$$(1-\mu^2)^m\frac{d^{m+1}P_n}{d\mu^{m+1}} - 2\mu m(1-\mu^2)^{m-1}\frac{d^mP_n}{d\mu^m}$$
$$+ (n-m+1)(n+m)(1-\mu^2)^{m-1}\frac{d^{m-1}P_n}{d\mu^{m-1}} = 0,$$

or $$\frac{d}{d\mu}\left\{(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}\right\} + (n-m+1)(n+m)(1-\mu^2)^{m-1}\frac{d^{m-1}P_n}{d\mu^{m-1}} = 0.$$

Hence integrating with respect to μ, we get

$$(n-m+1)(n+m)\int(1-\mu^2)^{m-1}\frac{d^{m-1}P_n}{d\mu^{m-1}}\,d\mu = -(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}.$$

Giving to m the several values 2, 3, &c. we get by integration:

$$(n-1)(n+2)\int(1-\mu^2)\frac{dP_n}{d\mu}\,d\mu = -(1-\mu^2)^2\frac{d^2P_n}{d\mu^2},$$

$$(n-2)(n+3)\int(1-\mu^2)^2\frac{d^2P_n}{d\mu^2}\,d\mu = -(1-\mu^2)^3\frac{d^3P_n}{d\mu^3}, \text{ \&c.}$$

Hence, employing $\int^m P_n d\mu^m$ to express the mth integral of P_n with regard to μ, we get by successive substitution

$$(-1)^m (n+m)(n+m-1) \dots (n-m+2)(n-m+1) \int^m P_n d\mu^m = (1-\mu^2)^m \frac{d^m P_n}{d\mu^m}.$$

10. Let
$$(1-\mu^2)^m \frac{d^m P_n}{d\mu^m} = S^m.$$

Then
$$\frac{dS^m}{d\mu} + (n-m+1)(n+m) S^{m-1} = 0.$$

Putting $m+1$ for m we have

$$\frac{dS^{m+1}}{d\mu} + (n-m)(n+m+1) S^m = 0.$$

Now
$$\frac{dS^m}{d\mu} = (1-\mu^2)^m \frac{d^{m+1} P_n}{d\mu^{m+1}} - 2m\mu (1-\mu^2)^{m-1} \frac{d^m P_n}{d\mu^m};$$

therefore
$$(1-\mu^2) \frac{dS^m}{d\mu} + 2m\mu S^m = S^{m+1},$$

and
$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dS^m}{d\mu} + 2m\mu S^m\right\} + (n-m)(n+m+1) S^m = 0,$$

or
$$(1-\mu^2)\frac{d^2 S^m}{d\mu^2} + 2\mu(m-1)\frac{dS^m}{d\mu} + (n+m)(n-m+1) S^m = 0.$$

Multiplying by $(1-\mu^2)^{-m}$ we get

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{-m+1}\frac{dS^m}{d\mu}\right\} + (n+m)(n-m+1)(1-\mu^2)^{-m} S^m = 0,$$

the differential equation for S^m.

11. From Art. 9 it appears that S_m and $\int^m P_n d\mu_m$ differ by merely a constant multiplier, so that the differential equation for $\int^m P_n d\mu^m$ is of the same form as the differential equation for S_m, hence we get

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{-m+1}\frac{d}{d\mu}\left[\int^m P_n d\mu^m\right]\right\} + (n+m)(n-m+1)(1-\mu^2)^{-m}\int^m P_n d\mu^m = 0.$$

When $m=n$, the equation

$$\frac{dS^{m+1}}{d\mu}+(n-m)(n+m+1)S^m=0 \quad \text{(see Art. 10)}$$

fails to give any value of S^m or of $\frac{d^mP_n}{d\mu^m}$, since the factor $n-m$ then vanishes. Hence we have

$$(1-\mu^2)^{n+1}\frac{d^{n+1}P_n}{d\mu^{n+1}}=\text{constant}.$$

This constant $=0$, since P_n is of the nth order.

Also when $m=n+1$, the factor $n-m+1$ vanishes and we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{-n}\int^n P_n d\mu^n\right\}=0,$$

or

$$\int^n P_n d\mu^n = c\,(1-\mu^2)^n,$$

and

$$P_n = c\,\frac{d^n(1-\mu^2)^n}{d\mu^n}.$$

The value of c is given above in Art. 2.

12. We have seen above that

$$(1-\mu^2)\frac{d^{m+1}P_n}{d\mu^{m+1}}-2\mu m\frac{d^mP_n}{d\mu^m}+(n-m+1)(n+m)\frac{d^{m-1}P_n}{d\mu^{m-1}}=0\,;$$

therefore by means of formulae (6) and (5) above, we get

$$(2n+1)(1-\mu^2)\frac{d^{m+1}P_n}{d\mu^{m+1}}=2m(2n+1)\mu\frac{d^mP_n}{d\mu^m}-(2n+1)(n-m+1)(n+m)\frac{d^{m-1}P_n}{d\mu^{m-1}}$$

$$=2m(n-m+1)\frac{d^mP_{n+1}}{d\mu^m}+2m(n+m)\frac{d^mP_{n-1}}{d\mu^m}$$

$$-(n-m+1)(n+m)\frac{d^mP_{n+1}}{d\mu^m}+(n-m+1)(n+m)\frac{d^mP_{n-1}}{d\mu^m}$$

$$=-(n-m)(n-m+1)\frac{d^mP_{n+1}}{d\mu^m}+(n+m)(n+m+1)\frac{d^mP_{n-1}}{d\mu^m}.$$

Multiply by $(1-\mu^2)^{\frac{m}{2}}$ and we get

$$(2n+1)(1-\mu^2)^{\frac{1}{2}} Q_n^{m+1} = -(n-m)(n-m+1)\, Q_{n+1}^m$$
$$+(n+m)(n+m+1)\, Q_{n-1}^m \text{.................}(11).$$

If P_n contain only even or only odd powers of μ, and if $\int P_n d\mu$ vanishes when $\mu=1$, then it also vanishes when $\mu=-1$, and $\int P_n d\mu$ will contain only odd or even powers of μ respectively.

First suppose n to be even and therefore P_n to contain only even powers of μ, then $\int P_n d\mu$ will contain odd powers of μ together with a constant term, but since $\int P_n d\mu$ vanishes both when $\mu=1$ and when $\mu=-1$ this constant is zero, because the value of the remaining terms, if finite, would change sign when μ is changed from 1 to -1.

Hence $\int P_n d\mu$ contains only odd powers of μ and is divisible by $(1-\mu^2)$.

Secondly, suppose n to be odd and P_n to contain only odd powers of μ, then $\int P_n d\mu$ will contain even powers of μ with a constant term and will be divisible by $(1-\mu^2)$.

13. We have taken $\quad Q_n^m = \dfrac{d^m P_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}}$,

hence
$$\frac{dQ_n^m}{d\mu} = \frac{d^{m+1} P_n}{d\mu^{m+1}}(1-\mu^2)^{\frac{m}{2}} - m\mu\frac{d^m P_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}-1}$$

and
$$\frac{d^2 Q_n^m}{d\mu^2} = \frac{d^{m+2} P_n}{d\mu^{m+2}}(1-\mu^2)^{\frac{m}{2}} - 2m\mu\frac{d^{m+1} P_n}{d\mu^{m+1}}(1-\mu^2)^{\frac{m}{2}-1}$$
$$+m(m-2)\mu^2\frac{d^m P_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}-2} - m\frac{d^m P_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}-1}.$$

From these equations we get

$$(1-\mu^2)\frac{d^2Q_n^m}{d\mu^2} - 2\mu\frac{dQ_n^m}{d\mu} = \frac{d^{m+2}P_n}{d\mu^{m+2}}(1-\mu^2)^{\frac{m}{2}+1}$$

$$-2\mu(m+1)\frac{d^{m+1}P_n}{d\mu^{m+1}}(1-\mu^2)^{\frac{m}{2}} + m^2\mu^2\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}-1} - m\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}}$$

$$= \frac{d^{m+2}P_n}{d\mu^{m+2}}(1-\mu^2)^{\frac{m}{2}+1} - 2\mu(m+1)\frac{d^{m+1}P_n}{d\mu^{m+1}}(1-\mu^2)^{\frac{m}{2}}$$

$$+ m^2\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}-1} - m(m+1)\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}}$$

$$= m^2\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}-1} - n(n+1)\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}},$$

or
$$(1-\mu^2)\frac{d^2Q_n^m}{d\mu^2} - 2\mu\frac{dQ_n^m}{d\mu} + n(n+1)Q_n^m - \frac{m^2}{1-\mu^2}Q_n^m = 0,$$

i.e.
$$(1-\mu^2)\frac{d^2Q_n^m}{d\mu^2} - 2\mu\frac{dQ_n^m}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}Q_n^m = 0 \text{............} (12),$$

the differential equation for Q_n^m.

Since
$$Q_n^m = \frac{1\,.\,3\,.\,5\ \ldots\ (2n-1)}{(n-m)!}H_n^m$$

equation (12) gives

$$(1-\mu^2)\frac{d^2H_n^m}{d\mu^2} - 2\mu\frac{dH_n^m}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}H_n^m = 0,$$

or
$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dH_n^m}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}H_n^m = 0.$$

Hence if we have a function of μ and λ of the form $R = Q_n^m\cos(m\lambda+\alpha)$ where m is a positive integer, we shall have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR}{d\mu}\right\} + n(n+1)R - \frac{m^2}{1-\mu^2}R = 0,$$

but
$$\frac{d^2R}{d\lambda^2} = -m^2R;$$

hence $$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR}{d\mu}\right\}+n(n+1)R+\frac{1}{1-\mu^2}\frac{d^2R}{d\lambda^2}=0,$$

where $$R=(1-\mu^2)^{\frac{m}{2}}\frac{d^mP_n}{d\mu^m}\cos(m\lambda+\alpha).$$

14. Since $$Q_n^m=\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}},$$

we have $$\frac{dQ_n^m}{d\mu}(1-\mu^2)^{\frac{1}{2}}=\frac{d^{m+1}P_n}{d\mu^{m+1}}(1-\mu^2)^{\frac{m+1}{2}}-m\mu\frac{d^mP_n}{d\mu^m}(1-\mu^2)^{\frac{m-1}{2}}.$$

Combining this with equation (10), we get

$$\frac{dQ_n^m}{d\mu}(1-\mu^2)^{\frac{1}{2}}=\frac{1}{2}\frac{d^{m+1}P_n}{d\mu^{m+1}}(1-\mu^2)^{\frac{m+1}{2}}-\frac{1}{2}(n-m+1)(n+m)\frac{d^{m-1}P_n}{d\mu^{m-1}}(1-\mu^2)^{\frac{m-1}{2}}$$

$$=\frac{1}{2}Q_n^{m+1}-\frac{1}{2}(n-m+1)(n+m)Q_n^{m-1}\quad\text{.........................(13)},$$

a very simple formula for expressing $\frac{dQ_n^m}{d\mu}(1-\mu^2)^{\frac{1}{2}}$ in terms of Q_n^{m+1} and Q_n^{m-1}.

15. Multiplying equation (4) by n and adding to equation (3), we get

$$(2n+1)P_{n+1}=(2n+1)\mu P_n+n(2n+1)\int P_n d\mu,$$

or $$P_{n+1}=\mu P_n+n\int P_n d\mu.$$

Multiplying equation (4) by $(n+1)$ and subtracting from equation (3), we get, similarly,

$$P_{n-1}=\mu P_n-(n+1)\int P_n d\mu.$$

Hence differentiating we get

$$\frac{dP_{n+1}}{d\mu}=\mu\frac{dP_n}{d\mu}+(n+1)P_n \text{ and } \frac{dP_{n-1}}{d\mu}=\mu\frac{dP_n}{d\mu}-nP_n,$$

therefore $$n\frac{dP_{n+1}}{d\mu}+(n+1)\frac{dP_{n-1}}{d\mu}=(2n+1)\mu\frac{dP_n}{d\mu}\quad\text{.................(14)}.$$

16. Differentiating the expressions for P_{n+1} and P_{n-1} (with respect to μ) m times, we get

$$\frac{d^m P_{n+1}}{d\mu^m} = \mu \frac{d^m P_n}{d\mu^m} + (m+n) \frac{d^{m-1} P_n}{d\mu^{m-1}},$$

and
$$\frac{d^m P_{n-1}}{d\mu^m} = \mu \frac{d^m P_n}{d\mu^m} - (n-m+1) \frac{d^{m-1} P_n}{d\mu^{m-1}}.$$

Multiplying by $(1-\mu^2)^{\frac{m}{2}}$ we have

$$Q_{n+1}^m = \mu (1-\mu^2)^{\frac{m}{2}} \frac{d^m P_n}{d\mu^m} + (m+n)(1-\mu^2)^{\frac{m}{2}} \frac{d^{m-1} P_n}{d\mu^{m-1}},$$

and
$$Q_{n-1}^m = \mu (1-\mu^2)^{\frac{m}{2}} \frac{d^m P_n}{d\mu^m} - (n-m+1)(1-\mu^2)^{\frac{m}{2}} \frac{d^{m-1} P_n}{d\mu^{m-1}};$$

or
$$Q_{n+1}^m = \mu Q_n^m + (m+n)(1-\mu^2)^{\frac{1}{2}} Q_n^{m-1},$$

and
$$Q_{n-1}^m = \mu Q_n^m - (n-m+1)(1-\mu^2)^{\frac{1}{2}} Q_n^{m-1}.$$

Hence, combining these two equations, we get

$$Q_{n+1}^m = \frac{2n+1}{n-m+1} \mu Q_n^m - \frac{n+m}{n-m+1} Q_{n-1}^m \text{..................(15)},$$

whence the quantities Q_{n+1}^m, &c., may be successively determined.

17. Putting
$$Q_n^m = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots (n-m)} H_n^m$$

in equation (13), we have

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots (n-m)} (1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu}$$
$$= \frac{1}{2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots (n-m-1)} H_n^{m+1}$$
$$- \frac{1}{2} (n-m+1)(n+m) \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{1 \cdot 2 \cdot 3 \ldots (n-m+1)} H_n^{m-1},$$

or
$$(1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu} = \frac{n-m}{2} H_n^{m+1} - \frac{n+m}{2} H_n^{m-1} \text{..................(16)},$$

which is a very elegant formula for finding

$$(1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu};$$

but when $m=0$,

$$(1-\mu^2)^{\frac{1}{2}} \frac{dH_n^0}{d\mu} = nH_n^1.$$

Making the same substitution in equation (7′), we have

$$\frac{1.3.5 \ldots (2n-1)}{1.2.3 \ldots (n-m)} (1-\mu^2)^{-\frac{1}{2}} H_n^m$$

$$= \frac{1.3.5 \ldots (2n-5)}{1.2.3 \ldots (n-m-2)} (1-\mu^2)^{-\frac{1}{2}} H_{n-2}^m + (2n-1) \frac{1.3.5 \ldots (2n-3)}{1.2.3 \ldots (n-m)} H_{n-1}^{m-1},$$

or $$(1-\mu^2)^{-\frac{1}{2}} H_n^m = \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} (1-\mu^2)^{-\frac{1}{2}} H_{n-2}^m + H_{n-1}^{m-1} \quad \ldots\ldots(17),$$

which is a convenient formula.

In order to complete the determination of the quantities H_n^m, &c. we must remember that, when m is greater than n, the quantity H_n^m vanishes; when $m=n$ we have

$$H_m^m = (1-\mu^2)^{\frac{m}{2}},$$

and when $n=m+1$, we have

$$H_{m+1}^m = \mu (1-\mu^2)^{\frac{m}{2}}.$$

Also $$(1-\mu^2)^{\frac{1}{2}} \frac{dH_m^m}{d\mu} = -m\mu (1-\mu^2)^{\frac{m-1}{2}},$$

and

$$(1-\mu^2)^{\frac{1}{2}} \frac{dH_{m+1}^m}{d\mu} = (1-\mu^2)^{\frac{m+1}{2}} - m\mu^2 (1-\mu^2)^{\frac{m-1}{2}} = (m+1)(1-\mu^2)^{\frac{m+1}{2}} - m(1-\mu^2)^{\frac{m-1}{2}}.$$

18. Let $$G_n^m (1-\mu^2)^{\frac{m}{2}} = H_n^m.$$

Then from equation (9) we get

$$G_{n+1}^m = \mu G_n^m - \frac{(n-m)(n+m)}{(2n-1)(2n+1)} G_{n-1}^m \quad \ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Also from equation (17) we get

$$G_n^m = \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} G_{n-2}^m + G_{n-1}^{m-1},$$

or putting $(n+1)$ for n and $(m+1)$ for m, we get

$$G_{n+1}^{m+1} = G_n^m + \frac{(n-m-1)(n-m)}{(2n-1)(2n+1)} G_{n-1}^{m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

We have

$$G_n^n = 1, \quad G_{n+1}^n = \mu, \text{ and } G_{n+1}^{n-1} = G_n^{n-2} + \frac{1 \,.\, 2}{(2n-1)(2n+1)}.$$

Equation (10) gives us

$$Q_n^{m+1} - 2m\mu(1-\mu^2)^{-\frac{1}{2}} Q_n^m + (n-m+1)(n+m) Q_n^{m-1} = 0.$$

Hence since

$$Q_n^m = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{1 \,.\, 2 \,.\, 3 \ldots (n-m)} H_n^m,$$

we get

$$(n-m) H_n^{m+1} - 2m\mu(1-\mu^2)^{-\frac{1}{2}} H_n^m + (n+m) H_n^{m-1} = 0 \quad \ldots\ldots\ldots (20).$$

Again since $$H_n^m = G_n^m (1-\mu^2)^{\frac{m}{2}},$$

we get $$(n-m)(1-\mu^2) G_n^{m+1} - 2m\mu G_n^m + (n+m) G_n^{m-1} = 0 \quad \ldots\ldots\ldots\ldots (21).$$

Putting $m=0$, we get

$$(1-\mu^2) G_n^1 + G_n^{-1} = 0.$$

We have also (see Art. 14)

$$(1-\mu^2)^{\frac{1}{2}} \frac{dQ_n^m}{d\mu} = \frac{d^{m+1} P_n}{d\mu^{m+1}} (1-\mu^2)^{\frac{m+1}{2}} - m\mu \frac{d^m P_n}{d\mu^m} (1-\mu^2)^{\frac{m-1}{2}};$$

therefore $$(1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu} = (n-m) H_n^{m+1} - m\mu(1-\mu^2)^{-\frac{1}{2}} H_n^m \quad \ldots\ldots\ldots\ldots\ldots (22);$$

hence from equation (16)

$$(1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu} = m\mu(1-\mu^2)^{-\frac{1}{2}} H_n^m - (n+m) H_n^{m-1}.$$

19. We have seen that

$$P_n = \frac{1}{2^n . 1 . 2 . 3 \dots n} \frac{d^n (\mu^2 - 1)^n}{d\mu^n}.$$

The coefficient of μ^n in this expression is

$$\frac{2n(2n-1)(2n-2)\dots(n+1)}{2^n . 1 . 2 . 3 \dots n} = \frac{2n(2n-1)\dots 3 . 2 . 1}{2^n (1 . 2 . 3 \dots n)^2} = \frac{1 . 3 . 5 \dots (2n-1)}{1 . 2 . 3 \dots n}.$$

Hence the coefficient of μ^{n-m} in $\frac{d^m P_n}{d\mu^m}$ is

$$n(n-1)\dots(n-m+1) \times \frac{1 . 3 . 5 \dots (2n-1)}{1 . 2 . 3 \dots n} = \frac{1 . 3 . 5 \dots (2n-1)}{1 . 2 . 3 \dots (n-m)}.$$

Also the coefficient of μ^{n-m} in the value of G_n^m is unity.

20.
$$2^n n! P_n = \frac{d^n}{d\mu^n} (\mu^2 - 1)^n.$$

Differentiating m times with respect to μ, then

$$2^n n! \frac{d^m P_n}{d\mu^m} = \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n,$$

hence
$$2^n n! Q_n^m = (1 - \mu^2)^{\frac{m}{2}} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n,$$

or
$$\frac{(2n)!}{1 . 3 . 5 \dots (2n-1)} Q_n^m = (1 - \mu^2)^{\frac{m}{2}} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n;$$

therefore
$$\frac{(2n)!}{(n-m)!} H_n^m = (1 - \mu^2)^{\frac{m}{2}} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n,$$

or
$$H_n^m = \frac{(n-m)!}{(2n)!} (1 - \mu^2)^{\frac{m}{2}} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n \quad \dots\dots\dots\dots\dots(23),$$

or
$$H_n^m = \left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)} \mu^{n-m-2} \right.$$
$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 . 4 . (2n-1)(2n-3)} \mu^{n-m-4} - \&c. \right\} (1 - \mu^2)^{\frac{m}{2}}.$$

This is the function represented by $P^{n,m}$ in Gauss's *Allgemeine Theorie des Erdmagnetismus.* (See Gauss's *Werke,* Band v. p. 142.)

Also
$$G_n^m = \mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4.(2n-1)(2n-3)}\mu^{n-m-4} - \&c.,$$

and
$$\frac{1.3.5\ldots(2n-1)}{(n-m)!}.G_n^m = \frac{d^mP_n}{d\mu^m}.$$

21. When $\mu=1$, we have
$$P_n = 1, \text{ and } \frac{d^mP_n}{d\mu^m} = \frac{(n+m)!}{(n-m)!\,m!}.\frac{1}{2^m},$$

and in particular
$$\frac{dP_n}{d\mu} = \frac{n(n+1)}{2}.$$

Hence, when $\mu=1$, H_n^0 or
$$G_n^0 = \frac{1.2.3\ldots n}{1.3.5\ldots(2n-1)}P_n$$
$$= \frac{1.2.3\ldots n}{1.3.5\ldots(2n-1)} = \frac{(n!)^2\,2^n}{(2n)!}.$$

And
$$G_n^m = \frac{1.2.3\ldots(n-m)}{1.3.5\ldots(2n-1)}\frac{d^mP_n}{d\mu^m} = \frac{(n-m)!\,n!\,2^n}{(2n)!}.\frac{(n+m)!}{(n-m)!\,m!\,2^m}$$
$$= \frac{2^{n-m}(n+m)!\,n!}{(2n)!\,m!} = \frac{(n+m)!}{2^m\,m!\,1.3.5\ldots(2n-1)}.$$

And in particular when $m=1$,
$$G_n^1 = \frac{2^{n-1}(n+1)!\,n!}{(2n)!} = \frac{n+1}{2}.\frac{2^n(n!)^2}{(2n)!} = \frac{n+1}{2}.G_n^0 = \frac{1}{2}.\frac{1.2.3\ldots(n+1)}{1.3.5\ldots(2n-1)}.$$

22. Again, when $\mu=0$, first, suppose $m=0$ and $n=2r$.

Then the coefficient of μ^n in $(\mu^2-1)^n$ is
$$\frac{n(n-1)\ldots(r+1)}{1.2.3\ldots r}(-1)^r;$$

therefore the constant term in P_n is

$$\frac{1}{2^n\, n!} \cdot \frac{n!\,(-1)^r\, n\,(n-1)\,\dots\,(r+1)}{1\,.\,2\,.\,3\,\dots\,r} = \frac{(-1)^r\, n!}{2^n\,(r!)^2}$$

$$= \frac{(-1)^{\frac{n}{2}}\, n!}{(2\,.\,4\,.\,6\,\dots\,n)^2} = \frac{(-1)^{\frac{n}{2}}\, 1\,.\,3\,.\,5\,\dots\,(n-1)}{2\,.\,4\,.\,6\,\dots\,n},$$

or when $\mu=0$, the value of G_n^0 is

$$\frac{1\,.\,2\,.\,3\,\dots\,n}{1\,.\,3\,.\,5\,\dots\,(2n-1)} P_n = \frac{(-1)^{\frac{n}{2}}\{1\,.\,3\,.\,5\,\dots\,(n-1)\}^2}{1\,.\,3\,.\,5\,\dots\,(2n-1)}.$$

Next, let $n-m=2r$, so that

$$n+m=2\,(n-r).$$

Then the coefficient of μ^{n+m} in $(\mu^2-1)^n$ is

$$\frac{n\,(n-1)\,\dots\,(n-r+1)}{1\,.\,2\,.\,3\,\dots\,r}(-1)^r;$$

therefore when $\mu=0$, the value of

$$\frac{d^m P_n}{d\mu^m} = \frac{1}{2^n\, n!}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n = \frac{(n+m)!}{2^n\, n!} \cdot \frac{n\,(n-1)\,\dots\,(n-r+1)}{1\,.\,2\,.\,3\,\dots\,r}(-1)^r.$$

And $$G_n^m = \frac{(n-m)!}{1\,.\,3\,.\,5\,\dots\,(2n-1)}\frac{d^m P_n}{d\mu^m}$$

$$= (-1)^{\frac{n-m}{2}} \frac{1\,.\,3\,.\,5\,\dots\,(n-m-1)\,.\,1\,.\,3\,.\,5\,\dots\,(n+m-1)}{1\,.\,3\,.\,5\,\dots\,(2n-1)};$$

when $m=0$, this reduces to

$$G_n^0 = (-1)^{\frac{n}{2}} \frac{\{1\,.\,3\,.\,5\,\dots\,(n-1)\}^2}{1\,.\,3\,.\,5\,\dots\,(2n-1)},$$

as before.

The same values for G_n^m and G_n^0, when $\mu=0$ and $n-m=2r$, may also be obtained from equation (18), thus

$$G_{m+2r}^m=(-1)^r.\frac{1\,(2m+1)}{(2m+1)\,(2m+3)}.\frac{3\,(2m+3)}{(2m+5)\,(2m+7)}\cdots\frac{(2r-1)\,(2m+2r-1)}{(2m+4r-3)\,(2m+4r-1)}$$

$$=(-1)^r.\frac{1.3.5\ldots(2r-1)\{(2m+1)\,(2m+3)\ldots(2m+2r-1)\}}{(2m+1)\,(2m+3)\ldots(2m+4r-1)}$$

$$=(-1)^{\frac{n-m}{2}}.\frac{1.3.5\ldots(n-m-1).1.3.5\ldots(n+m-1)}{1.3.5\ldots(2n-1)}.$$

23. We have seen above (in Art. 16) that

$$Q_{n+1}^m=\mu\,(1-\mu^2)^{\frac{m}{2}}\frac{d^mP_n}{d\mu^m}+(m+n)\,(1-\mu^2)^{\frac{m}{2}}\frac{d^{m-1}P_n}{d\mu^{m-1}},$$

and
$$Q_{n-1}^m=\mu\,(1-\mu^2)^{\frac{m}{2}}\frac{d^mP_n}{d\mu^m}-(n-m+1)\,(1-\mu^2)^{\frac{m}{2}}\frac{d^{m-1}P_n}{d\mu^{m-1}},$$

therefore

$$\frac{dQ_{n+1}^m}{d\mu}=\mu\,(1-\mu^2)^{\frac{m}{2}}\frac{d^{m+1}P_n}{d\mu^{m+1}}+\{(1-\mu^2)^{\frac{m}{2}}-m\mu^2\,(1-\mu^2)^{\frac{m}{2}-1}+(m+n)\,(1-\mu^2)^{\frac{m}{2}}\}\frac{d^mP_n}{d\mu^m}.$$
$$-m\,(m+n)\,\mu\,(1-\mu^2)^{\frac{m}{2}-1}\frac{d^{m-1}P_n}{d\mu^{m-1}}.$$

But $(1-\mu^2)\dfrac{d^{m+1}P_n}{d\mu^{m+1}}-2m\mu\dfrac{d^mP_n}{d\mu^m}+(n+m)\,(n-m+1)\dfrac{d^{m-1}P_n}{d\mu^{m-1}}=0,$

therefore

$$\mu\,(1-\mu^2)^{\frac{m}{2}}\frac{d^{m+1}P_n}{d\mu^{m+1}}-2m\mu^2\,(1-\mu^2)^{\frac{m}{2}-1}\frac{d^mP_n}{d\mu^m}$$
$$+(n+m)\,(n-m+1)\,\mu\,(1-\mu^2)^{\frac{m}{2}-1}\frac{d^{m-1}P_n}{d\mu^{m-1}}=0.$$

Combining these equations, we see that

$$\frac{dQ_{n+1}^m}{d\mu}=\{(1-\mu^2)+m\mu^2+(m+n)\,(1-\mu^2)\}\,(1-\mu^2)^{\frac{m}{2}-1}\frac{d^mP_n}{d\mu^m}$$
$$-(n+m)\,(n+1)\,\mu\,(1-\mu^2)^{\frac{m}{2}-1}\frac{d^{m-1}P_n}{d\mu^{m-1}}.$$

Similarly

$$\frac{dQ^m_{n-1}}{d\mu} = \mu(1-\mu^2)^{\frac{m}{2}} \frac{d^{m+1}P_n}{d\mu^{m+1}} + \{(1-\mu^2)^{\frac{m}{2}} - m\mu^2(1-\mu^2)^{\frac{m}{2}-1} - (n-m+1)(1-\mu^2)^{\frac{m}{2}}\} \frac{d^m P_n}{d\mu^m}$$

$$+ m(n-m+1)\mu(1-\mu^2)^{\frac{m}{2}-1} \frac{d^{m-1}P_n}{d\mu^{m-1}},$$

and reducing as before

$$\frac{dQ^m_{n-1}}{d\mu} = \{(1-\mu^2) + m\mu^2 - (n-m+1)(1-\mu^2)\}(1-\mu^2)^{\frac{m}{2}-1} \frac{d^m P_n}{d\mu^m}$$

$$- n(n-m+1)\mu(1-\mu^2)^{\frac{m}{2}-1} \frac{d^{m-1}P_n}{d\mu^{m-1}}.$$

Again, $$\frac{dQ^m_n}{d\mu} = (1-\mu^2)^{\frac{m}{2}} \frac{d^{m+1}P_n}{d\mu^{m+1}} - m\mu(1-\mu^2)^{\frac{m}{2}-1} \frac{d^m P_n}{d\mu^m},$$

or reducing by the expression

$$(1-\mu^2)^{\frac{m}{2}} \frac{d^{m+1}P_n}{d\mu^{m+1}} - 2m\mu(1-\mu^2)^{\frac{m}{2}-1} \frac{d^m P_n}{d\mu^m} + (n+m)(n-m+1)(1-\mu^2)^{\frac{m}{2}-1} \frac{d^{m-1}P_n}{d\mu^{m-1}} = 0,$$

we get

$$\frac{dQ^m_n}{d\mu} = m\mu(1-\mu^2)^{\frac{m}{2}-1} \frac{d^m P_n}{d\mu^m} - (n+m)(n-m+1)(1-\mu^2)^{\frac{m}{2}-1} \frac{d^{m-1}P_n}{d\mu^{m-1}}.$$

Test of the above,

$$(n-m+1)\frac{dQ^m_{n+1}}{d\mu} + (n+m)\frac{dQ^m_{n-1}}{d\mu} - (2n+1)\mu\frac{dQ^m_n}{d\mu}$$

$$= (1-\mu^2)^{\frac{m}{2}-1} \frac{d^m P_n}{d\mu^m} \{(n-m+1)(1-\mu^2) + m(n-m+1)\mu^2 + (n-m+1)(m+n)(1-\mu^2)$$

$$+ (n+m)(1-\mu^2) + m(n+m)\mu^2 - (n-m+1)(m+n)(1-\mu^2)$$

$$- (2n+1)m\mu^2\}$$

$$- \mu(1-\mu^2)^{\frac{m}{2}-1} \frac{d^{m-1}P_n}{d\mu^{m-1}} \{(n+1)(n-m+1)(n+m) + n(n-m+1)(n+m)$$

$$- (2n+1)(n-m+1)(n+m)\}$$

$$= (2n+1)(1-\mu^2)^{\frac{m}{2}} \frac{d^m P_n}{d\mu^m} = (2n+1)Q^m_n,$$

agreeing with the relation found in Art. 8.

24. To abridge, put

$$(1-\mu^2)^{\frac{m}{2}-1}\frac{d^m P_n}{d\mu^m}=X,$$

and

$$(1-\mu^2)^{\frac{m}{2}-1}\frac{d^{m-1} P_n}{d\mu^{m-1}}=Y.$$

Then we have

$$\frac{dQ^m_{n+1}}{d\mu}=\{m+n+1-(n+1)\mu^2\}X-(n+1)(n+m)\mu Y,$$

$$\frac{dQ^m_{n-1}}{d\mu}=\{-(n-m)+n\mu^2\}X-n(n-m+1)\mu Y,$$

$$\frac{dQ^m_n}{d\mu}=m\mu X-(n+m)(n-m+1)Y.$$

Hence $(n-m+1)n\dfrac{dQ^m_{n+1}}{d\mu}-(n+1)(n+m)\dfrac{dQ^m_{n-1}}{d\mu}$

$$=\{n(n+1)(1-\mu^2)-m^2\}(2n+1)X,$$

and similarly $\{n(1-\mu^2)-m\}\dfrac{dQ^m_{n+1}}{d\mu}+\{(n+1)(1-\mu^2)+m\}\dfrac{dQ^m_{n-1}}{d\mu}$

$$=-\{n(n+1)(1-\mu^2)-m^2\}(2n+1)\mu Y.$$

Substitute for X and Y in the equation for $\dfrac{dQ^m_n}{d\mu}$, then

$$(2n+1)\{n(n+1)(1-\mu^2)-m^2\}\mu\frac{dQ^m_n}{d\mu}$$

$$=(n-m+1)nm\mu^2\frac{dQ^m_{n+1}}{d\mu}-(n+1)(n+m)m\mu^2\frac{dQ^m_{n-1}}{d\mu}$$

$$+\{n(n+m)(n-m+1)(1-\mu^2)-m(n+m)(n-m+1)\}\frac{dQ^m_{n+1}}{d\mu}$$

$$+\{(n+1)(n+m)(n-m+1)(1-\mu^2)+m(n+m)(n-m+1)\}\frac{dQ^m_{n-1}}{d\mu}.$$

Or,

$$=\{n^2(1-\mu^2)-m^2\}(n-m+1)\frac{dQ^m_{n+1}}{d\mu}$$

$$+\{(n+1)^2(1-\mu^2)-m^2\}(n+m)\frac{dQ^m_{n-1}}{d\mu}.$$

When $m=0$, then Q_n^m is reduced to P_n and we should have

$$n(n+1)(2n+1)\mu\frac{dP_n}{d\mu}=n^2(n+1)\frac{dP_{n+1}}{d\mu}+n(n+1)^2\frac{dP_{n-1}}{d\mu},$$

or
$$(2n+1)\mu\frac{dP_n}{d\mu}=n\frac{dP_{n+1}}{d\mu}+(n+1)\frac{dP_{n-1}}{d\mu},$$

which is identical with equation (14).

25. From equation (19) we may derive a scheme of calculation, by means of which the numerical values of G_n^m for different values of n and m may be obtained.

This is the equation employed by Mr Graham in the calculation of these functions.

From equation (18) we may also derive a scheme of calculation by means of which the numerical values of G_n^m for different values of n and m may be obtained.

In the latter case the value of each function is derived from the values of the two previous functions with the same value of m.

This is the equation employed by Mr Wright in 1873—74 in the calculation of all the values of G_n^m up to G_{10}^{10} to ten places of decimals for all values of μ differing by ·01 from 0 to 1.

The functions G_n^m, H_n^m, &c. are functions of μ, the cosine of the geocentric co-latitude.

The values of G_n^m for different values of n and m up to G_{10}^{10} have been determined for every degree of latitude on a sphere of radius unity. They have also been determined for every degree of the geographical co-latitude, taking into account the spheroidal figure of the Earth.

The values of these functions are given in the tables.

Collection of the values of the quantities G_n^m in powers of μ for the several values of n and m.

$m=0,\qquad G_0{}^0=1,\qquad G_1{}^0=\mu,\qquad G_2{}^0=\mu^2-\frac{1}{3}$

$$G_3{}^0=\mu^3-\frac{3}{5}\mu$$

$$G_4{}^0=\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}$$

$$G_5{}^0=\mu^5-\frac{10}{9}\mu^3+\frac{5}{21}\mu$$

$$G_6{}^0=\mu^6-\frac{15}{11}\mu^4+\frac{5}{11}\mu^2-\frac{5}{231}$$

$$G_7{}^0=\mu^7-\frac{21}{13}\mu^5+\frac{105}{143}\mu^3-\frac{35}{429}\mu$$

$$G_8{}^0=\mu^8-\frac{28}{15}\mu^6+\frac{14}{13}\mu^4-\frac{28}{143}\mu^2+\frac{7}{1287}$$

$$G_9{}^0=\mu^9-\frac{36}{17}\mu^7+\frac{126}{85}\mu^5-\frac{84}{221}\mu^3+\frac{63}{2431}\mu$$

$$G_{10}{}^0=\mu^{10}-\frac{45}{19}\mu^8+\frac{630}{323}\mu^6-\frac{210}{323}\mu^4+\frac{315}{4199}\mu^2-\frac{63}{46189}$$

$m=1,\qquad G_2{}^1=\mu,\qquad G_3{}^1=\mu^2-\frac{1}{5}$

$$G_4{}^1=\mu^3-\frac{3}{7}\mu$$

$$G_5{}^1=\mu^4-\frac{2}{3}\mu^2+\frac{1}{21}$$

$$G_6{}^1=\mu^5-\frac{10}{11}\mu^3+\frac{5}{33}\mu$$

$$G_7{}^1=\mu^6-\frac{15}{13}\mu^4+\frac{45}{143}\mu^2-\frac{5}{429}$$

$$G_8{}^1=\mu^7-\frac{7}{5}\mu^5+\frac{7}{13}\mu^3-\frac{7}{143}\mu$$

$$G_9{}^1=\mu^8-\frac{28}{17}\mu^6+\frac{14}{17}\mu^4-\frac{28}{221}\mu^2+\frac{7}{2431}$$

$$G_{10}{}^1=\mu^9-\frac{36}{19}\mu^7+\frac{378}{323}\mu^5-\frac{84}{323}\mu^3+\frac{63}{4199}\mu$$

$m=2,\quad G_3^2=\mu,$

$$G_4^2=\mu^2-\frac{1}{7}$$
$$G_5^2=\mu^3-\frac{1}{3}\mu$$
$$G_6^2=\mu^4-\frac{6}{11}\mu^2+\frac{1}{33}$$
$$G_7^2=\mu^5-\frac{10}{13}\mu^3+\frac{15}{143}\mu$$
$$G_8^2=\mu^6-\mu^4+\frac{3}{13}\mu^2-\frac{1}{143}$$
$$G_9^2=\mu^7-\frac{21}{17}\mu^5+\frac{7}{17}\mu^3-\frac{7}{221}\mu$$
$$G_{10}^2=\mu^8-\frac{28}{19}\mu^6+\frac{210}{323}\mu^4-\frac{28}{323}\mu^2+\frac{7}{4199}$$

$m=3,\quad G_4^3=\mu,$

$$G_5^3=\mu^2-\frac{1}{9}$$
$$G_6^3=\mu^3-\frac{3}{11}\mu$$
$$G_7^3=\mu^4-\frac{6}{13}\mu^2+\frac{3}{143}$$
$$G_8^3=\mu^5-\frac{2}{3}\mu^3+\frac{1}{13}\mu$$
$$G_9^3=\mu^6-\frac{15}{17}\mu^4+\frac{3}{17}\mu^2-\frac{1}{221}$$
$$G_{10}^3=\mu^7-\frac{21}{19}\mu^5+\frac{105}{323}\mu^3-\frac{7}{323}\mu$$

$m=4,\quad G_5^4=\mu,$

$$G_6^4=\mu^2-\frac{1}{11}$$
$$G_7^4=\mu^3-\frac{3}{13}\mu$$
$$G_8^4=\mu^4-\frac{2}{5}\mu^2+\frac{1}{65}$$
$$G_9^4=\mu^5-\frac{10}{17}\mu^3+\frac{1}{17}\mu$$
$$G_{10}^4=\mu^6-\frac{15}{19}\mu^4+\frac{45}{323}\mu^2-\frac{1}{323}$$

$m=5,\quad G_6^5=\mu,$

$$G_7^5=\mu^2-\frac{1}{13}$$
$$G_8^5=\mu^3-\frac{1}{5}\mu$$
$$G_9^5=\mu^4-\frac{6}{17}\mu^2+\frac{1}{85}$$
$$G_{10}^5=\mu^5-\frac{10}{19}\mu^3+\frac{15}{323}\mu$$

$m=6,\quad G_7^6=\mu,$

$$G_8^6=\mu^2-\frac{1}{15}$$
$$G_9^6=\mu^3-\frac{3}{17}\mu$$
$$G_{10}^6=\mu^4-\frac{6}{19}\mu^2+\frac{3}{323}$$

$m=7,\quad G_8^7=\mu,$

$$G_9^7=\mu^2-\frac{1}{17}$$
$$G_{10}^7=\mu^3-\frac{3}{19}\mu$$

$m=8,\quad G_9^8=\mu$

$$G_{10}^8=\mu^2-\frac{1}{19}$$

$m=9$

$$G_{10}^9=\mu$$

$m=10$

$$G_{10}^{10}=1$$

Formula employed by *Mr Wright* for the determination of the numerical values of G_n^m for values of n and m from 0 to 10

$$G_{n+1}^m=\mu G_n^m-\frac{(n-m)(n+m)}{(2n-1)(2n+1)}G_{n-1}^m.$$

Scheme of Calculation employed by Mr Graham.

Let G_n^m denote the number in the $\overline{n-m}$th vertical line and in the nth horizontal line of the following scheme, where m varies from $n-1$ to -1 and n takes successive integral values

$$G_{n+1}^{m+1} = G_n^m + \frac{(n-m-1)(n-m)}{(2n-1)(2n+1)} G_{n-1}^{m+1} \text{ and } (\mu^2-1)\, G_n^1 = G_n^{-1}.$$

n	$n-m=1$	$n-m=2$	$n-m=3$	$n-m=4$	$n-m=5$	$n-m=6$	$n-m=7$	$n-m=8$	$n-m=9$	$n-m=10$
1	$\mu=G_1^0$	$\mu^2-1=G_1^{-1}$								
		$\frac{1 . 2}{1 . 3}$								
2	$\mu=G_2^1$	sum $=G_2^0$	G_2^{-1}							
		$\frac{1 . 2}{3 . 5}$	$\frac{2 . 3}{3 . 5} G_1^0$							
3	$\mu=G_3^2$	sum $=G_3^1$	sum $=G_3^0$	G_3^{-1}						
		$\frac{1 . 2}{5 . 7}$	$\frac{2 . 3}{5 . 7} G_2^1$	$\frac{3 . 4}{5 . 7} G_2^0$						
4	$\mu=G_4^3$	sum $=G_4^2$	sum $=G_4^1$	sum $=G_4^0$	G_4^{-1}					
		$\frac{1 . 2}{7 . 9}$	$\frac{2 . 3}{7 . 9} G_3^2$	$\frac{3 . 4}{7 . 9} G_3^1$	$\frac{4 . 5}{7 . 9} G_3^0$					
5	$\mu=G_5^4$	sum $=G_5^3$	sum $=G_5^2$	sum $=G_5^1$	sum $=G_5^0$	G_5^{-1}				
		$\frac{1 . 2}{9 . 11}$	$\frac{2 . 3}{9 . 11} G_4^3$	$\frac{3 . 4}{9 . 11} G_4^2$	$\frac{4 . 5}{9 . 11} G_4^1$	$\frac{5 . 6}{9 . 11} G_4^0$				
6	$\mu=G_6^5$	sum $=G_6^4$	sum $=G_6^3$	sum $=G_6^2$	sum $=G_6^1$	sum $=G_6^0$	G_6^{-1}			
		$\frac{1 . 2}{11 . 13}$	$\frac{2 . 3}{11 . 13} G_5^4$	$\frac{3 . 4}{11 . 13} G_5^3$	$\frac{4 . 5}{11 . 13} G_5^2$	$\frac{5 . 6}{11 . 13} G_5^1$	$\frac{6 . 7}{11 . 13} G_5^0$			
7	$\mu=G_7^6$	sum $=G_7^5$	sum $=G_7^4$	sum $=G_7^3$	sum $=G_7^2$	sum $=G_7^1$	sum $=G_7^0$	G_7^{-1}		
		$\frac{1 . 2}{13 . 15}$	$\frac{2 . 3}{13 . 15} G_6^5$	$\frac{3 . 4}{13 . 15} G_6^4$	$\frac{4 . 5}{13 . 15} G_6^3$	$\frac{5 . 6}{13 . 15} G_6^2$	$\frac{6 . 7}{13 . 15} G_6^1$	$\frac{7 . 8}{13 . 15} G_6^0$		
8	$\mu=G_8^7$	sum $=G_8^6$	sum $=G_8^5$	sum $=G_8^4$	sum $=G_8^3$	sum $=G_8^2$	sum $=G_8^1$	sum $=G_8^0$	G_8^{-1}	
		$\frac{1 . 2}{15 . 17}$	$\frac{2 . 3}{15 . 17} G_7^6$	$\frac{3 . 4}{15 . 17} G_7^5$	$\frac{4 . 5}{15 . 17} G_7^4$	$\frac{5 . 6}{15 . 17} G_7^3$	$\frac{6 . 7}{15 . 17} G_7^2$	$\frac{7 . 8}{15 . 17} G_7^1$	$\frac{8 . 9}{15 . 17} G_7^0$	
9	$\mu=G_9^8$	sum $=G_9^7$	sum $=G_9^6$	sum $=G_9^5$	sum $=G_9^4$	sum $=G_9^3$	sum $=G_9^2$	sum $=G_9^1$	sum $=G_9^0$	G_9^{-1}
		$\frac{1 . 2}{17 . 19}$	$\frac{2 . 3}{17 . 19} G_8^7$	$\frac{3 . 4}{17 . 19} G_8^6$	$\frac{4 . 5}{17 . 19} G_8^5$	$\frac{5 . 6}{17 . 19} G_8^4$	$\frac{6 . 7}{17 . 19} G_8^3$	$\frac{7 . 8}{17 . 19} G_8^2$	$\frac{8 . 9}{17 . 19} G_8^1$	$\frac{9 . 10}{17 . 19} G_8^0$
10	$\mu=G_{10}^9$	sum $=G_{10}^8$	sum $=G_{10}^7$	sum $=G_{10}^6$	sum $=G_{10}^5$	sum $=G_{10}^4$	sum $=G_{10}^3$	sum $=G_{10}^2$	sum $=G_{10}^1$	sum $=G_{10}^0$

COMPUTATION OF THE VALUES OF G_0^0, G_1^0, G_2^0, ... G_{10}^0, WHEN μ IS ·00, ·01, ·02, ETC., TO 1·00.

$m = 0$.

μ	G_0^0	G_1^0	G_2^0	G_3^0	G_4^0	G_5^0	G_6^0	G_7^0	G_8^0	G_9^0	G_{10}^0
·00	1	·00	− ·3333333333	·0000000000	+ ·0857142857	·0000000000	− ·0216450216	·0000000000	+ ·0054390054	·0000000000	− ·0013639611
·01		·01	·3332333333	− ·0059990000	·0856285814	+ ·0023798414	·0215995807	− ·0008151167	·0054194357	+ ·0002587727	·0013564658
·02		·02	·3329333333	·0119920000	·0853715886	·0047530191	·0214634216	·0016258326	·0053608559	·0005152692	·0013340578
·03		·03	·3324333333	·0179730000	·0849436671	·0071128814	·0212370346	·0024277665	·0052636526	·0007672313	·0012969703
·04		·04	·3317333333	·0239360000	·0843454171	·0094528008	·0209212358	·0032165755	·0051284681	·0010124361	·0012455890
·05		·05	·3308333333	·0298750000	·0835776786	·0117661855	·0205171651	·0039879749	·0049561966	·0012487133	·0011804495
·06		·06	·3297333333	·0357840000	·0826415314	·0140464919	·0200262841	·0047377568	·0047479803	·0014739629	·0011022322
·07		·07	·3284333333	·0416570000	·0815382957	·0162872362	·0194503722	·0054618093	·0045052027	·0016861712	·0010117557
·08		·08	·3269333333	·0474880000	·0802695314	·0184820069	·0187915231	·0061561347	·0042294817	·0018834276	·0009099698
·09		·09	·3252333333	·0532710000	·0788370386	·0206244763	·0180521402	·0068168684	·0039226606	·0020639398	·0007979467
·10		·10	− ·3233333333	− ·0590000000	+ ·0772428571	+ ·0227084127	− ·0172349307	− ·0074402963	+ ·0035867991	+ ·0022260484	− ·0006768711
·11		·11	− ·3212333333	− ·0646690000	+ ·0754892671	+ ·0247276924	− ·0163429000	− ·0080228723	+ ·0032241614	+ ·0023682414	− ·0005480292
·12		·12	·3189333333	·0702720000	·0735787886	·0266763117	·0153793447	·0085612362	·0028372050	·0024891670	·0004127972
·13		·13	·3164333333	·0758030000	·0715141814	·0285483991	·0143478448	·0090522294	·0024285661	·0025876457	·0002726273
·14		·14	·3137333333	·0812560000	·0692984457	·0303382268	·0132522557	·0094929114	·0020010464	·0026626811	− ·0001290350
·15		·15	·3108333333	·0866250000	·0669348214	·0320402232	·0120966992	·0098805751	·0015575971	·0027134702	+ ·0000164157
·16		·16	·3077333333	·0919040000	·0644267886	·0336489846	·0108855536	·0102127616	·0011013023	·0027394113	·0001621278
·17		·17	·3044333333	·0970870000	·0617780671	·0351592873	·0096234432	·0104872744	·0006353620	·0027401117	·0003064867
·18		·18	·3009333333	·1021680000	·0589926171	·0365660997	·0083152276	·0107021927	+ ·0001630728	·0027153936	·0004478764
·19		·19	·2972333333	·1071410000	·0560746386	·0378645940	·0069659893	·0108558833	− ·0003121897	·0026652978	·0005846957
·20		·20	− ·2933333333	− ·1120000000	+ ·0530285714	+ ·0390501587	− ·0055810217	− ·0109470135	− ·0007869921	+ ·0025900873	+ ·0007153747
·21		·21	− ·2892333333	− ·1167390000	+ ·0498590957	+ ·0401184101	− ·0041658146	− ·0109745607	− ·0012578633	+ ·0024902482	+ ·0008383915
·22		·22	·2849333333	·1213520000	·0465711314	·0410652044	·0027260417	·0109378226	·0017213156	·0023664896	·0009522889
·23		·23	·2804333333	·1258330000	·0431698386	·0418866501	− ·0012675449	·0108364262	·0021738667	·0022197412	·0010556897
·24		·24	·2757333333	·1301760000	·0396606171	·0425791195	+ ·0002036814	·0106703354	·0026120620	·0020511501	·0011473132
·25		·25	·2708333333	·1343750000	·0360491071	·0431392609	·0016815053	·0104398572	·0030324964	·0018620754	·0012259901
·26		·26	·2657333333	·1384240000	·0323411886	·0435640106	·0031596760	·0101456477	·0034318383	·0016540806	·0012906768
·27		·27	·2604333333	·1423170000	·0285429814	·0438506050	·0046318398	·0097887164	·0038068516	·0014289260	·0013404694
·28		·28	·2549333333	·1460480000	·0246608457	·0439965923	·0060915595	·0093704286	·0041544196	·0011885563	·0013746162
·29		·29	·2492333333	·1496110000	·0207013814	·0439998450	·0075323335	·0088925074	·0044715673	·0009350905	·0013925290
·30		·30	− ·2433333333	− ·1530000000	+ ·0166714286	+ ·0438585714	+ ·0089476147	− ·0083570343	− ·0047554853	+ ·0006708061	+ ·0013937938
·31		·31	− ·2372333333	− ·1562090000	+ ·0125780671	+ ·0435713278	+ ·0103308320	− ·0077664477	− ·0050035515	+ ·0003981251	+ ·0013781794
·32		·32	·2309333333	·1592320000	·0084286171	·0431370305	·0116754111	·0071235404	·0052133542	·0001195957	·0013456443
·33		·33	·2244333333	·1620630000	+ ·0042306386	·0425549678	·0129747963	·0064314560	·0053827139	− ·0001621262	·0012963431
·34		·34	·2177333333	·1646960000	− ·0000080686	·0418248122	·0142224736	·0056936824	·0055097043	·0004442968	·0012306297
·35		·35	·2108333333	·1671250000	·0042794643	·0409466319	·0154119940	·0049140451	·0055926733	·0007241067	·0011490597
·36		·36	·2037333333	·1693440000	·0085752686	·0399209033	·0165369971	·0040966987	·0056302621	·0009987034	·0010523901
·37		·37	·1964333333	·1713470000	·0128869614	·0387485227	·0175912366	·0032461154	·0056214247	·0012652158	·0009415773
·38		·38	·1889333333	·1731280000	·0172057829	·0374308184	·0185686057	·0023670729	·0055654450	·0015207802	·0008177724
·39		·39	·1812333333	·1746810000	·0215227329	·0359695628	·0194631631	·0014646410	·0054619535	·0017625657	·0006823153
·40		·40	− ·1733333333	− ·1760000000	− ·0258285714	+ ·0343669841	+ ·0202691601	− ·0005441642	− ·0053109418	− ·0019878022	+ ·0005367252
·41		·41	− ·1652333333	− ·1770790000	− ·0301138186	+ ·0326257788	+ ·0209810690	+ ·0003887555	− ·0051127763	− ·0021938083	+ ·0003826899
·42		·42	·1569333333	·1779120000	·0343687543	·0307491232	·0215936101	·0013282782	·0048682098	·0023780198	·0002220521
·43		·43	·1484333333	·1784930000	·0385834186	·0287406855	·0221017823	·0022683491	·0045783911	·0025380193	+ ·0000567931
·44		·44	·1397333333	·1788160000	·0427476114	·0266046382	·0225008922	·0032027214	·0042448729	·0026715644	− ·0001109847
·45		·45	·1308333333	·1788750000	·0468508929	·0243456696	·0227865849	·0041249834	·0038696173	·0027766178	·0002790786

·46	·46	·1217333333	·1786640000	·0508825829	·0219689960	·0229548753	·0050285863	·0034549985	·0028513759	·0004452091
·47	·47	·1124333333	·1781770000	·0548317614	·0194803737	·0230021800	·0059068746	·0030038039	·0028942975	·0006070439
·48	·48	·1029333333	·1774080000	·0586872686	·0168861111	·0229253506	·0067531194	·0025192318	·0029041319	·0007622255
·49	·49	·0932333333	·1763510000	·0624377043	·0141930804	·0227217064	·0075605529	·0020048861	·0028799447	·0009083996
·50	·50	− ·0833333333	− ·1750000000	− ·0660714286	+ ·0114087301	+ ·0223890693	+ ·0083224067	− ·0014647679	− ·0028211449	− ·0010432467
·51	·51	− ·0732333333	− ·1733490000	− ·0695765614	+ ·0085410966	+ ·0219257980	+ ·0090319508	·0009032646	− ·0027275075	− ·0011645135
·52	·52	·0629333333	·1713920000	·0729409829	·0055988159	·0213308244	·0096825366	− ·0003251343	·0025991967	·0012700471
·53	·53	·0524333333	·1691230000	·0761523329	+ ·0025911366	·0206036895	·0102676413	+ ·0002645125	·0024367850	·0013578290
·54	·54	·0417333333	·1665360000	·0791980114	− ·0004720691	·0197445805	·0107809161	·0008602360	·0022412712	·0014260112
·55	·55	·0308333333	·1636250000	·0820651786	·0035802927	·0187543689	·0112162353	·0014562931	·0020140940	·0014729521
·56	·56	·0197333333	·1603840000	·0847407543	·0067223780	·0176346487	·0115677502	·0020466694	·0017571437	·0014972520
·57	·57	− ·0084333333	·1568070000	·0872114186	·0098865086	·0163877756	·0118299434	·0026251138	·0014727688	·0014977885
·58	·58	+ ·0030666667	·1528880000	·0894636114	·0130601962	·0150169073	·0119976878	·0031851796	·0011637802	·0014737527
·59	·59	·0147666667	·1486210000	·0914835329	·0162302685	·0135260439	·0120663076	·0037202694	·0008334477	·0014246809
·60	·60	+ ·0266666667	− ·1440000000	− ·0932571429	− ·0193828571	+ ·0119200693	+ ·0120316420	+ ·0042236857	− ·0004854948	− ·0013504874
·61	·61	+ ·0387666667	− ·1390190000	− ·0947701614	− ·0225033858	+ ·0102047936	+ ·0118901121	+ ·0046886869	− ·0001240860	− ·0012514932
·62	·62	·0510666667	·1336720000	·0960080686	·0255765581	·0083869958	·0116387912	·0051085490	+ ·0002461920	·0011284522
·63	·63	·0635666667	·1279530000	·0969561043	·0285863457	·0064744669	·0112754767	·0054766330	·0006203552	·0009825733
·64	·64	·0762666667	·1218560000	·0975992686	·0315159764	·0044760550	·0107987672	·0057864588	·0009930548	·0008155383
·65	·65	·0891666667	·1153750000	·0979223214	·0343479216	·0024017099	·0102081406	·0060317848	·0013586170	·0006295137
·66	·66	·1022666667	·1085040000	·0979097829	·0370638853	+ ·0002625284	·0095040370	·0062066957	·0017110923	·0004271570
·67	·67	·1155666667	·1012370000	·0975459329	·0396447909	− ·0019291986	·0086879437	·0063056953	·0020443124	− ·0002116151
·68	·68	·1290666667	·0935680000	·0968148114	·0420707702	·0041599390	·0077624843	·0063238073	·0023519576	+ ·0000134848
·69	·69	·1427666667	·0854910000	·0957002186	·0443211508	·0064148722	·0067315104	·0062566844	·0026276351	·0002440545
·70	·70	+ ·1566666667	− ·0770000000	− ·0941857143	− ·0463744445	− ·0086778398	+ ·0056001974	+ ·0061007236	+ ·0028649668	+ ·0004755740
·71	·71	+ ·1707666667	− ·0680890000	− ·0922546186	− ·0482083348	− ·0109312968	+ ·0043751433	+ ·0058531904	+ ·0030576900	+ ·0007031320
·72	·72	·1850666667	·0587520000	·0898900114	·0497996654	·0131562613	·0030644706	·0055123511	·0031997708	·0009214807
·73	·73	·1995666667	·0489830000	·0870747329	·0511244279	·0153322635	·0016779329	·0050776136	·0032855296	·0011251031
·74	·74	·2142666667	·0387760000	·0837913829	·0521577503	·0174372951	+ ·0002270241	·0045496771	·0033097824	·0013082983
·75	·75	·2291666667	·0281250000	·0800223214	·0528738840	·0194477561	− ·0012749092	·0039306901	·0032679948	·0014652812
·76	·76	·2442666667	·0170240000	·0757496686	·0532461926	·0213384022	·0028125498	·0032244197	·0031564539	·0015903044
·77	·77	·2595666667	− ·0054670000	·0709553043	·0532471399	·0230822916	·0043684901	·0024364282	·0029724551	·0016777976
·78	·78	·2750666667	+ ·0065520000	·0656208686	·0528482775	·0246507301	·0059231080	·0015742618	·0027145082	·0017225325
·79	·79	·2907666667	·0190390000	·0597277614	·0520202331	·0260132161	·0074544380	+ ·0006476483	·0023825600	·0017198090
·80	·80	+ ·3066666667	+ ·0320000000	− ·0532571429	− ·0507326984	− ·0271373852	− ·0089380401	− ·0003312943	+ ·0019782376	+ ·0016656701
·81	·81	+ ·3227666667	+ ·0454410000	− ·0461899329	− ·0489544170	− ·0279889533	− ·0103468633	·0013478377	+ ·0015051113	+ ·0015571428
·82	·82	·3390666667	·0593680000	·0385068114	·0466531726	·0285316592	·0116511059	·0023844129	·0009689805	·0013925127
·83	·83	·3555666667	·0737870000	·0301882186	·0437957769	·0287272073	·0128180718	·0034203680	+ ·0003781793	·0011716281
·84	·84	·3722666667	·0887040000	·0212143543	·0403480576	·0285352082	·0138120219	·0044317128	− ·0002560921	·0008962409
·85	·85	·3891666667	·1041250000	·0115651786	·0362748462	·0279131197	·0145940226	·0053908532	·0009194117	·0005703859
·86	·86	·4062666667	·1200560000	− ·0012204114	·0315399665	·0268161865	·0151217890	·0062663121	·0015937559	+ ·0002007980
·87	·87	·4235666667	·1365030000	+ ·0098404671	·0261062222	·0251973797	·0153495245	·0070224370	·0022570905	− ·0002026241
·88	·88	·4410666667	·1534720000	·0216382171	·0199353848	·0230073348	·0152277563	·0076190952	·0028829356	·0006263124
·89	·89	·4587666667	·1709690000	·0341938386	·0129881820	·0201942897	·0147031657	·0080113550	·0034398997	·0010524712
·90	·90	+ ·4766666667	+ ·1890000000	+ ·0475285714	− ·0052242857	− ·0167040216	− ·0137184146	− ·0081491523	− ·0038911840	− ·0014584701
·91	·91	+ ·4947666667	+ ·2075710000	+ ·0616638957	+ ·0033977007	− ·0124797832	− ·0122119679	− ·0079769453	− ·0041940557	− ·0018161803
·92	·92	·5130666667	·2266880000	·0766215314	·0129202534	·0074622385	·0101179106	·0074333512	·0042992859	·0020912518
·93	·93	·5315666667	·2463570000	·0924234386	·0233869408	− ·0015893973	·0073657610	·0064507707	·0041505552	·0022423308
·94	·94	·5502666667	·2665840000	·1090918171	·0348424351	+ ·0052034503	− ·0038802788	·0049549958	·0036838222	·0022202088
·95	·95	·5691666667	·2873750000	·1266491071	·0473325247	·0129838007	+ ·0004187304	·0028648022	·0028266552	·0019669045
·96	·96	·5882666667	·3087360000	·1451179886	·0609041262	·0218220045	·0056166170	− ·0000915258	− ·0014975255	·0014146722
·97	·97	·6075666667	·3306730000	·1645213814	·0756052956	·0317913333	·0118040923	+ ·0034613781	+ ·0003949411	− ·0004849307
·98	·98	·6270666667	·3531920000	·1848824457	·0914852413	·0429680502	·0190774396	·0078987910	·0029527519	+ ·0009128855
·99	·99	·6467666667	·3762990000	·2062245814	·1085943356	·0554314777	·0275387288	·0133344061	·0062893810	·0028825649
1·00	1·00	+ ·6666666667	+ ·4000000000	+ ·2285714286	+ ·1269841270	+ ·0692640693	+ ·0372960373	+ ·0198912199	+ ·0105306458	+ ·0055424451

COMPUTATION OF THE VALUES OF G_1^1, G_2^1, G_3^1, ... G_{10}^1, WHEN μ IS ·00, ·01, ·02, ETC., TO 1·00.

$m = 1$.

μ	G_1^1	G_2^1	G_3^1	G_4^1	G_5^1	G_6^1	G_7^1	G_8^1	G_9^1	G_{10}^1
·00	1	·00	− ·20000	− ·0000000000	+ ·0476190476	·0000000000	− ·0116550117	·0000000000	+ ·0028794735	·0000000000
·01		·01	·19990	·0042847143	·0475523910	+ ·0015142425	·0116235547	− ·0004889709	·0028668121	+ ·0001497754
·02		·02	·19960	·0085634286	·0473525409	·0030230335	·0115293220	·0009747177	·0028289263	·0002979946
·03		·03	·19910	·0128301429	·0470198576	·0045209334	·0113727288	·0014540270	·0027661122	·0004431139
·04		·04	·19840	·0170788571	·0465549409	·0060025266	·0111544649	·0019237236	·0026788601	·0005836184
·05		·05	·19750	·0213035714	·0459586309	·0074624337	·0108754943	·0023806815	·0025678527	·0007180351
·06		·06	·19640	·0254982857	·0452320077	·0088953231	·0105370517	·0028218411	·0024339611	·0008449457
·07		·07	·19510	·0296570000	·0443763910	·0102959232	·0101406398	·0032442259	·0022782387	·0009630001
·08		·08	·19360	·0337737143	·0433933410	·0116590344	·0096880250	·0036449582	·0021019154	·0010709292
·09		·09	·19190	·0378424286	·0422846576	·0129795413	·0091812330	·0040212750	·0019063898	·0011675565
·10		·10	− ·19000	− ·0418571429	+ ·0410523809	+ ·0142524243	− ·0086225431	− ·0043705434	+ ·0016932211	+ ·0012518096
·11		·11	− ·18790	− ·0458118571	+ ·0396987909	+ ·0154727718	− ·0080144824	− ·0046902754	+ ·0014641183	+ ·0013227311
·12		·12	·18560	·0497005714	·0382264076	·0166357923	·0073598187	·0049781425	·0012209311	·0013794882
·13		·13	·18310	·0535172857	·0366379910	·0177368262	·0066615531	·0052319899	·0009656368	·0014213817
·14		·14	·18040	·0572560000	·0349365410	·0187713581	·0059229115	·0054498496	·0007003287	·0014478540
·15		·15	·17750	·0609107143	·0331252977	·0197350285	·0051473360	·0056299536	·0004272018	·0014584961
·16		·16	·17440	·0644754286	·0312077409	·0206236454	·0043384750	·0057707456	+ ·0001485392	·0014530531
·17		·17	·17110	·0679441429	·0291875909	·0214331979	·0035001724	·0058708934	− ·0001333034	·0014314296
·18		·18	·16760	·0713108571	·0270688076	·0221598659	·0026364567	·0059292984	·0004159138	·0013936924
·19		·19	·16390	·0745695714	·0248555909	·0228000342	·0017515297	·0059451067	·0006968394	·0013400728
·20		·20	− ·16000	− ·0777142857	+ ·0225523810	+ ·0233503030	− ·0008497529	− ·0059177175	− ·0009736045	+ ·0012709677
·21		·21	− ·15590	− ·0807390000	+ ·0201638576	+ ·0238075010	+ ·0000643653	− ·0058467912	− ·0012437283	+ ·0011869388
·22		·22	·15160	·0836377143	·0176949410	·0241686965	·0009861836	·0057322572	·0015047420	·0010887109
·23		·23	·14710	·0864044286	·0151507909	·0244312100	·0019109428	·0055743195	·0017542088	·0009771684
·24		·24	·14240	·0890331429	·0125368076	·0245926260	·0028337808	·0053734621	·0019897415	·0008533504
·25		·25	·13750	·0915178571	·0098586309	·0246508049	·0037497496	·0051304530	·0022090220	·0007184449
·26		·26	·13240	·0938525714	·0071221409	·0246038951	·0046538314	·0048463472	·0024098204	·0005737804
·27		·27	·12710	·0960312857	·0043334577	·0244503453	·0055409567	·0045224882	·0025900140	·0004208171
·28		·28	·12160	·0980480000	+ ·0014989410	·0241889156	·0064060227	·0041605082	·0027476067	·0002611365
·29		·29	·11590	·0998967143	− ·0013748090	·0238186907	·0072439120	·0037623278	·0028807475	+ ·0000964291
·30		·30	− ·11000	− ·1015714286	− ·0042809524	+ ·0233390909	+ ·0080495129	− ·0033301531	− ·0029877491	− ·0000715190
·31		·31	− ·10390	− ·1030661429	− ·0072124091	+ ·0227498848	+ ·0088177393	− ·0028664724	− ·0030671064	− ·0002408408
·32		·32	·09760	·1043748571	·0101618591	·0220512008	·0095435526	·0023740511	·0031175153	·0004096046
·33		·33	·09110	·1054915714	·0131217424	·0212435393	·0102219833	·0018559244	·0031378863	·0005758308
·34		·34	·08440	·1064102857	·0160842590	·0203277848	·0108481536	·0013153902	·0031273648	·0007375108
·35		·35	·07750	·1071250000	·0190413690	·0193052178	·0114173011	·0007559982	·0030853444	·0008926264
·36		·36	·07040	·1076297143	·0219847923	·0181775267	·0119248028	− ·0001815391	·0030114819	·0010391703
·37		·37	·06310	·1079184286	·0249060091	·0169468199	·0123661998	+ ·0004039690	·0029057103	·0011751670
·38		·38	·05560	·1079851429	·0277962591	·0156156380	·0127372226	·0009962952	·0027682510	·0012986958
·39		·39	·04790	·1078238571	·0306465424	·0141869654	·0130338185	·0015910131	·0025996248	·0014079127
·40		·40	− ·04000	− ·1074285714	− ·0334476191	+ ·0126642424	+ ·0132521772	+ ·0021835189	− ·0024006597	− ·0015010735
·41		·41	− ·03190	− ·1067932857	− ·0361900090	+ ·0110513777	+ ·0133887594	+ ·0027690523	− ·0021724998	− ·0015765583
·42		·42	·02360	·1059120000	·0388639924	·0093527596	·0134403250	·0033427188	·0019166090	·0016328938
·43		·43	·01510	·1047787143	·0414596090	·0075732685	·0134039622	·0038995145	·0016347759	·0016687776
·44		·44	− ·00640	·1033874286	·0439666591	·0057182890	·0132771176	·0044343529	·0013291138	·0016831018
·45		·45	+ ·00250	·1017321429	·0463747024	·0037937216	·0130576263	·0049420926	·0010020601	·0016749747

·46		·46	·01160	·0998068571	·0486730591	+ ·0018059945	·0127437440	·0054175697	·0006563723	·0016437442
·47		·47	·02090	·0976055714	·0508508091	− ·0002379236	·0123341781	·0058556295	− ·0002951216	·0015890179
·48		·48	·03040	·0951222857	·0528967923	·0023305123	·0118281214	·0062511629	+ ·0000783164	·0015106837
·49		·49	·04010	·0923510000	·0547996090	·0044636872	·0112252850	·0065991435	·0004602746	·0014089282
·50		·50	+ ·05000	− ·0892857143	− ·0565476191	− ·0066287880	+ ·0105259324	+ ·0068946679	+ ·0008468095	− ·0012842529
·51		·51	+ ·06010	− ·0859204286	− ·0581289424	− ·0088165658	+ ·0097309149	+ ·0071329982	+ ·0012337207	− ·0011374893
·52		·52	·07040	·0822491429	·0595314591	·0110171725	·0088417071	·0073096071	·0016165739	·0009698106
·53		·53	·08090	·0782658571	·0607428091	·0132201477	·0078604428	·0074202249	·0019907274	·0007827411
·54		·54	·09160	·0739645714	·0617503924	·0154144067	·0067899528	·0074608900	·0023513628	·0005781626
·55		·55	·10250	·0693392857	·0625413690	·0175882292	·0056338020	·0074280014	·0026935203	·0003583165
·56		·56	·11360	·0643840000	·0631026590	·0197292466	·0043963287	·0073183740	·0030121376	− ·0001258035
·57		·57	·12490	·0590927143	·0634209424	·0218244307	·0030826828	·0071292968	·0033020952	+ ·0001164242
·58		·58	·13640	·0534594286	·0634826591	·0238600808	·0016988669	·0068585935	·0035582641	·0003650703
·59		·59	·14810	·0474781429	·0632740091	·0258218126	+ ·0002517762	·0065046865	·0037755615	·0006165134
·60		·60	+ ·16000	− ·0411428571	− ·0627809524	− ·0276945454	− ·0012507598	+ ·0060666630	+ ·0039490090	+ ·0008668263
·61		·61	+ ·17210	− ·0344475714	− ·0619892091	− ·0294624912	− ·0027999356	+ ·0055443448	+ ·0040737991	+ ·0011118052
·62		·62	·18440	·0273862857	·0608842590	·0311091410	·0043859257	·0049383608	·0041453653	·0013470031
·63		·63	·19690	·0199530000	·0594513424	·0326172548	·0059978426	·0042502219	·0041594597	·0015677731
·64		·64	·20960	·0121417143	·0576754591	·0339688479	·0076236916	·0034824000	·0041122363	·0017693173
·65		·65	·22250	− ·0039464286	·0555413691	·0351451799	·0092503255	·0026384096	·0040003407	·0019467454
·66		·66	·23560	+ ·0046388571	·0530335924	·0361267424	·0108634001	·0017228925	·0038210080	·0020951427
·67		·67	·24890	·0136201429	·0501364091	·0368932469	·0124473263	+ ·0007417060	·0035721648	·0022096461
·68		·68	·26240	·0230034286	·0468338591	·0374236130	·0139852242	− ·0002979862	·0032525424	·0022855334
·69		·69	·27610	·0327947143	·0431097423	·0376959560	·0154588741	·0013876185	·0028617945	·0023183208
·70		·70	+ ·29000	+ ·0430000000	− ·0389476190	− ·0376875757	− ·0168486690	− ·0025171266	+ ·0024006237	+ ·0023038735
·71		·71	+ ·30410	+ ·0536252857	− ·0343308091	− ·0373749438	− ·0181335645	− ·0036748446	+ ·0018709174	+ ·0022385296
·72		·72	·31840	·0646765714	·0292423924	·0367336913	·0192910288	·0048474013	·0012758900	·0021192355
·73		·73	·33290	·0761598571	·0236652091	·0357385983	·0202969928	·0060196113	+ ·0006202350	·0019436970
·74		·74	·34760	·0880811429	·0175818591	·0343635800	·0211257970	·0071743624	− ·0000897136	·0017105438
·75		·75	·36250	·1004464286	·0109747023	·0325816761	·0217501411	·0082925009	·0008458114	·0014195116
·76		·76	·37760	·1132617143	− ·0038258590	·0303650381	·0221410299	·0093527118	·0016379242	·0010716387
·77		·77	·39290	·1265330000	+ ·0038827910	·0276849176	·0222677201	·0103313955	·0024537378	·0006694815
·78		·78	·40840	·1402662857	·0121696076	·0245116541	·0220976655	·0112025412	·0032785589	+ ·0002173473
·79		·79	·42410	·1544675714	·0210531909	·0208146632	·0215964628	·0119375962	·0040951043	− ·0002784522
·80		·80	+ ·44000	+ ·1691428571	+ ·0305523809	− ·0165624243	− ·0207277949	− ·0125053315	− ·0048832806	− ·0008093288
·81		·81	+ ·45610	+ ·1842981429	+ ·0406862576	− ·0117224690	− ·0194533748	− ·0128717028	− ·0056199514	− ·0013641227
·82		·82	·47240	·1999394286	·0514741410	·0062613689	·0177328885	·0129997086	·0062786945	·0019287874
·83		·83	·48890	·2160727143	·0629355910	− ·0001447236	·0155239366	·0128492431	·0068295463	·0024860483
·84		·84	·50560	·2327040000	·0750904076	+ ·0066628515	·0127819758	·0123769462	·0072387349	·0030150398
·85		·85	·52250	·2498392857	·0879586309	·0141977367	·0094602600	·0115360485	·0074684005	·0034909148
·86		·86	·53960	·2674845714	·1015605409	·0224973206	·0055097794	·0102762123	·0074763030	·0038844287
·87		·87	·55690	·2856458571	·1159166576	·0316000116	− ·0008791998	·0085433682	·0072155162	·0041614946
·88		·88	·57440	·3043291429	·1310477410	·0415452502	+ ·0044851983	·0062795486	·0066341106	·0042827111
·89		·89	·59210	·3235404286	·1469747910	·0523735207	·0106395825	− ·0034227152	·0056748192	·0042028578
·90		·90	+ ·61000	+ ·3432857143	+ ·1637190477	+ ·0641263637	+ ·0176426317	+ ·0000934174	− ·0042746921	− ·0038703603
·91		·91	+ ·62810	+ ·3635710000	+ ·1813019910	+ ·0768463876	+ ·0255555995	+ ·0043395616	− ·0023647353	− ·0032267231
·92		·92	·64640	·3844022857	·1997453409	·0905772807	·0344423784	·0093910421	+ ·0001304652	·0022059267
·93		·93	·66490	·4057855714	·2190710576	·1053638239	·0443695659	·0153279858	·0032931340	− ·0007337906
·94		·94	·68360	·4277268571	·2393013409	·1212519011	·0554065287	·0222355152	·0072127125	+ ·0012727014
·95		·95	·70250	·4502321429	·2604586310	·1382885133	·0676254716	·0302039486	·0119862817	·0039061135
·96		·96	·72160	·4733074286	·2825656077	·1565217886	·0811015026	·0393290022	·0177190003	·0072693110
·97		·97	·74090	·4969587143	·3056451910	·1760009955	·0959127021	·0497119990	·0245245597	·0114762535
·98		·98	·76040	·5211920000	·3297205410	·1967765544	·1121401916	·0614600821	·0325256567	·0166528323
·99		·99	·78010	·5460132857	·3548150576	·2189000499	·1298682031	·0746864319	·0418544821	·0229377498
1·00		1·00	+ ·80000	+ ·5714285714	+ ·3809523809	+ ·2424242424	+ ·1491841492	+ ·0895104901	+ ·0526532297	+ ·0304834489

COMPUTATION OF THE VALUES OF G_2^2, G_3^2, G_4^2, ... G_{10}^2, WHEN μ IS ·00, ·01, ·02, ETC., TO 1·00.

$m = 2.$

μ	G_2^2	G_3^2	G_4^2	G_5^2	G_6^2	G_7^2	G_8^2	G_9^2	G_{10}^2
·00	1	·00	− ·1428571429	·0000000000	+ ·0303030303	·0000000000	− ·0069930069	·0000000000	+ ·0016670636
·01		·01	·1427571429	− ·0033323333	·0302484949	+ ·0010481819	·0069699401	− ·0003163304	·0016584014
·02		·02	·1424571429	·0066586667	·0300850085	·0020917515	·0069008593	·0006301940	·0016324926
·03		·03	·1419571429	·0099730000	·0298129312	·0031261082	·0067861240	·0009391386	·0015895705
·04		·04	·1412571429	·0132693333	·0294328631	·0041466758	·0066263322	·0012407417	·0015300223
·05		·05	·1403571429	·0165416667	·0289456440	·0051489139	·0064223183	·0015326251	·0014543858
·06		·06	·1392571429	·0197840000	·0283523539	·0061283300	·0061751511	·0018124691	·0013633467
·07		·07	·1379571429	·0229903333	·0276543131	·0070804919	·0058861302	·0020780272	·0012577332
·08		·08	·1364571429	·0261546667	·0268530812	·0080010391	·0055567818	·0023271399	·0011385105
·09		·09	·1347571429	·0292710000	·0259504584	·0088856951	·0051888547	·0025577487	·0010067742
·10		·10	− ·1328571429	− ·0323333333	+ ·0249484849	+ ·0097302797	− ·0047843147	− ·0027679091	+ ·0008637423
·11		·11	− ·1307571429	− ·0353356667	+ ·0238494403	+ ·0105307205	− ·0043453377	− ·0029558037	+ ·0007107471
·12		·12	·1284571429	·0382720000	·0226558448	·0112830650	·0038743041	·0031197553	·0005492251
·13		·13	·1259571429	·0411363333	·0213704585	·0119834929	·0033737902	·0032582381	·0003807072
·14		·14	·1232571429	·0439226667	·0199962812	·0126283279	·0028465605	·0033698898	·0002068072
·15		·15	·1203571429	·0466250000	·0185365530	·0132140494	·0022955587	·0034535219	+ ·0000292102
·16		·16	·1172571429	·0492373333	·0169947540	·0137373051	·0017238975	·0035081307	− ·0001503408
·17		·17	·1139571429	·0517536667	·0153746040	·0141949228	·0011348486	·0035329061	·0003300573
·18		·18	·1104571429	·0541680000	·0136800630	·0145839218	− ·0005318317	·0035272407	·0005081199
·19		·19	·1067571429	·0564743333	·0119153312	·0149015259	+ ·0000815981	·0034907378	·0006826924
·20		·20	− ·1028571429	− ·0586666667	+ ·0100848485	+ ·0151451748	+ ·0007017623	− ·0034232180	− ·0008519368
·21		·21	− ·0987571429	− ·0607390000	+ ·0081932948	+ ·0153125360	+ ·0013248723	− ·0033247265	− ·0010140290
·22		·22	·0944571429	·0626853333	·0062455903	·0154015171	·0019470437	·0031955368	·0011671740
·23		·23	·0899571429	·0644996667	·0042468949	·0154102776	·0025643111	·0030361561	·0013096223
·24		·24	·0852571429	·0661760000	·0022026084	·0153372414	·0031726437	·0028473282	·0014396856
·25		·25	·0803571429	·0677083333	+ ·0001183712	·0151811080	·0037679606	·0026300352	·0015557533
·26		·26	·0752571429	·0690906667	− ·0019999369	·0149408649	·0043461488	·0023854989	·0016563085
·27		·27	·0699571429	·0703170000	·0041461961	·0146157998	·0049030804	·0021151800	·0017399444
·28		·28	·0644571429	·0713813333	·0063140460	·0142055023	·0054346309	·0018207768	·0018053797
·29		·29	·0587571429	·0722776667	·0084968869	·0137099261	·0059366987	·0015042224	·0018514747
·30		·30	− ·0528571429	− ·0730000000	− ·0106878788	+ ·0131293007	+ ·0064052238	− ·0011676801	− ·0018772459
·31		·31	− ·0467571429	− ·0735423333	− ·0128799415	+ ·0124642437	+ ·0068362097	− ·0008135382	− ·0018818815
·32		·32	·0404571429	·0738986667	·0150657551	·0117157230	·0072257441	·0004444026	·0018647546
·33		·33	·0339571429	·0740630000	·0172377597	·0108850778	·0075700203	− ·0000630881	·0018254369
·34		·34	·0272571429	·0740293333	·0193881551	·0099740320	·0078653605	+ ·0003273915	·0017637109
·35		·35	·0203571429	·0737916667	·0215089015	·0089847050	·0081082395	·0007238356	·0016795815
·36		·36	·0132571429	·0733440000	·0235917188	·0079196246	·0082953077	·0011228697	·0015732861
·37		·37	− ·0059571429	·0726803333	·0256280869	·0067817383	·0084234171	·0015209612	·0014453036
·38		·38	+ ·0015428571	·0717946667	·0276092460	·0055744259	·0084896463	·0019144360	·0012963619
·39		·39	·0092428571	·0706810000	·0295261961	·0043015108	·0084913268	·0022994973	·0011274443
·40		·40	+ ·0171428571	− ·0693333333	− ·0313696969	+ ·0029672727	+ ·0084260699	+ ·0026722462	− ·0009397931
·41		·41	+ ·0252428571	− ·0677456667	− ·0331302687	+ ·0015764593	+ ·0082917949	+ ·0030287042	− ·0007349131
·42		·42	·0335428571	·0659120000	·0347981915	+ ·0001342981	·0080867571	·0033648384	·0005145707
·43		·43	·0420428571	·0638263333	·0363635051	− ·0013534914	·0078095768	·0036765866	·0002807935
·44		·44	·0507428571	·0614826667	·0378160097	·0028806853	·0074592700	·0039598871	− ·0000358658
·45		·45	·0596428571	·0588750000	·0391452652	·0044405441	·0070352779	·0042107090	+ ·0002176785

·46	·46	·0687428571	·0559973333	·0403405915	·0060258003	·0065374992	·0044250850	·0004770641
·47	·47	·0780428571	·0528436667	·0413910688	·0076286461	·0059663214	·0045991467	·0007392870
·48	·48	·0875428571	·0494080000	·0422855370	·0092407221	·0053226542	·0047291616	·0010011296
·49	·49	·0972428571	·0456843333	·0430125960	·0108531044	·0046079625	·0048115732	·0012591783
·50	·50	+ ·1071428571	− ·0416666667	− ·0435606061	− ·0124562938	+ ·0038243007	+ ·0048430431	+ ·0015098462
·51	·51	+ ·1172428571	− ·0373490000	− ·0439176870	− ·0140402022	+ ·0029743478	+ ·0048204944	+ ·0017493971
·52	·52	·1275428571	·0327253333	·0440717188	·0155941423	·0020614426	·0047411602	·0019739758
·53	·53	·1380428571	·0277896667	·0440103416	·0171068143	·0010896211	·0046026320	·0021796401
·54	·54	·1487428571	·0225360000	·0437209552	·0185662948	+ ·0000636520	·0044029121	·0023623985
·55	·55	·1596428571	·0169583333	·0431907197	·0199600240	− ·0010109240	·0041404680	·0025182517
·56	·56	·1707428571	·0110506667	·0424065552	·0212747944	·0021277568	·0038142902	·0026432386
·57	·57	·1820428571	− ·0048070000	·0413551415	·0224967384	·0032796467	·0034239516	·0027334877
·58	·58	·1935428571	+ ·0017786667	·0400229188	·0236113162	·0044585052	·0029696708	·0027852726
·59	·59	·2052428571	·0087123333	·0383960870	·0246033044	·0056553141	·0024523775	·0027950736
·60	·60	+ ·2171428571	+ ·0160000000	− ·0364606061	− ·0254567833	− ·0068600840	+ ·0018737810	+ ·0027596447
·61	·61	+ ·2292428571	+ ·0236476667	− ·0342021960	− ·0261551252	− ·0080618119	+ ·0012364418	+ ·0026760856
·62	·62	·2415428571	·0316613333	·0316063370	·0266809825	·0092484391	+ ·0005438460	·0025419208
·63	·63	·2540428571	·0400470000	·0286582688	·0270162757	·0104068070	− ·0001995176	·0023551836
·64	·64	·2667428571	·0488106667	·0253429915	·0271421813	·0115226133	·0009880769	·0021145077
·65	·65	·2796428571	·0579583333	·0216452652	·0270391194	·0125803664	·0018150925	·0018192246
·66	·66	·2927428571	·0674960000	·0175496097	·0266867424	·0135633401	·0026725710	·0014694690
·67	·67	·3060428571	·0774296667	·0130403051	·0260639228	·0144535271	·0035511755	·0010662901
·68	·68	·3195428571	·0877653333	·0081013916	·0251487412	·0152315921	·0044401317	·0006117711
·69	·69	·3332428571	·0985090000	− ·0027166688	·0239184735	·0158768231	·0053271318	+ ·0001091564
·70	·70	+ ·3471428571	+ ·1096666667	+ ·0031303031	− ·0223495803	− ·0163670838	− ·0061982339	− ·0004370131
·71	·71	+ ·3612428571	+ ·1212443333	+ ·0094562039	− ·0204176943	− ·0166787639	− ·0070377590	− ·0010207568
·72	·72	·3755428571	·1332480000	·0162779539	·0180976075	·0167867283	·0078281838	·0016345026
·73	·73	·3900428571	·1456836667	·0236127131	·0153632602	·0166642675	·0085500305	·0022689260
·74	·74	·4047428571	·1585573333	·0314778811	·0121877293	·0162830461	·0091817531	·0029127804
·75	·75	·4196428571	·1718750000	·0398910985	·0085432146	·0156130491	·0096996187	·0035527178
·76	·76	·4347428571	·1856426667	·0488702449	− ·0044010288	·0146225307	·0100775871	·0041730997
·77	·77	·4500428571	·1998663333	·0584334402	+ ·0002684157	·0132779599	·0102871857	·0047557989
·78	·78	·4655428571	·2145520000	·0685990448	·0054956185	·0115439664	·0102973805	·0052799896
·79	·79	·4812428571	·2297056667	·0793856585	·0113120035	·0093832845	·0100744427	·0057219276
·80	·80	+ ·4971428571	+ ·2453333333	+ ·0908121211	+ ·0177499300	− ·0067566993	− ·0095818135	− ·0060547206
·81	·81	+ ·5132428571	+ ·2614410000	+ ·1028975130	+ ·0248427058	− ·0036229882	− ·0087799629	− ·0062480854
·82	·82	·5295428571	·2780346667	·1156611540	·0326245985	+ ·0000611353	·0076262452	·0062680951
·83	·83	·5460428571	·2951203333	·1291226039	·0411308475	·0043410794	·0060747506	·0060769133
·84	·84	·5627428571	·3127040000	·1433016630	·0503976766	·0092644338	·0040761524	·0056325172
·85	·85	·5796428571	·3307916667	·1582183712	·0604623055	·0148810279	− ·0015775511	·0048884080
·86	·86	·5967428571	·3493893333	·1738930084	·0713629616	·0212429912	+ ·0014776873	·0037933076
·87	·87	·6140428571	·3685030000	·1903460948	·0831388927	·0284048147	·0051500964	·0022908425
·88	·88	·6315428571	·3881386667	·2075983903	·0958303784	·0364234122	·0095042784	− ·0003192156
·89	·89	·6492428571	·4083023333	·2256708948	·1094787421	·0453581817	·0146090777	+ ·0021891381
·90	·90	+ ·6671428571	+ ·4290000000	+ ·2445848485	+ ·1241263637	+ ·0552710700	+ ·0205377598	+ ·0053079083
·91	·91	+ ·6852428571	+ ·4502376667	+ ·2643617312	+ ·1398166905	+ ·0662266350	+ ·0273681931	+ ·0091172820
·92	·92	·7035428571	·4720213333	·2850232630	·1565942505	·0782921113	·0351830364	·0137043298
·93	·93	·7220428571	·4943570000	·3065914039	·1745046630	·0915374742	·0440699303	·0191634082
·94	·94	·7407428571	·5172506667	·3290883540	·1935946519	·1060355065	·0541216933	·0255965775
·95	·95	·7596428571	·5407083333	·3525365530	·2139120564	·1218618644	·0654365226	·0331140353
·96	·96	·7787428571	·5647360000	·3769586812	·2355058445	·1390951458	·0781182001	·0418345674
·97	·97	·7980428571	·5893396667	·4023776585	·2584261236	·1578169572	·0922763018	·0518860136
·98	·98	·8175428571	·6145253333	·4288166448	·2827241534	·1781119830	·1080264131	·0634057526
·99	·99	·8372428571	·6402990000	·4562990403	·3084523576	·2000680555	·1254903496	·0765412038
1·00	1·00	+ ·8571428571	+ ·6666666667	+ ·4848484848	+ ·3356643357	+ ·2237762238	+ ·1447963801	+ ·0914503453

Computation of the Values of G_3^3, G_4^3, G_5^3, ... G_{10}^3, when μ is ·00, ·01, ·02, etc., to 1·00.

$m = 3$.

μ	G_3^3	G_4^3	G_5^3	G_6^3	G_7^3	G_8^3	G_9^3	G_{10}^3
·00	1	·00	− ·1111111111	·0000000000	+ ·0209790210	·0000000000	− ·0045248869	·0000000000
·01		·01	·1110111111	− ·0027262727	·0209328771	+ ·0007685642	·0045072487	− ·0002163933
·02		·02	·1107111111	·0054465455	·0207945656	·0015331314	·0044544398	·0004308395
·03		·03	·1102111111	·0081548182	·0205644464	·0022897166	·0043667773	·0006414045
·04		·04	·1095111111	·0108450909	·0202431195	·0030343588	·0042447886	·0008461811
·05		·05	·1086111111	·0135113636	·0198314248	·0037631330	·0040892094	·0010433013
·06		·06	·1075111111	·0161476364	·0193304425	·0044721622	·0039009814	·0012309496
·07		·07	·1062111111	·0187479091	·0187414925	·0051576294	·0036812486	·0014073757
·08		·08	·1047111111	·0213061818	·0180661348	·0058157896	·0034313541	·0015709072
·09		·09	·1030111111	·0238164545	·0173061694	·0064429818	·0031528348	·0017199616
·10		·10	− ·1011111111	− ·0262727273	+ ·0164636364	+ ·0070356410	− ·0028474163	− ·0018530579
·11		·11	− ·0990111111	− ·0286690000	+ ·0155408156	+ ·0075903102	− ·0025170065	− ·0019688284
·12		·12	·0967111111	·0309992727	·0145402271	·0081036525	·0021636891	·0020660296
·13		·13	·0942111111	·0332575455	·0134646310	·0085724626	·0017897160	·0021435526
·14		·14	·0915111111	·0354378182	·0123170272	·0089936798	·0013974985	·0022004329
·15		·15	·0886111111	·0375340909	·0111006556	·0093643990	·0009895992	·0022358601
·16		·16	·0855111111	·0395403636	·0098189963	·0096818832	·0005687214	·0022491861
·17		·17	·0822111111	·0414506364	·0084757694	·0099435755	− ·0001376993	·0022399335
·18		·18	·0787111111	·0432589091	·0070749348	·0101471107	+ ·0003005136	·0022078023
·19		·19	·0750111111	·0449591818	·0056206925	·0102903279	·0007428561	·0021526765
·20		·20	− ·0711111111	− ·0465454545	+ ·0041174825	+ ·0103712820	+ ·0011861719	− ·0020746303
·21		·21	− ·0670111111	− ·0480117273	+ ·0025699848	+ ·0103882563	+ ·0016272234	− ·0019739316
·22		·22	·0627111111	·0493520000	+ ·0009831194	·0103397735	·0020627048	·0018510461
·23		·23	·0582111111	·0505602727	− ·0006379536	·0102246087	·0024892578	·0017066404
·24		·24	·0535111111	·0516305455	·0022878344	·0100418008	·0029034867	·0015415832
·25		·25	·0486111111	·0525568182	·0039608829	·0097906651	·0033019744	·0013569457
·26		·26	·0435111111	·0533330909	·0056512190	·0094708043	·0036812995	·0011540011
·27		·27	·0382111111	·0539533636	·0073527229	·0090821214	·0040380542	·0009342231
·28		·28	·0327111111	·0544116364	·0090590344	·0086248317	·0043688623	·0006992817
·29		·29	·0270111111	·0547019091	·0107635536	·0080994739	·0046703982	·0004510400
·30		·30	− ·0211111111	− ·0548181818	− ·0124594405	+ ·0075069230	+ ·0049394072	− ·0001915476
·31		·31	− ·0150111111	− ·0547544545	− ·0141396152	+ ·0068484023	+ ·0051727256	+ ·0000769661
·32		·32	·0087111111	·0545047273	·0157967575	·0061254945	·0053673020	·0003521013
·33		·33	− ·0022111111	·0540630000	·0174233075	·0053401547	·0055202194	·0006312974
·34		·34	+ ·0044888889	·0534232727	·0190114651	·0044947219	·0056287175	·0009118446
·35		·35	·0113888889	·0525795455	·0205531906	·0035919311	·0056902170	·0011908979
·36		·36	·0184888889	·0515258182	·0220402036	·0026349253	·0057023425	·0014654915
·37		·37	·0257888889	·0502560909	·0234639844	·0016272675	·0056629484	·0017325563
·38		·38	·0332888889	·0487643636	·0248157729	+ ·0005729527	·0055701436	·0019889376
·39		·39	·0409888889	·0470446364	·0260865690	− ·0005235801	·0054223187	·0022314157
·40		·40	+ ·0488888889	− ·0450909091	− ·0272671328	− ·0016574358	+ ·0052181720	+ ·0024567282
·41		·41	+ ·0569888889	− ·0428971818	− ·0283479844	− ·0028232517	+ ·0049567379	+ ·0026615941
·42		·42	·0652888889	·0404574545	·0293194036	·0040151845	·0046374154	·0028427401
·43		·43	·0737888889	·0377657273	·0301714305	·0052268992	·0042599967	·0029969278
·44		·44	·0824888889	·0348160000	·0308938652	·0064515571	·0038246976	·0031209849
·45		·45	·0913888889	·0316022727	·0314762675	·0076818029	·0033321876	·0032118368

·46	·46	·1004888889	·0281185455	·0319079575	·0089097537	·0027836218	·0032665411
·47	·47	·1097888889	·0243588182	·0321780152	·0101269864	·0021806726	·0032823248
·48	·48	·1192888889	·0203170909	·0322752805	·0113245262	·0015255624	·0032566226
·49	·49	·1289888889	·0159873636	·0321883537	·0124928341	·0008210974	·0031871181
·50	·50	+ ·1388888889	− ·0113636364	− ·0319055944	− ·0136217949	+ ·0000707013	+ ·0030717880
·51	·51	+ ·1489888889	− ·0064399091	− ·0314151228	− ·0147007056	− ·0007215491	+ ·0029089475
·52	·52	·1592888889	− ·0012101818	·0307048190	·0157182635	·0015508890	·0026972992
·53	·53	·1697888889	+ ·0043315455	·0297623229	·0166625533	·0024118286	·0024359842
·54	·54	·1804888889	·0101912727	·0285750344	·0175210361	·0032981168	·0021246349
·55	·55	·1913888889	·0163750000	·0271301136	·0182805369	·0042027022	·0017634322
·56	·56	·2024888889	·0228887273	·0254144805	·0189272327	·0051176957	·0013531639
·57	·57	·2137888889	·0297384545	·0234148152	·0194466405	·0060343308	·0008952863
·58	·58	·2252888889	·0369301818	·0211175575	·0198236053	·0069429238	+ ·0003919884
·59	·59	·2369888889	·0444699091	·0185089074	·0200422880	·0078328326	− ·0001537404
·60	·60	+ ·2488888889	+ ·0523636364	− ·0155748252	− ·0200861539	− ·0086924163	− ·0007380409
·61	·61	+ ·2609888889	+ ·0606173636	− ·0123010306	− ·0199379597	− ·0095089919	− ·0013561102
·62	·62	·2732888889	·0692370909	·0086730036	·0195797424	·0102687925	·0020021268
·63	·63	·2857888889	·0782288182	·0046759844	·0189928073	·0109569229	·0026691706
·64	·64	·2984888889	·0875985455	− ·0002949729	·0181577151	·0115573161	·0033491421
·65	·65	·3113888889	·0973522727	+ ·0044852710	·0170542708	·0120526874	·0040326756
·66	·66	·3244888889	·1074960000	·0096802271	·0156615116	·0124244898	·0047090524
·67	·67	·3377888889	·1180357273	·0153056156	·0139576944	·0126528664	·0053661078
·68	·68	·3512888889	·1289774545	·0213773964	·0119202841	·0127166042	·0059901378
·69	·69	·3649888889	·1403271818	·0279117694	·0095259420	·0125930856	·0065657993
·70	·70	+ ·3788888889	+ ·1520909091	+ ·0349251749	− ·0067505128	− ·0122582399	− ·0070760096
·71	·71	+ ·3929888889	+ ·1642746364	+ ·0424342925	− ·0035690136	− ·0116864942	− ·0075018413
·72	·72	·4072888889	·1768843636	·0504560425	+ ·0000443786	·0108507232	·0078224131
·73	·73	·4217888889	·1899260909	·0590075849	·0041163389	·0097221987	·0080147791
·74	·74	·4364888889	·2034058182	·0681063195	·0086744060	·0082705379	·0080538117
·75	·75	·4513888889	·2173295455	·0777698864	·0137469952	·0064636507	·0079120837
·76	·76	·4664888889	·2317032727	·0880161656	·0193634094	·0042676878	·0075597445
·77	·77	·4817888889	·2465330000	·0988632771	·0255538516	− ·0016469862	·0069643940
·78	·78	·4972888889	·2618247273	·1103295810	·0323494368	+ ·0014359844	·0060909522
·79	·79	·5129888889	·2775844545	·1224336771	·0397822040	·0050206775	·0049015245
·80	·80	+ ·5288888889	+ ·2938181818	+ ·1351944055	+ ·0478851281	+ ·0091485248	− ·0033552645
·81	·81	+ ·5449888889	+ ·3105319091	+ ·1486308464	+ ·0566921324	+ ·0138629937	− ·0014082306
·82	·82	·5612888889	·3277316364	·1627623194	·0662380996	·0192096434	+ ·0009867585
·83	·83	·5777888889	·3454233636	·1776083848	·0765588848	·0252361836	·0038802748
·84	·84	·5944888889	·3636130909	·1931888426	·0876913271	·0319925331	·0073264351
·85	·85	·6113888889	·3823068182	·2095237326	·0996732613	·0395308789	·0113830588
·86	·86	·6284888889	·4015105455	·2266333348	·1125435304	·0479057364	·0161118305
·87	·87	·6457888889	·4212302727	·2445381694	·1263419976	·0571740112	·0215784676
·88	·88	·6632888889	·4414720000	·2632589964	·1411095578	·0673950587	·0278528896
·89	·89	·6809888889	·4622417273	·2828168156	·1568881500	·0786307482	·0350093944
·90	·90	+ ·6988888889	+ ·4835454545	+ ·3032328672	+ ·1737207693	+ ·0909455250	+ ·0431268382
·91	·91	+ ·7169888889	+ ·5053891818	+ ·3245286309	+ ·1916514783	+ ·1044064739	+ ·0522888186
·92	·92	·7352888889	·5277789091	·3467258272	·2107254207	·1190833850	·0625838650
·93	·93	·7537888889	·5507206364	·3698464157	·2309888309	·1350488172	·0741056296
·94	·94	·7724888889	·5742203636	·3939125963	·2524890480	·1523781647	·0869530864
·95	·95	·7913888889	·5982840909	·4189468095	·2752745273	·1711497243	·1012307336
·96	·96	·8104888889	·6229178182	·4449717349	·2993948515	·1914447615	·1170487998
·97	·97	·8297888889	·6481275455	·4720102925	·3249007436	·2133475798	·1345234572
·98	·98	·8492888889	·6739192727	·5000856424	·3518440788	·2369455881	·1537770364
·99	·99	·8689888889	·7002990000	·5292211848	·3802778961	·2623293714	·1749382495
1·00	1·00	+ ·8888888889	+ ·7272727273	+ ·5594405595	+ ·4102564103	+ ·2895927602	+ ·1981424149

COMPUTATION OF THE VALUES OF G_4^4, G_5^4, G_6^4, ... G_{10}^4, WHEN μ IS ·00, ·01, ·02, ETC., TO 1·00.

$m = 4$.

μ	G_4^4	G_5^4	G_6^4	G_7^4	G_8^4	G_9^4	G_{10}^4
·00	1	·00	− ·0909090909	·0000000000	+ ·0153846154	·0000000000	− ·0030959752
·01		·01	·0908090909	− ·0023066923	·0153446254	+ ·0005876472	·0030820512
·02		·02	·0905090909	·0046073846	·0152247753	·0011717679	·0030403739
·03		·03	·0900090909	·0068960769	·0150254254	·0017488479	·0029712270
·04		·04	·0893090909	·0091667692	·0147471754	·0023153965	·0028750819
·05		·05	·0884090909	·0114134615	·0143908654	·0028679596	·0027525966
·06		·06	·0873090909	·0136301538	·0139575754	·0034031305	·0026046122
·07		·07	·0860090909	·0158108462	·0134486254	·0039175631	·0024321503
·08		·08	·0845090909	·0179495385	·0128655754	·0044079827	·0022364091
·09		·09	·0828090909	·0200402308	·0122102253	·0048711990	·0020187582
·10		·10	− ·0809090909	− ·0220769231	+ ·0114846154	+ ·0053041176	− ·0017807337
·11		·11	− ·0788090909	− ·0240536154	+ ·0106910253	+ ·0057037522	− ·0015240320
·12		·12	·0765090909	·0259643077	·0098319754	·0060672361	·0012505026
·13		·13	·0740090909	·0278030000	·0089102254	·0063918352	·0009621408
·14		·14	·0713090909	·0295636923	·0079287754	·0066749589	·0006610798
·15		·15	·0684090909	·0312403846	·0068908654	·0069141728	·0003495808
·16		·16	·0653090909	·0328270769	·0057999754	·0071072106	− ·0000300240
·17		·17	·0620090909	·0343177692	·0046598254	·0072519857	+ ·0002951018
·18		·18	·0585090909	·0357064615	·0034743754	·0073466039	·0006232110
·19		·19	·0548090909	·0369871538	·0022478254	·0073893746	·0009516324
·20		·20	− ·0509090909	− ·0381538462	+ ·0009846154	+ ·0073788236	+ ·0012776223
·21		·21	− ·0468090909	− ·0392005385	− ·0003105746	+ ·0073137042	+ ·0015983774
·22		·22	·0425090909	·0401212308	·0016328246	·0071930103	·0019110493
·23		·23	·0380090909	·0409099231	·0029769746	·0070159872	·0022127587
·24		·24	·0333090909	·0415606154	·0043376246	·0067821448	·0025006114
·25		·25	·0284090909	·0420673077	·0057091346	·0064912683	·0027717141
·26		·26	·0233090909	·0424240000	·0070856246	·0061434317	·0030231919
·27		·27	·0180090909	·0426246923	·0084609746	·0057390084	·0032522052
·28		·28	·0125090909	·0426633846	·0098288246	·0052786838	·0034559683
·29		·29	·0068090909	·0425340769	·0111825745	·0047634679	·0036317689
·30		·30	− ·0009090909	− ·0422307692	− ·0125153846	+ ·0041947059	+ ·0037769877
·31		·31	+ ·0051909091	− ·0417474615	− ·0138201746	+ ·0035740916	+ ·0038891181
·32		·32	·0114909091	·0410781538	·0150896246	·0029036785	·0039657889
·33		·33	·0179909091	·0402168462	·0163161746	·0021858923	·0040047852
·34		·34	·0246909091	·0391575385	·0174920246	·0014235423	·0040040713
·35		·35	·0315909091	·0378942308	·0186091347	+ ·0006198346	·0039618144
·36		·36	·0386909091	·0364209231	·0196592246	− ·0002216177	·0038764083
·37		·37	·0459909091	·0347316154	·0206337746	·0010967808	·0037464987
·38		·38	·0534909091	·0328203077	·0215240246	·0020011890	·0035710083
·39		·39	·0611909091	·0306810000	·0223209746	·0029299330	·0033491631
·40		·40	+ ·0690909091	− ·0283076923	− ·0230153846	− ·0038776470	+ ·0030805201
·41		·41	+ ·0771909091	− ·0256943846	− ·0235977746	− ·0048384976	+ ·0027649942
·42		·42	·0854909091	·0228350769	·0240584246	·0058061709	·0024028868
·43		·43	·0939909091	·0197237692	·0243873747	·0067738616	·0019949155
·44		·44	·1026909091	·0163544615	·0245744246	·0077342599	·0015422432
·45		·45	·1115909091	·0127211538	·0246091346	·0086795405	·0010465094

·46	·46	·1206909091	·0088178462	·0244808247	·0096013495	+·0005098610
·47	·47	·1299909091	·0046385385	·0241785746	·0104907934	−·0000650155
·48	·48	·1394909091	−·0001772308	·0236912247	·0113384268	·0006748610
·49	·49	·1491909091	+·0045720769	·0230073746	·0121342398	·0013158105
·50	·50	+·1590909091	+·0096153846	−·0221153846	−·0128676471	−·0019833592
·51	·51	+·1691909091	+·0149586923	−·0210033746	−·0135274748	−·0026723268
·52	·52	·1794909091	·0206080000	·0196592246	·0141019497	·0033768231
·53	·53	·1899909091	·0265693077	·0180705746	·0145786860	·0040902103
·54	·54	·2006909091	·0328486154	·0162248246	·0149446741	·0048050664
·55	·55	·2115909091	·0394519231	·0141091346	·0151862684	·0055131481
·56	·56	·2226909091	·0463852308	·0117104247	·0152891754	·0062053512
·57	·57	·2339909091	·0536545385	·0090153746	·0152384413	·0068716720
·58	·58	·2454909091	·0612658462	·0060104246	·0150184409	·0075011675
·59	·59	·2571909091	·0692251538	−·0026817747	·0146128643	·0080819139
·60	·60	+·2690909091	+·0775384615	+·0009846154	−·0140047059	−·0086009659
·61	·61	+·2811909091	+·0862117692	+·0050030253	−·0131762523	−·0090443147
·62	·62	·2934909091	·0952510769	·0093879754	·0121090698	·0093968444
·63	·63	·3059909091	·1046623846	·0141542254	·0107839928	·0096422890
·64	·64	·3186909091	·1144516923	·0193167754	·0091811117	·0097631883
·65	·65	·3315909091	·1246250000	·0248908654	·0072797610	·0097408424
·66	·66	·3446909091	·1351883077	·0308919754	·0050585071	·0095552661
·67	·67	·3579909091	·1461476154	·0373358254	−·0024951364	·0091851428
·68	·68	·3714909091	·1575089231	·0442383754	+·0004333568	·0086077769
·69	·69	·3851909091	·1692782308	·0516158255	·0037507820	·0077990460
·70	·70	+·3990909091	+·1814615385	+·0594846155	+·0074817648	−·0067333529
·71	·71	+·4131909091	+·1940648462	+·0678614254	+·0116517586	−·0053835754
·72	·72	·4274909091	·2070941538	·0767631753	·0162870573	·0037210165
·73	·73	·4419909091	·2205554615	·0862070254	·0214148063	−·0017153544
·74	·74	·4566909091	·2344547692	·0962103754	·0270630154	+·0006654103
·75	·75	·4715909091	·2487980769	·1067908654	·0332605699	·0034550056
·76	·76	·4866909091	·2635913846	·1179663753	·0400372434	·0066889415
·77	·77	·5019909091	·2788406923	·1297550254	·0474237099	·0104045642
·78	·78	·5174909091	·2945520000	·1421751754	·0554515545	·0146411092
·79	·79	·5331909091	·3107313077	·1552454254	·0641532870	·0194397572
·80	·80	+·5490909091	+·3273846154	+·1689846154	+·0735623529	+·0248436903
·81	·81	+·5651909091	+·3445179231	+·1834118254	+·0837131460	+·0308981478
·82	·82	·5814909091	·3621372308	·1985463754	·0946410196	·0376504832
·83	·83	·5979909091	·3802485385	·2144078255	·1063822997	·0451502232
·84	·84	·6146909091	·3988578462	·2310159754	·1189742951	·0534491248
·85	·85	·6315909091	·4179711538	·2483908653	·1324553124	·0626012377
·86	·86	·6486909091	·4375944615	·2665527754	·1468646646	·0726629602
·87	·87	·6659909091	·4577337692	·2855222253	·1622426853	·0836931032
·88	·88	·6834909091	·4783950769	·3053199754	·1786307404	·0957529513
·89	·89	·7011909091	·4995843846	·3259670254	·1960712390	·1089063233
·90	·90	+·7190909091	+·5213076923	+·3474846154	+·2146076471	+·1232196378
·91	·91	+·7371909091	+·5435710000	+·3698942254	+·2342844980	+·1387619748
·92	·92	·7554909091	·5663803077	·3932175754	·2551474056	·1556051414
·93	·93	·7739909091	·5897416154	·4174766254	·2772430752	·1738237359
·94	·94	·7926909091	·6136609231	·4426935754	·3006193166	·1934952152
·95	·95	·8115909091	·6381442308	·4688908654	·3253250551	·2146999594
·96	·96	·8306909091	·6631975385	·4960911755	·3514103448	·2375213415
·97	·97	·8499909091	·6888268462	·5243174254	·3789263786	·2620457926
·98	·98	·8694909091	·7150381538	·5535927753	·4079255026	·2883628736
·99	·99	·8891909091	·7418374615	·5839406254	·4384612263	·3165653426
1·00	1·00	+·9090909091	+·7692307692	+·6153846153	+·4705882352	+·3467492259

COMPUTATION OF THE VALUES OF G_5^5, G_6^5, G_7^5, …… G_{10}^5, WHEN μ IS ·00, ·01, ·02, ETC., TO 1·00.

$m = 5$.

μ	G_5^5	G_6^5	G_7^5	G_8^5	G_9^5	G_{10}^5
·00	1	·00	− ·0769230769	·0000000000	+ ·0117647059	·0000000000
·01		·01	·0768230769	− ·0019990000	·0117294218	+ ·0004638701
·02		·02	·0765230769	·0039920000	·0116236894	·0009245853
·03		·03	·0760230769	·0059730000	·0114478688	·0013790027
·04		·04	·0753230769	·0079360000	·0112025600	·0018240033
·05		·05	·0744230769	·0098750000	·0108886029	·0022565044
·06		·06	·0733230769	·0117840000	·0105070776	·0026734711
·07		·07	·0720230769	·0136570000	·0100593041	·0030719284
·08		·08	·0705230769	·0154880000	·0095468424	·0034489734
·09		·09	·0688230769	·0172710000	·0089714924	·0038017872
·10		·10	− ·0669230769	− ·0190000000	+ ·0083352941	+ ·0041276471
·11		·11	− ·0648230769	− ·0206690000	+ ·0076405276	+ ·0044239379
·12		·12	·0625230769	·0222720000	·0068897129	·0046881649
·13		·13	·0600230769	·0238030000	·0060856100	·0049179652
·14		·14	·0573230769	·0252560000	·0052312188	·0051111198
·15		·15	·0544230769	·0266250000	·0043297794	·0052655660
·16		·16	·0513230769	·0279040000	·0033847718	·0053794087
·17		·17	·0480230769	·0290870000	·0023999159	·0054509331
·18		·18	·0445230769	·0301680000	·0013791718	·0054786162
·19		·19	·0408230769	·0311410000	+ ·0003267394	·0054611393
·20		·20	− ·0369230769	− ·0320000000	− ·0007529412	+ ·0053973994
·21		·21	− ·0328230769	− ·0327390000	− ·0018551900	+ ·0052865216
·22		·22	·0285230769	·0333520000	·0029750871	·0051278709
·23		·23	·0240230769	·0338330000	·0041074724	·0049210646
·24		·24	·0193230769	·0341760000	·0052469459	·0046659838
·25		·25	·0144230769	·0343750000	·0063878676	·0043627854
·26		·26	·0093230769	·0344240000	·0075243576	·0040119147
·27		·27	− ·0040230769	·0343170000	·0086502959	·0036141167
·28		·28	+ ·0014769231	·0340480000	·0097593224	·0031704485
·29		·29	·0071769231	·0336110000	·0108448371	·0026822913
·30		·30	+ ·0130769231	− ·0330000000	− ·0119000000	+ ·0021513622
·31		·31	+ ·0191769231	− ·0322090000	− ·0129177312	+ ·0015797262
·32		·32	·0254769231	·0312320000	·0138907106	·0009698085
·33		·33	·0319769231	·0300630000	·0148113782	+ ·0003244062
·34		·34	·0386769231	·0286960000	·0156719341	− ·0003532997
·35		·35	·0455769231	·0271250000	·0164643382	·0010597320
·36		·36	·0526769231	·0253440000	·0171803106	·0017909056
·37		·37	·0599769231	·0233470000	·0178113312	·0025424154
·38		·38	·0674769231	·0211280000	·0183486400	·0033094244
·39		·39	·0751769231	·0186810000	·0187832371	·0040866513
·40		·40	+ ·0830769231	− ·0160000000	− ·0191058824	− ·0048683592
·41		·41	+ ·0911769231	− ·0130790000	− ·0193070959	− ·0056483427
·42		·42	·0994769231	·0099120000	·0193771576	·0064199170
·43		·43	·1079769231	·0064930000	·0193061076	·0071759049
·44		·44	·1166769231	− ·0028160000	·0190837459	·0079086253
·45		·45	·1255769231	+ ·0011250000	·0186996324	·0086098810

·46		·46	·1346769231	·0053360000	·0181430871	·0092709470
·47		·47	·1439769231	·0098230000	·0174031900	·0098825581
·48		·48	·1534769231	·0145920000	·0164687812	·0104348974
·49		·49	·1631769231	·0196490000	·0153284606	·0109175835
·50		·50	+·1730769231	+·0250000000	−·0139705882	−·0113196594
·51		·51	+·1831769231	+·0306510000	−·0123832841	−·0116295802
·52		·52	·1934769231	·0366080000	·0105544282	·0118352005
·53		·53	·2039769231	·0428770000	·0084716606	·0119237634
·54		·54	·2146769231	·0494640000	·0061223812	·0118818877
·55		·55	·2255769231	·0563750000	·0034937500	·0116955563
·56		·56	·2366769231	·0636160000	−·0005726871	·0113501042
·57		·57	·2479769231	·0711930000	+·0026541276	·0108302061
·58		·58	·2594769231	·0791120000	·0062002541	·0101198650
·59		·59	·2711769231	·0873790000	·0100794924	·0092023998
·60		·60	+·2830769231	+·0960000000	+·0143058824	−·0080604335
·61		·61	+·2951769231	+·1049810000	+·0188937041	−·0066758808
·62		·62	·3074769231	·1143280000	·0238574776	·0050299367
·63		·63	·3199769231	·1240470000	·0292119629	·0031030640
·64		·64	·3326769231	·1341440000	·0349721600	−·0008749817
·65		·65	·3455769231	·1446250000	·0411533088	+·0016753473
·66		·66	·3586769231	·1554960000	·0477708894	·0045697282
·67		·67	·3719769231	·1667630000	·0548406218	·0078307460
·68		·68	·3854769231	·1784320000	·0623784659	·0114817779
·69		·69	·3991769231	·1905090000	·0704006218	·0155470048
·70		·70	+·4130769231	+·2030000000	+·0789235294	+·0200514242
·71		·71	+·4271769231	+·2159110000	+·0879638688	+·0250208607
·72		·72	·4414769231	·2292480000	·0975385600	·0304819799
·73		·73	·4559769231	·2430170000	·1076647629	·0364622986
·74		·74	·4706769231	·2572240000	·1183598776	·0429901979
·75		·75	·4855769231	·2718750000	·1296415441	·0500949352
·76		·76	·5006769231	·2869760000	·1415276423	·0578066552
·77		·77	·5159769231	·3025330000	·1540362924	·0661564033
·78		·78	·5314769231	·3185520000	·1671858541	·0751761365
·79		·79	·5471769231	·3350390000	·1809949276	·0848987358
·80		·80	+·5630769231	+·3520000000	+·1954823529	+·0953580185
·81		·81	+·5791769231	+·3694410000	+·2106672100	+·1065887497
·82		·82	·5954769231	·3873680000	·2265688188	·1186266543
·83		·83	·6119769231	·4057870000	·2432067394	·1315084296
·84		·84	·6286769231	·4247040000	·2606007718	·1452717566
·85		·85	·6455769231	·4441250000	·2787709559	·1599553125
·86		·86	·6626769231	·4640560000	·2977375718	·1755987823
·87		·87	·6799769231	·4845030000	·3175211394	·1922428712
·88		·88	·6974769231	·5054720000	·3381424188	·2099293161
·89		·89	·7151769231	·5269690000	·3596224100	·2287008984
·90		·90	+·7330769231	+·5490000000	+·3819823529	+·2486014551
·91		·91	+·7511769231	+·5715710000	+·4052437276	+·2696758912
·92		·92	·7694769231	·5946880000	·4294282541	·2919701918
·93		·93	·7879769231	·6183570000	·4545578924	·3155314343
·94		·94	·8066769231	·6425840000	·4806548424	·3404077996
·95		·95	·8255769231	·6673750000	·5077415441	·3666485845
·96		·96	·8446769231	·6927360000	·5358406776	·3943042146
·97		·97	·8639769231	·7186730000	·5649751629	·4234262548
·98		·98	·8834769231	·7451920000	·5951681600	·4540674222
·99		·99	·9031769231	·7722990000	·6264430688	·4862815978
1·00		1·00	+·9230769231	+·8000000000	+·6588235294	+·5201238390

$m = 6.$

μ	G_6^6	G_7^6	G_8^6	G_9^6	G_{10}^6	μ	G_6^6	G_7^6	G_8^6	G_9^6	G_{10}^6
·00	1	·00	·0666666667	·0000000000	+ ·0092879257						
·01		·01	·0665666667	·0017637059	·0092563567	·51		·51	+ ·1934333333	+ ·0426510000	·0051969664
·02		·02	·0662666667	·0035214118	·0091617699	·52		·52	·2037333333	·0488432941	·0029853880
·03		·03	·0657666667	·0052071176	·0090045252	·53		·53	·2142333333	·0553475882	·0005125275
·04		·04	·0650666667	·0069948235	·0087852220	·54		·54	·2249333333	·0621698824	+ ·0022342752
·05		·05	·0641666667	·0086985294	·0085047020	·55		·55	·2358333333	·0693161765	·0052678599
·06		·06	·0630666667	·0103722353	·0081640436	·56		·56	·2469333333	·0767924706	·0086013967
·07		·07	·0617666667	·0120099412	·0077645673	·57		·57	·2582333333	·0846047647	·0122479357
·08		·08	·0602666667	·0136050471	·0073078330	·58		·58	·2697333333	·0927590588	·0162213967
·09		·09	·0585666667	·0151533529	·0067950409	·59		·59	·2814333333	·1012613529	·0205352199
·10		·10	·0566666667	·0166470588	+ ·0062300309	·60		·60	+ ·2933333333	+ ·1101176471	+ ·0252037152
·11		·11	·0545666667	·0180807647	+ ·0056132831	·61		·61	+ ·3054333333	+ ·1193339412	+ ·0302410725
·12		·12	·0522666667	·0194484706	·0049479172	·62		·62	·3177333333	·1289162353	·0356618120
·13		·13	·0497666667	·0207441765	·0042366936	·63		·63	·3302333333	·1388705294	·0414806936
·14		·14	·0470666667	·0219618824	·0034826120	·64		·64	·3429333333	·1492028235	·0477127172
·15		·15	·0441666667	·0230955882	·0026889120	·65		·65	·3558333333	·1599191176	·0543731230
·16		·16	·0410666667	·0241392941	·0018590751	·66		·66	·3689333333	·1710254118	·0614773910
·17		·17	·0377666667	·0250870000	·0009968199	·67		·67	·3822333333	·1825277059	·0690412410
·18		·18	·0342666667	·0259327059	+ ·0001061067	·68		·68	·3957333333	·1944320000	·0770866331
·19		·19	·0305666667	·0266704118	·0008088643	·69		·69	·4094333333	·2067442941	·0856117672
·20		·20	− ·0266666667	− ·0272941176	·0017436532	·70		·70	+ ·4233333333	+ ·2194705882	+ ·0946510835
·21		·21	− ·0225666667	·0277978235	·0026935800	·71		·71	+ ·4374333333	+ ·2326168824	+ ·1042152620
·22		·22	·0182666667	·0281755294	·0036537249	·72		·72	·4517333333	·2461891765	·1143212226
·23		·23	·0137666667	·0284212353	·0046189274	·73		·73	·4662333333	·2601934706	·1249861251
·24		·24	·0090666667	·0285289412	·0055837880	·74		·74	·4809333333	·2746357647	·1362273699
·25		·25	− ·0041666667	·0284926471	·0065420664	·75		·75	·4958333333	·2895220588	·1480625967
·26		·26	+ ·0009333333	·0283063529	·0074896828	·76		·76	·5109333333	·3048583529	·1605096856
·27		·27	·0062333333	·0279640588	·0084187170	·77		·77	·5262333333	·3206506471	·1735867568
·28		·28	·0117333333	·0274597647	·0093234090	·78		·78	·5417333333	·3369049412	·1873121699
·29		·29	·0174333333	·0267874706	·0101971591	·79		·79	·5574333333	·3536272353	·2017045252
·30		·30	+ ·0233333333	− ·0259411765	·0110331270	·80		·80	+ ·5733333333	+ ·3708235294	+ ·2167826615
·31		·31	+ ·0294333333	− ·0249148824	− ·0118242327	·81		·81	+ ·5894333333	+ ·3884998235	+ ·2325656619
·32		·32	·0357333333	·0237025882	·0125631564	·82		·82	·6057333333	·4066621176	·2490728435
·33		·33	·0422333333	·0222982941	·0132423380	·83		·83	·6222333333	·4253164118	·2663237673
·34		·34	·0489333333	·0206960000	·0138539775	·84		·84	·6389333333	·4444687059	·2843382331
·35		·35	·0558333333	·0188897059	·0143900349	·85		·85	·6558333333	·4641250000	·3031362810
·36		·36	·0629333333	·0168734118	·0148422303	·86		·86	·6729333333	·4842912941	·3227381909
·37		·37	·0702333333	·0146411176	·0152020432	·87		·87	·6902333333	·5049735882	·3431644830
·38		·38	·0777333333	·0121868235	·0154607143	·88		·88	·7077333333	·5261778824	·3644359173
·39		·39	·0854333333	·0095045294	·0156092433	·89		·89	·7254333333	·5479101765	·3865734936
·40		·40	+ ·0933333333	− ·0065882353	− ·0156383901	·90		·90	+ ·7433333333	+ ·5701764706	+ ·4095984520
·41		·41	+ ·1014333333	− ·0034319412	− ·0155386748	·91		·91	+ ·7614333333	+ ·5929827647	+ ·4335322726
·42		·42	·1097333333	− ·0000296471	·0153003775	·92		·92	·7797333333	·6163350588	·4583966752
·43		·43	·1182333333	+ ·0036246471	·0149135379	·93		·93	·7982333333	·6402393529	·4842136199
·44		·44	·1269333333	·0075369412	·0143679564	·94		·94	·8169333333	·6647016471	·5110053068
·45		·45	·1358333333	·0117132353	·0136531927	·95		·95	·8358333333	·6897279412	·5387941757
·46		·46	·1449333333	·0161595294	·0127585670	·96		·96	·8549333333	·7153242353	·5676029068
·47		·47	·1542333333	·0208818235	·0116731591	·97		·97	·8792333333	·7414965294	·5974544199
·48		·48	·1637333333	·0258861176	·0103858091	·98		·98	·8937333333	·7682508235	·6283718751
·49		·49	·1734333333	·0311784118	·0088851169	·99		·99	·9134333333	·7955931176	·6603786725
·50		·50	+ ·1833333333	+ ·0367647059	− ·0071594427	1·00		1·00	+ ·9333333333	+ ·8235294118	+ ·6934984520

COMPUTATION OF THE VALUES OF G_n^7, G_m^7 AND G_m^8, WHEN μ IS ·00, ·01, ·02, ETC., TO 1·00.

$\mu = G_n^7$	G_n^7 ($m = 7$)	G_m^7 ($m = 7$)	G_m^8 ($m = 8$)	$\mu = G_n^7$	G_n^7 ($m = 7$)	G_m^7 ($m = 7$)	G_m^8 ($m = 8$)
[illegible]	[illegible]		[illegible]				
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

TABLE OF LOG G_0^0, G_1^0, G_2^0, ... G_{10}^0, FOR VALUES OF θ FROM 0° TO 90°.

$\mu = \cos\theta$. $m = 0$.

θ	G_0^0 Log	G_1^0 Log	Log G_2^0	Log G_3^0	Log G_4^0	Log G_5^0	Log G_6^0	Log G_7^0	Log G_8^0	Log G_9^0	Log G_{10}^0
0°	0	L. μ	9·8239087,4	9·6020599,9	9·3590219,4	9·1037494,4	8·8405080,0	8·5716626,9	8·2986614,2	8·0224550,1	7·7437014,1
1			9·8237102,8	9·6016630,2	9·3583602,3	9·1027566,7	8·8391178,3	8·5698085,7	8·2962767,9	8·0194731,5	7·7400554,2
2			9·8231145,8	9·6004709,4	9·3563719,2	9·0997713,6	8·8349335,5	8·5642220,7	8·2890832,5	8·0104657,5	7·7290254,1
3			9·8221207,4	9·5984803,0	9·3530475,5	9·0947723,0	8·8279136,7	8·5548289,6	8·2769569,5	7·9952375,7	7·7103162,6
4			9·8207272,2	9·5956852,3	9·3483710,9	9·0877234,4	8·8179870,4	8·5415012,4	8·2596827,4	7·9734456,0	7·6834037,0
5			9·8189319,0	9·5920775,1	9·3423197,2	9·0785727,4	8·8050493,4	8·5240480,3	8·2369345,9	7·9445591,6	7·6474563,2
6			9·8167319,9	9·5876463,5	9·3348631,8	9·0672503,9	8·7889584,4	8·5022039,1	8·2082462,0	7·9077965,7	7·6012101,1
7			9·8141240,4	9·5823782,1	9·3259631,5	9·0536662,7	8·7695264,2	8·4756082,9	8·1729638,2	7·8620198,1	7·5427633,3
8			9·8111039,5	9·5762568,0	9·3155724,8	9·0377069,0	8·7465102,0	8·4437775,8	8·1301722,5	7·8055477,6	7·4689835,4
9			9·8076668,7	9·5692625,4	9·3036335,3	9·0192300,3	8·7195942,7	8·4060569,2	8·0785612,2	7·7357960,5	7·3748344,7
10			9·8038072,6	9·5613727,6	9·2900776,3	8·9980600,6	8·6883714,0	8·3615510,6	8·0162040,4	7·6485805,2	7·2507980,2
11			9·7995187,8	9·5525608,1	9·2748220,7	8·9739777,3	8·6523071,7	8·3090025,4	7·9401248,4	7·5364902,7	7·0760904,0
12			9·7947942,5	9·5427960,6	9·2577683,8	8·9467096,2	8·6106918,8	8·2465872,0	7·8454069,0	7·3844788,6	6·7882178,8
13			9·7896256,1	9·5320431,2	9·2387988,4	8·9159113,0	8·5625605,0	8·1715254,9	7·7230655,4	7·1536502,7	5·7962273,1
14			9·7840039,1	9·5202615,1	9·2177724,9	8·8811434,7	8·5065611,8	8·0792937,0	7·5538065,9	6·6632296,3	6·6539742,7 n
15			9·7779191,1	9·5074045,9	9·1945193,9	8·8418367,5	8·4407249,4	7·9617868,8	7·2817671,8	6·6535694,5 n	6·9613309,0 n
16			9·7713601,5	9·4934189,0	9·1688336,2	8·7972377,6	8·3620287,2	7·8021447,5	6·4937228,7	7·1111952,8 n	7·1205476,1 n
17			9·7643147,4	9·4782429,2	9·1404629,8	8·7463235,5	8·2654848,0	7·5545557,9	7·0798394,3 n	7·3115864,7 n	7·2196578,7 n
18			9·7567693,2	9·4618057,6	9·1090954,6	8·6876579,8	8·1419618,5	6·9656202,7	7·4159227,5 n	7·4329757,0 n	7·2841960,4 n
19			9·7487089,1	9·4440253,6	9·0743394,1	8·6191380,1	7·9717650,9	7·2078416,4 n	7·5888876,9 n	7·5137581,3 n	7·3249200,5 n
20			9·7401169,6	9·4248063,6	9·0356952,4	8·5375061,6	7·6972555,0	7·6019633,5 n	7·6997848,4 n	7·5686229,0 n	7·3471386,9 n
21			9·7309752,3	9·4040374,1	8·9925124,3	8·4373146,2	6·8654356,4	7·7928215,7 n	7·7765748,0 n	7·6046115,7 n	7·3535585,0 n
22			9·7212635,4	9·3815874,5	8·9439231,6	8·3083784,9	7·5228991,5 n	7·9144299,1 n	7·8309766,1 n	7·6254853,1 n	7·3453756,9 n
23			9·7109596,2	9·3573014,6	8·8887346,2	8·1279356,5	7·8565343,6 n	7·9997984,4 n	7·8689305,4 n	7·6333037,9 n	7·3226802,3 n
24			9·7000387,7	9·3309939,4	8·8252432,8	7·8240353,2	8·0332125,9 n	8·0622171,0 n	7·8937650,9 n	7·6290761,0 n	7·2844623,1 n
25			9·6884736,4	9·3024411,5	8·7508956,2	6·0479754,1	8·1501000,5 n	8·1083285,3 n	7·9074323,3 n	7·6130475,3 n	7·2283413,2 n
26			9·6762338,4	9·2713699,7	8·6616086,5	7·7927592,8 n	8·2345856,1 n	8·1418750,8 n	7·9110408,9 n	7·5847517,4 n	7·1496655,0 n
27			9·6632855,6	9·2374422,2	8·5502332,4	8·0879364,5 n	8·2982975,3 n	8·1651214,1 n	7·9051274,7 n	7·5429212,4 n	7·0392832,1 n
28			9·6495910,0	9·2002327,0	8·4024226,8	8·2545509,8 n	8·3472382,7 n	8·1794802,6 n	7·8897570,5 n	7·4851277,7 n	6·8765584,4 n
29			9·6351079,0	9·1591969,6	8·1817821,5	8·3682762,9 n	8·3848857,8 n	8·1858211,2 n	7·8645413,4 n	7·4069929,7 n	6·5985935,5 n
30			9·6197887,6	9·1136218,9	7·7289329,9	8·4525840,1 n	8·4134067,3 n	8·1846349,4 n	7·8285735,5 n	7·3002366,9 n	5·5911759,5 n
31			9·6035800,2	9·0625485,2	7·6255768,6 n	8·5178622,7 n	8·4342142,9 n	8·1761109,4 n	7·7802308,8 n	7·1469988,6 n	6·4905624,2
32			9·5864210,2	9·0046441,4	8·1304177,2 n	8·5696163,9 n	8·4482487,5 n	8·1601677,0 n	7·7167685,3 n	6·8967796,7 n	6·8045959,9
33			9·5682428,0	8·9379788,7	8·3511362,5 n	8·6111037,1 n	8·4561316,6 n	8·1364421,5 n	7·6334310,8 n	6·2297842,0 n	6·9712201,7
34			9·5489664,3	8·8596049,6	8·4915996,5 n	8·6443982,1 n	8·4582538,4 n	8·1042374,5 n	7·5213120,3 n	6·6406106,1	7·0774769,2
35			9·5285011,5	8·7646845,0	8·5930941,4 n	8·6708894,9 n	8·4548227,6 n	8·0624074,8 n	7·3610172,8 n	7·0071885,5	7·1488061,8
36			9·5067419,6	8·6444219,4	8·6712804,3 n	8·6915400,5 n	8·4458894,3 n	8·0091320,8 n	7·0957654,4 n	7·1915991,1	7·1959771,0
37			9·4835664,0	8·4800551,1	8·7337969,9 n	8·7070297,9 n	8·4313520,2 n	7·9414806,3 n	6·2903685,6 n	7·3091104,1	7·2243284,3
38			9·4588305,9	8·2179432,7	8·7849412,9 n	8·7178416,9 n	8·4109477,1 n	7·8544760,8 n	6·9225022,9	7·3899524,2	7·2366229,1
39			9·4323639,3	7·4877434,2	8·8273659,8 n	8·7243134,0 n	8·3842230,9 n	7·7388431,6 n	7·2618264,9	7·4464905,7	7·2341146,9
40			9·4039621,3	8·0040344,7 n	8·8628185,5 n	8·7266698,4 n	8·3504773,3 n	7·5742686,2 n	7·4405076,5	7·4848718,7	7·2169624,8

41	9·3733777, 4	8·3608453, 3 n	8·8925059, 3 n	8·7250431, 6 n	8·3086649, 2 n	7·3000822, 1 n	7·5573435, 6	7·5084421, 8	7·1842654, 4
42	9·3403070, 7	8·5499172, 6 n	8·9172915, 7 n	8·7194836, 8 n	8·2572202, 9 n	6·3810970, 2 n	7·6399534, 1	7·5190347, 8	7·1337847, 2
43	9·3043718, 3	8·6778535, 9 n	8·9378088, 6 n	8·7099642, 8 n	8·1937333, 0 n	7·1717293, 7	7·6999738, 0	7·5175284, 2	7·0611621, 0
44	9·2650925, 2	8·7736524, 1 n	8·9545310, 7 n	8·6963786, 8 n	8·1142986, 7 n	7·4987873, 2	7·7433333, 4	7·5040710, 4	6·9578274, 9
45	9·2218487, 7	8·8494850, 2 n	8·9678153, 5 n	8·6785334, 0 n	8·0120518, 4 n	7·6756656, 4	7·7733731, 6	7·4781161, 0	6·8048274, 9
46	9·1738180, 7	8·9116233, 6 n	8·9779314, 3 n	8·6561318, 5 n	7·8732671, 1 n	7·7935543, 2	7·7920609, 4	7·4382812, 2	6·5466290, 2
47	9·1198772, 8	8·9637252, 2 n	8·9850816, 4 n	8·6287483, 8 n	7·6636180, 1 n	7·8787945, 8	7·8005342, 5	7·3819846, 0	5·7728349, 3
48	9·0584352, 8	9·0081088, 0 n	8·9894138, 2 n	8·5957856, 9 n	7·2397748, 2 n	7·9426671, 8	7·7993571, 0	7·3045486, 9	6·3632169, 9 n
49	8·9871303, 1	9·0463343, 4 n	8·9910301, 6 n	8·5564061, 2 n	7·0463195, 4	7·9910044, 4	7·7886307, 2	7·1970433, 6	6·7064275, 9 n
50	8·9022345, 7	9·0795003, 7 n	8·9899930, 6 n	8·5094176, 0 n	7·5916199, 3	8·0271693, 3	7·7680161, 8	7·0397828, 2	6·8839395, 2 n
51	8·7973424, 8	9·1084079, 5 n	8·9863289, 8 n	8·4530759, 0 n	7·8200220, 5	8·0532225, 5	7·7366665, 6	6·7744512, 7	6·9970279, 7 n
52	8·6599705, 5	9·1336575, 0 n	8·9800294, 3 n	8·3847131, 7 n	7·9632101, 2	8·0704619, 8	7·6930556, 4	5·9282421, 8	7·0736157, 2 n
53	8·4601155, 5	9·1557089, 8 n	8·9710508, 9 n	8·2999877, 5 n	8·0650816, 5	8·0796872, 6	7·6345921, 3	6·6240447, 7 n	7·1252697, 7 n
54	8·0848682, 1	9·1749210, 0 n	8·9593129, 6 n	8·1911490, 8 n	8·1419618, 5	8·0813430, 0	7·5568161, 1	6·9582101, 9 n	7·1576712, 9 n
55	7·6378298, 3 n	9·1915767, 3 n	8·9446934, 0 n	8·0422261, 2 n	8·2017124, 9	8·0755849, 2	7·4514594, 4	7·1351103, 8 n	7·1737833, 7 n
56	8·3146387, 8 n	9·2059022, 8 n	8·9270223, 7 n	7·8107572, 5 n	8·2487031, 0	8·0623044, 0	7·3008953, 5	7·2501588, 6 n	7·1749780, 4 n
57	8·5646853, 9 n	9·2180789, 7 n	8·9060713, 7 n	7·2808173, 1 n	8·2855897, 1	8·0411130, 7	7·0557833, 8	7·3303569, 5 n	7·1615267, 1 n
58	8·7203156, 2 n	9·2282529, 6 n	8·8815390, 1 n	7·4175812, 9	8·3140798, 9	8·0112823, 7	6·4144886, 7	7·3870746, 7 n	7·1326438, 5 n
59	8·8329500, 7 n	9·2365414, 7 n	8·8530282, 5 n	7·8492670, 5	8·3353084, 3	7·9716237, 0	6·7879985, 8 n	7·4260507, 0 n	7·0862815, 1 n
60	8·9208187, 4 n	9·2430380, 3 n	8·8200137, 1 n	8·0572369, 2	8·3500358, 6	7·9202485, 9	7·1657694, 4 n	7·4504261, 8 n	7·0183842, 3 n
61	8·9925224, 1 n	9·2478160, 6 n	8·7817906, 9 n	8·1932008, 6	8·3587603, 7	7·8541073, 7	7·3570444, 0 n	7·4619170, 1 n	6·9210618, 9 n
62	9·0528084, 8 n	9·2509314, 5 n	8·7373939, 9 n	8·2925792, 4	8·3617799, 5	7·7679865, 4	7·4812290, 1 n	7·4613259, 2 n	6·7771657, 3 n
63	9·1045757, 3 n	9·2524246, 2 n	8·6854633, 4 n	8·3694210, 8	8·3592287, 5	7·6520702, 6	7·5692263, 9 n	7·4487341, 4 n	6·5387496, 3 n
64	9·1497241, 7 n	9·2523219, 2 n	8·6240055, 4 n	8·4307372, 9	8·3510907, 8	7·4844692, 5	7·6336805, 0 n	7·4235260, 3 n	5·9221023, 7 n
65	9·1895664, 8 n	9·2506364, 0 n	8·5499415, 1 n	8·4805220, 4	8·3371986, 1	7·1971730, 1	7·6809380, 3 n	7·3842369, 0 n	6·2532168, 1
66	9·2250471, 6 n	9·2473683, 4 n	8·4581579, 7 n	8·5212564, 5	8·3172152, 2	5·9206450, 0	7·7145468, 5 n	7·3281471, 0 n	6·6372595, 1
67	9·2568680, 7 n	9·2425052, 6 n	8·3392134, 6 n	8·5545757, 7	8·2905943, 8	7·1454730, 1 n	7·7365972, 6 n	7·2503297, 2 n	6·8273628, 1
68	9·2855645, 8 n	9·2360214, 8 n	8·1724896, 7 n	8·5816023, 4	8·2565106, 5	7·4539922, 0 n	7·7483058, 9 n	7·1412866, 9 n	6·9474729, 3
69	9·3115541, 3 n	9·2278775, 6 n	7·8961325, 3 n	8·6031265, 0	8·2137317, 9	7·6265959, 4 n	7·7502985, 2 n	6·9797167, 8 n	7·0290427, 1
70	9·3351680, 4 n	9·2180187, 9 n	6·9388148, 3 n	8·6197121, 8	8·1603986, 0	7·7434650, 8 n	7·7427306, 3 n	6·6999330, 7 n	7·0847228, 0
71	9·3566733, 9 n	9·2063735, 7 n	7·7850891, 2	8·6317602, 4	8·0935939, 7	7·8289194, 9 n	7·7253210, 1 n	5·5687882, 1 n	7·1206528, 9
72	9·3762882, 4 n	9·1928508, 6 n	8·1133767, 3	8·6395468, 5	8·0084494, 3	7·8935881, 2 n	7·6973017, 2 n	6·6286239, 0	7·1400238, 3
73	9·3941926, 3 n	9·1773366, 0 n	8·2956047, 0	8·6432499, 9	7·8960045, 3	7·9430307, 9 n	7·6572480, 9 n	6·9411634, 6	7·1443718, 1
74	9·4105366, 3 n	9·1596892, 8 n	8·4210175, 6	8·6429622, 4	7·7369292, 8	7·9804737, 9 n	7·6027499, 8 n	7·1123234, 3	7·1340750, 7
75	9·4254462, 3 n	9·1397337, 9 n	8·5156606, 1	8·6386988, 7	7·4749879, 8	8·0079051, 7 n	7·5296715, 6 n	7·2252007, 0	7·1084973, 7
76	9·4390280, 0 n	9·1172526, 3 n	8·5907781, 0	8·6303987, 8	6·6878677, 0	8·0265749, 4 n	7·4304540, 1 n	7·3045680, 7	7·0657552, 0
77	9·4513724, 3 n	9·0919743, 5 n	8·6522556, 3	8·6179205, 1	7·3023092, 8 n	8·0372555, 4 n	7·2893548, 1 n	7·3610929, 2	7·0021098, 9
78	9·4625568, 0 n	9·0635568, 7 n	8·7035573, 0	8·6010297, 1	7·6496833, 2 n	8·0403689, 2 n	7·0644654, 8 n	7·4002097, 7	6·9103041, 7
79	9·4726471, 5 n	9·0315636, 2 n	8·7468902, 7	8·5793806, 5	7·8360201, 3 n	8·0360533, 2 n	6·5448862, 3 n	7·4249044, 9	6·7748963, 5
80	9·4817000, 3 n	8·9954279, 6 n	8·7837429, 1	8·5524827, 8	7·9614813, 7 n	8·0241850, 8 n	6·6661621, 6	7·4368064, 5	6·5544891, 6
81	9·4897639, 1 n	8·9543991, 2 n	8·8151616, 5	8·5196495, 3	8·0539803, 0 n	8·0043566, 4 n	7·1021832, 1	7·4366459, 9	6·0423392, 0
82	9·4968801, 0 n	8·9074554, 3 n	8·8419055, 6	8·4799133, 2	8·1253244, 4 n	7·9758199, 8 n	7·3089995, 6	7·4244553, 0	6·1491883, 1 n
83	9·5030837, 8 n	8·8531600, 6 n	8·8645375, 4	8·4318823, 6	8·1816026, 6 n	7·9373528, 7 n	7·4412834, 2	7·3995543, 2	6·5876211, 8 n
84	9·5084046, 4 n	8·7894028, 6 n	8·8834818, 3	8·3734850, 4	8·2263436, 4 n	7·8870188, 9 n	7·5347556, 7	7·3604040, 5	6·7919782, 1 n
85	9·5128673, 7 n	8·7129139, 1 n	8·8990597, 1	8·3014812, 1	8·2617499, 6 n	7·8216825, 0 n	7·6034659, 2	7·3041457, 1	6·9194860, 3 n
86	9·5164922, 2 n	8·6181993, 9 n	8·9115152, 4	8·2104041, 4	8·2892631, 1 n	7·7359970, 9 n	7·6543190, 4	7·2256426, 3	7·0060722, 3 n
87	9·5192953, 3 n	8·4949642, 5 n	8·9210308, 0	8·0899778, 4	8·3098474, 7 n	7·6196369, 0 n	7·6911460, 5	7·1147443, 9	7·0656992, 0 n
88	9·5212889, 6 n	8·3200878, 9 n	8·9277387, 4	7·9170967, 5	8·3241467, 9 n	7·4496557, 8 n	7·7161804, 1	6·9485841, 1	7·1050194, 2 n
89	9·5224817, 2 n	8·0197861, 2 n	8·9317288, 3	7·6179884, 4	8·3325737, 8 n	7·1522730, 5 n	7·7307394, 9	6·6534633, 7	7·1274838, 3 n
90	9·5228787, 5 n	− ∞	8·9330532, 1	− ∞	8·3353580, 2 n	− ∞	7·7355194, 9	− ∞	7·1348019, 9 n

TABLE OF LOG G_1^1, G_2^1, G_3^1, ... G_{10}^1, FOR VALUES OF θ FROM 0 TO 90°.

$\mu = \cos\theta$. $m = 1$.

θ	G_1^1 Log	Log G_2^1	Log G_3^1	Log G_4^1	Log G_5^1	Log G_6^1	Log G_7^1	Log G_8^1	Log G_9^1	Log G_{10}^1
0°	0	0·0000000, 0	9·9030899, 9	9·7569619, 5	9·5808706, 9	9·3845760, 5	9·1737226, 8	8·9518739, 3	8·7214250, 1	8·4840641, 0
1		9·9999338, 5	9·9029246, 1	9·7566642, 7	9·5804075, 9	9·3839144, 2	9·1728294, 2	8·9507158, 9	8·7199690, 0	8·4822770, 2
2		9·9997353, 6	9·9024282, 8	9·7557706, 3	9·5790170, 5	9·3819271, 8	9·1701454, 4	8·9472349, 1	8·7155904, 0	8·4768997, 1
3		9·9994044, 1	9·9016005, 0	9·7542796, 6	9·5766958, 5	9·3786078, 6	9·1656591, 0	8·9414114, 4	8·7082584, 6	8·4678863, 5
4		9·9989407, 9	9·9004403, 4	9·7521887, 0	9·5734380, 5	9·3739448, 9	9·1593498, 7	8·9332116, 4	8·6979198, 3	8·4551563, 7
5		9·9983442, 3	9·8989466, 0	9·7494943, 0	9·5692356, 7	9·3679223, 4	9·1511889, 9	8·9225870, 0	8·6844976, 1	8·4385928, 2
6		9·9976143, 5	9·8971176, 2	9·7461918, 0	9·5640781, 3	9·3605191, 2	9·1411381, 3	8·9094728, 6	8·6678880, 6	8·4180363, 2
7		9·9967507, 1	9·8949514, 5	9·7422755, 2	9·5579522, 4	9·3517086, 5	9·1291486, 6	8·8937859, 5	8·6479561, 7	8·3932764, 5
8		9·9957527, 8	9·8924457, 1	9·7377386, 1	9·5508420, 9	9·3414586, 2	9·1151607, 1	8·8754226, 8	8·6245310, 4	8·3640417, 4
9		9·9946199, 3	9·8895976, 3	9·7325729, 9	9·5427287, 0	9·3297300, 3	9·0991011, 0	8·8542540, 4	8·5973957, 9	8·3299800, 7
10		9·9933514, 6	9·8864040, 3	9·7267693, 3	9·5335899, 5	9·3164767, 6	9·0808817, 2	8·8301217, 9	8·5662781, 2	8·2906368, 6
11		9·9919465, 8	9·8828613, 3	9·7203170, 1	9·5234002, 6	9·3016446, 7	9·0603970, 9	8·8028317, 2	8·5308340, 3	8·2454176, 5
12		9·9904043, 9	9·8789654, 8	9·7132038, 7	9·5121300, 0	9·2851699, 2	9·0375202, 5	8·7721440, 0	8·4906244, 4	8·1935308, 7
13		9·9887239, 3	9·8747119, 6	9·7054162, 5	9·4997453, 5	9·2669781, 3	9·0120990, 6	8·7377610, 7	8·4450814, 9	8·1338992, 8
14		9·9869041, 2	9·8700958, 1	9·6969388, 3	9·4862076, 4	9·2469822, 2	8·9839503, 4	8·6993113, 1	8·3934595, 6	8·0650113, 6
15		9·9849437, 8	9·8651115, 0	9·6877543, 9	9·4714725, 7	9·2250797, 7	8·9528508, 8	8·6563200, 6	8·3347492, 2	7·9846461, 1
16		9·9828416, 4	9·8597529, 9	9·6778437, 7	9·4554896, 8	9·2011506, 1	8·9185282, 9	8·6081785, 6	8·2675488, 1	7·8893633, 4
17		9·9805963, 2	9·8540136, 5	9·6671855, 5	9·4382012, 3	9·1750525, 5	8·8806448, 7	8·5540778, 1	8·1898239, 8	7·7733885, 2
18		9·9782063, 3	9·8478862, 4	9·6557558, 7	9·4195411, 1	9·1466164, 0	8·8387768, 2	8·4929233, 2	8·0984616, 8	7·6258349, 3
19		9·9756700, 7	9·8413628, 4	9·6435281, 4	9·3994333, 1	9·1156392, 0	8·7923830, 0	8·4231767, 0	7·9883319, 4	7·4219525, 4
20		9·9729858, 2	9·8344348, 3	9·6304727, 0	9·3777902, 5	9·0818752, 0	8·7407577, 9	8·3425780, 1	7·8500351, 0	7·0799695, 0
21		9·9701517, 4	9·8270928, 5	9·6165564, 4	9·3545103, 6	9·0450232, 5	8·6829572, 9	8·2476062, 9	7·6631966, 3	5·9105710, 5 n
22		9·9671658, 6	9·8193266, 7	9·6017423, 4	9·3294753, 8	9·0047094, 0	8·6176786, 4	8·1323125, 4	7·3673410, 1	7·0786960, 1 n
23		9·9640260, 8	9·8111252, 0	9·5859889, 8	9·3025467, 0	8·9604617, 8	8·5430499, 8	7·9853748, 3	6·4368143, 9	7·3310404, 1 n
24		9·9607301, 6	9·8024763, 2	9·5692496, 3	9·2735600, 9	8·9116723, 8	8·4562378, 6	7·7806312, 2	7·1906510, 0 n	7·4641346, 9 n
25		9·9572757, 1	9·7933669, 2	9·5514719, 5	9·2423201, 7	8·8575406, 7	8·3526459, 1	7·4284782, 1	7·4960892, 9 n	7·5446397, 0 n
26		9·9536601, 9	9·7837826, 6	9·5325964, 4	9·2085910, 8	8·7969806, 2	8·2240428, 1	6·5657061, 8 n	7·6502249, 3 n	7·5939204, 3 n
27		9·9498808, 8	9·7737079, 3	9·5125557, 4	9·1720853, 8	8·7284667, 7	8·0532721, 1	7·4913336, 7 n	7·7446801, 6 n	7·6213580, 6 n
28		9·9459349, 3	9·7631257, 1	9·4912730, 1	9·1324476, 9	8·6497569, 7	7·7934117, 8	7·7405197, 9 n	7·8057278, 0 n	7·6316730, 4 n
29		9·9418192, 6	9·7520174, 1	9·4686600, 1	9·0892309, 1	8·5573542, 7	7·1773777, 5	7·8789357, 8 n	7·8444846, 3 n	7·6273115, 7 n
30		9·9375306, 3	9·7403626, 9	9·4446151, 2	9·0418624, 5	8·4453455, 8	7·4478539, 0 n	7·9681260, 2 n	7·8665561, 0 n	7·6093510, 6 n
31		9·9330656, 0	9·7281392, 4	9·4190202, 0	8·9895915, 7	8·3024707, 7	7·8257895, 0 n	8·0285179, 8 n	7·8750155, 2 n	7·5778593, 0 n
32		9·9284204, 8	9·7153226, 1	9·3917372, 3	8·9314064, 6	8·1027121, 6	8·0067419, 4 n	8·0693520, 7 n	7·8715495, 9 n	7·5319283, 0 n
33		9·9235914, 0	9·7018858, 8	9·3626032, 1	8·8658961, 2	7·7571454, 0	8·1198785, 1 n	8·0955156, 1 n	7·8569595, 7 n	7·4694096, 1 n
34		9·9185742, 1	9·6877993, 5	9·3314241, 2	8·7910071, 7	6·8824619, 2 n	8·1975283, 8 n	8·1098147, 0 n	7·8313654, 7 n	7·3861920, 9 n
35		9·9133645, 2	9·6730301, 9	9·2979663, 7	8·7035794, 6	7·8292651, 7 n	8·2527831, 5 n	8·1139122, 8 n	7·7942488, 7 n	7·2743726, 4 n
36		9·9079576, 4	9·6575420, 1	9·2619451, 4	8·5983682, 6	8·0872566, 9 n	8·2922062, 1 n	8·1087550, 8 n	7·7443336, 8 n	7·1169031, 5 n
37		9·9023486, 2	9·6412942, 9	9·2230083, 9	8·4656693, 6	8·2349351, 8 n	8·3195089, 3 n	8·0947830, 3 n	7·6792652, 3 n	6·8664055, 0 n
38		9·8965321, 4	9·6242418, 1	9·1807135, 5	8·2841504, 5	8·3343192, 2 n	8·3369449, 9 n	8·0720170, 7 n	7·5948619, 5 n	6·2472857, 0 n
39		9·8905025, 9	9·6063339, 0	9·1344935, 9	7·9887586, 7	8·4060182, 0 n	8·3459308, 0 n	8·0400680, 5 n	7·4833290, 6 n	6·5455050, 4
40		9·8842539, 7	9·5875135, 1	9·0836049, 5	6·8839452, 0	8·4594276, 3 n	8·3473540, 2 n	7·9980687, 5 n	7·3281960, 6 n	6·9215824, 4

41	9·8777798,6	9·5677161,6	9·0270455,8	7·8851621,3 n	8·4995993,9 n	8·3417341,8 n	7·9445034,0 n	7·0846048,8 n	7·1012795,5
42	9·8710734,6	9·5468685,4	8·9634179,1	8·1920464,3 n	8·5295375,0 n	8·3293067,4 n	7·8768474,6 n	6·5091487,4 n	7·2109148,3
43	9·8641274,6	9·5248869,3	8·8906885,6	8·3592986,3 n	8·5511483,3 n	8·3100587,2 n	7·7908022,1 n	6·7219342,1	7·2822866,9
44	9·8569340,9	9·5016749,9	8·8057299,0	8·4711920,7 n	8·5656902,4 n	8·2837310,2 n	7·6785019,9 n	7·1196220,5	7·3281409,8
45	9·8494850,0	9·4771212,5	8·7033569,0	8·5528420,2 n	8·5740091,4 n	8·2497820,0 n	7·5234499,0 n	7·3077248,9	7·3546576,9
46	9·8417712,7	9·4510956,9	8·5739945,5	8·6151752,0 n	8·5766682,7 n	8·2073103,5 n	7·2809173,9 n	7·4241708,4	7·3650590,9
47	9·8337833,3	9·4234453,7	8·3966745,4	8·6639258,4 n	8·5740240,9 n	8·1548942,2 n	6·7093038,9 n	7·5024462,4	7·3609094,6
48	9·8255109,0	9·3939887,2	8·1080048,8	8·7024662,9 n	8·5662686,9 n	8·0903003,8 n	6·9187849,8	7·5558909,3	7·3426516,1
49	9·8169429,2	9·3625078,2	7·0822394,7	8·7329363,7 n	8·5534514,3 n	8·0098870,9 n	7·3205388,5	7·5910512,4	7·3097217,0
50	8·8080675,0	9·3287381,3	7·9954614,5 n	8·7567670,6 n	8·5354865,3 n	7·9072850,3 n	7·5124989,5	7·6114758,3	7·2604053,3
51	9·7988718,0	9·2923538,8	8·3111195,4 n	8·7749504,6 n	8·5121441,3 n	7·7699471,5 n	7·6334219,9	7·6191350,9	7·1912609,3
52	9·7893419,8	9·2529477,7	8·4842310,6 n	8·7881899,4 n	8·4830267,1 n	7·5674957,1 n	7·7169602,4	7·6150163,1	7·0956748,7
53	9·7794630,2	9·2100008,4	8·6015663,6 n	8·7969893,5 n	8·4475253,2 n	7·1843364,8 n	7·7765646,3	7·5993819,2	6·9599043,5
54	9·7692186,9	9·1628376,2	8·6887148,3 n	8·8017071,3 n	8·4047391,6 n	6·7614316,5	7·8189051,8	7·5718299,1	6·7487925,2
55	9·7585913,0	9·1105558,1	8·7567700,3 n	8·8025904,7 n	8·3533400,6 n	7·4142193,6	7·8476929,3	7·5312054,8	6·3113087,5
56	9·7475616,5	9·0519112,1	8·8115503,5 n	8·7997966,0 n	8·2913237,9 n	7·6531256,5	7·8651116,8	7·4752962,9	6·1612781,4 n
57	9·7361087,6	8·9851195,3	8·8564844,3 n	8·7934065,4 n	8·2155352,6 n	7·7968531,4	7·8724470,6	7·4001406,4	6·6793642,9 n
58	9·7242097,1	8·9074889,1	8·8937577,7 n	8·7834306,7 n	8·1206525,6 n	7·8959040,9	7·8703810,8	7·2983617,7	6·8944932,9 n
59	9·7118393,4	8·8146751,5	8·9248446,9 n	8·7698103,7 n	7·9966994,5 n	7·9682297,7	7·8591359,0	7·1545426,5	7·0234747,0 n
60	9·6989700,0	8·6989700,0	8·9507819,5 n	8·7524142,6 n	7·8214342,6 n	8·0222604,8	7·8385128,6	6·9277859,8	7·1086529,7 n
61	9·6855712,3	8·5445686,9	8·9723219,0 n	8·7310289,0 n	7·5252962,3 n	8·0625917,6	7·8078575,8	6·4169731,7	7·1657308,9 n
62	9·6716092,9	8·3097057,4	8·9900229,7 n	8·7053412,2 n	6·1093422,7 n	8·0919562,7	7·7659292,0	6·4976896,9 n	7·2020794,4 n
63	9·6570467,6	7·7858542,3	9·0043066,3 n	8·6749127,4 n	7·4782374,1	8·1120611,3	7·7106241,7	6·9376131,7 n	7·2214950,9 n
64	9·6418419,6	7·8938028,1 n	9·0154934,2 n	8·6391354,3 n	7·7799815,6	8·1239838,7	7·6383844,1	7·1400143,9 n	7·2258982,8 n
65	9·6259482,6	8·3302879,3 n	9·0238278,5 n	8·5971638,4 n	7·9491191,0	8·1283805,2	7·5428677,4	7·2660552,2 n	7·2159863,4 n
66	9·6093133,0	8·5386403,3 n	9·0294947,3 n	8·5478006,6 n	8·0639845,0	8·1255912,1	7·4114580,8	7·3521979,6 n	7·1914398,4 n
67	9·5918780,1	8·6751290,7 n	9·0326306,9 n	8·4892990,9 n	8·1485650,3	8·1156930,4	7·2133742,9	7·4126453,1 n	7·1508454,4 n
68	9·5735754,2	8·7757553,1 n	9·0333316,7 n	8·4189939,5 n	8·2134091,0	8·0985090,9	6·8301072,8	7·4542806,7 n	7·0912201,2 n
69	9·5543291,6	8·8547456,4 n	9·0316585,8 n	8·3325535,9 n	8·2640777,1	8·0735853,0	6·4411607,7 n	7·4808287,4 n	7·0068595,1 n
70	9·5340516,8	8·9191943,4 n	9·0276402,7 n	8·2222567,1 n	8·3038390,3	8·0401226,9	7·0805471,3 n	7·4943536,0 n	6·8861747,6 n
71	9·5126419,2	8·9731527,1 n	9·0212750,7 n	8·0722197,4 n	8·3347528,4	7·9968390,2	7·3198386,4 n	7·4959091,0 n	6·7012311,3 n
72	9·4899823,6	9·0191516,2 n	9·0125308,5 n	7·8403144,0 n	8·3581761,2	7·9417111,4	7·4643956,5 n	7·4858023,1 n	6·3506162,4 n
73	9·4659353,4	9·0588767,6 n	9·0013436,7 n	7·3141601,6 n	8·3750203,9	7·8714614,2	7·5638144,0 n	7·4636765,4 n	5·7545012,3
74	9·4403380,8	9·0935059,2 n	8·9876148,0 n	7·4378710,1	8·3858973,1	7·7804368,5	7·6357852,4 n	7·4284182,9 n	6·5205097,3
75	9·4129962,3	9·1238931,1 n	8·9712055,9 n	7·8720484,2	8·3911991,4	7·6578061,4	7·6885977,8 n	7·3778778,9 n	6·7718225,7
76	9·3836751,8	9·1506760,1 n	8·9519299,7 n	8·0801537,0	8·3911447,9	7·4786386,3	7·7267272,1 n	7·3081822,4 n	6·9181562,0
77	9·3520880,3	9·1743419,4 n	8·9295435,7 n	8·2160177,7	8·3858056,8	7·1595581,8	7·7527757,6 n	7·2121476,4 n	7·0149781,2
78	9·3178789,1	9·1952705,2 n	8·9037276,6 n	8·3153291,4	8·3751121,3	6·1048966,3 n	7·7682828,3 n	7·0748091,3 n	7·0812356,6
79	9·2805988,4	9·2137618,7 n	8·8740658,2 n	8·3922140,4	8·3588468,6	7·2250894,9 n	7·7741160,6 n	6·8569825,1 n	7·1254975,4
80	9·2396702,3	9·2300561,2 n	8·8400088,7 n	8·4537133,1	8·3366218,1	7·5034910,7 n	7·7706377,6 n	6·3737393,2 n	7·1521415,2
81	9·1943324,4	9·2443470,5 n	8·8008209,6 n	8·5038372,4	8·3078311,6	7·6656241,4 n	7·7577861,1 n	6·3951341,1	7·1634148,4
82	9·1435553,0	9·2567919,2 n	8·7554933,5 n	8·5450774,1	8·2715730,6	7·7770960,8 n	7·7350588,9 n	6·8588979,6	7·1602463,8
83	9·0858944,7	9·2675187,0 n	8·7026002,6 n	8·5790767,2	8·2265086,6	7·8592884,4 n	7·7014160,0 n	7·0696903,0	7·1425333,8
84	9·0192345,7	9·2766313,5 n	8·6400420,5 n	8·6069646,9	8·1706096,0	7·9218497,6 n	7·6550557,9 n	7·2017629,8	7·1091160,0
85	8·9402960,1	9·2842138,3 n	8·5645525,6 n	8·6295387,5	8·1006661,8	7·9699170,2 n	7·5929402,7 n	7·2931127,4	7·0573960,1
86	8·8435845,2	9·2903330,1 n	8·4706486,0 n	8·6473706,5	8·0112421,7	8·0065071,2 n	7·5097655,3 n	7·3583670,2	6·9823797,9
87	8·7188001,6	9·2950411,2 n	8·3480387,6 n	8·6608703,2	7·8920812,4	8·0334925,2 n	7·3953718,3 n	7·4045925,7	6·8741337,8
88	8·5428191,6	9·2983771,3 n	8·1736063,7 n	8·6703251,9	7·7200939,5	8·0520568,5 n	7·2267416,4 n	7·4355863,8	6·7097955,5
89	8·2418553,2	9·3003680,8 n	7·8735699,4 n	8·6759257,3	7·4215178,8	8·0629304,1 n	6·9301642,5 n	7·4534651,2	6·4157577,1
90	− ∝	9·3010300,0 n	− ∝	8·6777807,1	− ∝	8·0665127,1 n	− ∝	7·4593130,8	− ∝

TABLE OF LOG G_3^2, G_4^2, G_5^2, ... G_{10}^2, FOR VALUES OF θ FROM 0° TO 90°.

$\mu = \cos\theta$. $m = 2$.

θ	G_3^2 Log	Log G_4^2	Log G_5^2	Log G_6^2	Log G_7^2	Log G_8^2	Log G_9^2	Log G_{10}^2
0°	0	9·9330532, 1	9·8239087, 4	9·6856060, 4	9·5259052, 0	9·3498139, 4	9·1607577, 0	8·9611853, 5
1		9·9328988, 5	9·8236441, 1	9·6852090, 9	9·5253538, 5	9·3490861, 2	9·1598313, 3	8·9600383, 1
2		9·9324356, 5	9·8228499, 3	9 6840176, 0	9·5236986, 4	9·3469006, 8	9·1570490, 6	8·9565925, 3
3		9·9316631, 8	9·8215251, 5	9·6820294, 2	9·5209356, 7	9·3432511, 6	9·1524007, 6	8·9508327, 4
4		9·9305807, 1	9·8196680, 0	9·6792410, 3	9·5170585, 8	9·3381268, 2	9·1458694, 0	8·9427331, 4
5		9·9291872, 8	9·8172761, 3	9·6756475, 6	9·5120584, 0	9·3315125, 7	9·1374310, 3	8·9322575, 8
6		9·9274815, 7	9·8143463, 4	9·6712424, 9	9·5059232, 9	9·3233883, 5	9·1270537, 2	8·9193574, 1
7		9·9254620, 1	9·8108747, 5	9·6660178, 6	9·4986385, 8	9·3137293, 0	9·1146975, 0	8·9039711, 9
8		9·9231267, 2	9·8068567, 1	9·6599640, 3	9·4901864, 5	9·3025048, 5	9·1003129, 1	8·8860224, 0
9		9·9204735, 0	9·8022868, 1	9·6530697, 6	9·4805458, 8	9·2896786, 3	9·0838406, 1	8·8654181, 8
10		9·9174998, 3	9·7971587, 3	9·6453218, 7	9·4696922, 5	9·2752078, 5	9·0652098, 7	8·8420462, 3
11		9·9142028, 4	9·7914653, 6	9·6367053, 5	9·4575970, 2	9·2590420, 3	9·0443361, 7	8·8157705, 5
12		9·9105793, 6	9·7851986, 3	9·6272030, 1	9·4442272, 9	9·2411225, 6	9·0211200, 2	8·7864280, 7
13		9·9066258, 5	9·7783495, 5	9·6167955, 5	9·4295455, 3	9·2213813, 3	8·9954435, 8	8·7538215, 1
14		9·9023383, 6	9·7709080, 2	9·6054610, 5	9·4135087, 7	9·1997391, 7	8·9671672, 0	8·7177115, 2
15		9·8977126, 0	9·7628628, 8	9·5931749, 3	9·3960678, 3	9·1761037, 7	8·9361242, 6	8·6778043, 5
16		9·8927438, 4	9·7542017, 9	9·5799097, 2	9·3771669, 4	9·1503680, 4	8·9021161, 3	8·6337375, 3
17		9·8874269, 2	9·7449110, 7	9·5656344, 4	9·3567419, 5	9·1224061, 1	8·8649024, 2	8·5850553, 5
18		9·8817562, 3	9·7349756, 6	9·5503144, 8	9·3347197, 3	9·0920702, 6	8·8241904, 8	8·5311771, 1
19		9·8757256, 5	9·7243789, 7	9·5339109, 9	9·3110161, 9	9·0591853, 8	8·7796195, 0	8·4713457, 1
20		9·8693285, 9	9·7131027, 8	9·5163804, 0	9·2855344, 5	9·0235424, 9	8·7307373, 1	8·4045475, 7
21		9·8625578, 7	9·7011269, 7	9·4976736, 3	9·2581621, 7	8·9848896, 3	8·6769679, 9	8·3293828, 3
22		9·8554057, 3	9·6884294, 1	9·4777353, 7	9·2287686, 6	8·9429197, 6	8·6175616, 3	8·2438334, 4
23		9·8478637, 6	9·6749857, 0	9·4565030, 3	9·1972004, 0	8·8972527, 8	8·5515140, 4	8·1448300, 4
24		9·8399229, 0	9·6607689, 2	9·4339055, 0	9·1632757, 5	8·8474111, 9	8·4774373, 1	8·0273455, 3
25		9·8315733, 1	9·6457493, 3	9·4098618, 4	9·1267781, 8	8·7927836, 4	8·3933314, 0	7·8822398, 5
26		9·8228044, 0	9·6298940, 3	9·3842791, 7	9·0874460, 3	8·7325677, 7	8·2961507, 5	7·6899150, 3
27		9·8136046, 6	9·6131664, 2	9·3570504, 5	9·0449593, 6	8·6656804, 1	8·1809071, 4	7·3934242, 7
28		9·8039616, 6	9·5955259, 2	9·3280516, 9	8·9989217, 4	8·5906085, 8	8·0385307, 6	6·5577116, 7
29		9·7938619, 4	9·5769271, 6	9·2971377, 7	8·9488321, 0	8·5051387, 2	7·8495926, 1	7·1590435, 6 n
30		9·7832909, 0	9·5573194, 0	9·2641378, 4	8·8940449, 8	8·4058399, 8	7·5578102, 6	7·4684389, 0 n
31		9·7722326, 7	9·5366456, 3	9·2288486, 5	8·8337080, 1	8·2869513, 4	6·7421593, 2	7·6172436, 5 n
32		9·7606700, 4	9·5148415, 0	9·1910256, 8	8·7666606, 3	8·1376022, 1	7·3236954, 6 n	7·7041110, 3 n
33		9·7485842, 8	9·4918341, 9	9·1503713, 3	8·6912633, 1	7·9330350, 5	7·6404297, 5 n	7·7564695, 1 n
34		9·7359549, 4	9·4675406, 2	9·1065177, 4	8·6050860, 1	7·5912817, 0	7·7953731, 6 n	7·7857304, 8 n
35		9·7227596, 9	9·4418656, 7	9·0590029, 0	8·5042957, 5	6·4373224, 0 n	7·8886764, 3 n	7·7975472, 8 n
36		9·7089741, 3	9·4146996, 1	9·0072345, 6	8·3822962, 8	7·5971146, 4 n	7·9481032, 8 n	7·7949193, 4 n
37		9·6945714, 5	9·3859150, 1	8·9504359, 3	8·2261785, 5	7·8539426, 7 n	7·9852616, 2 n	7·7793663, 0 n
38		9·6795222, 0	9·3553627, 1	8·8875585, 0	8·0046003, 5	7·9933899, 4 n	8·0059985, 5 n	7·7514184, 6 n
39		9·6637939, 0	9·3228665, 1	8·8171370, 1	7·5960536, 2	8·0820921, 0 n	8·0135416, 5 n	7·7107670, 5 n
40		9·6473506, 4	9·2882161, 1	8·7370309, 8	7·2157882, 1 n	8·1415864, 7 n	8·0097202, 3 n	7·6562026, 9 n

41	9·6301525, 8	9·2511576, 0	8·6439290, 5	7·8232270, 1 n	8·1815148, 9 n	7·9954921, 7 n	7·5852768, 2 n
42	9·6121553, 6	9·2113805, 1	8·5322927, 1	8·0450682, 0 n	8·2069591, 1 n	7·9711862, 1 n	7·4934775, 7 n
43	9·5933094, 4	9·1684992, 8	8·3917462, 7	8·1749260, 8 n	8·2208517, 7 n	7·9365800, 9 n	7·3721384, 2 n
44	9·5735592, 1	9·1220266, 1	8·1989854, 3	8·2615785, 4 n	8·2249578, 3 n	7·8908617, 9 n	7·2022812, 1 n
45	9·5528419, 7	9·0713337, 6	7·8794256, 4	8·3225380, 7 n	8·2203278, 0 n	7·8324520, 7 n	6·9286109, 0 n
46	9·5310867, 4	9·0155893, 4	5·7113853, 8 n	8·3660720, 1 n	8·2075236, 9 n	7·7585947, 6 n	6·1466860, 6 n
47	9·5082126, 3	8·9536606, 0	7·8488969, 6 n	8·3967017, 3 n	8·1867255, 8 n	7·6644501, 6 n	6·7118493, 9
48	9·4841269, 9	8·8839462, 2	8·1286948, 3 n	8·4171286, 1 n	8·1577679, 5 n	7·5408573, 2 n	7·0427369, 2
49	9·4587229, 0	8·8040731, 6	8·2835773, 3 n	8·4290528, 2 n	8·1201130, 7 n	7·3676466, 0 n	7·2078416, 4
50	9·4318762, 0	8·7103020, 5	8·3865223, 3 n	8·4335669, 5 n	8·0727644, 2 n	7·0846013, 1 n	7·3091381, 2
51	9·4034414, 3	8·5962143, 1	8·4604479, 8 n	8·4313621, 6 n	8·0140692, 1 n	6·1687625, 8 n	7·3746695, 4
52	9·3732466, 3	8·4493125, 2	8·5155419, 1 n	8·4228387, 6 n	7·9413270, 4 n	6·9298530, 0	7·4160544, 0
53	9·3410865, 2	8·2395787, 4	8·5572075, 4 n	8·4081644, 6 n	7·8499578, 0 n	7·2460970, 9	7·4390372, 7
54	9·3067130, 9	7·8540867, 1	8·5886359, 1 n	8·3873004, 7 n	7·7315243, 0 n	7·4103233, 1	7·4466458, 8
55	9·2698228, 9	7·3964208, 0 n	8·6118469, 0 n	8·3599958, 9 n	7·5680373, 7 n	7·5143883, 5	7·4403769, 0
56	9·2300388, 6	8·0622005, 4 n	8·6281802, 1 n	8·3257604, 5 n	7·3088225, 6 n	7·5846703, 8	7·4206800, 6
57	9·1868844, 3	8·3007943, 3 n	8·6385462, 6 n	8·2837942, 8 n	6·6326293, 6 n	7·6323601, 0	7·3870853, 6
58	9·1397446, 5	8·4445253, 9 n	8·6435685, 4 n	8·2328640, 3 n	7·0404836, 6	7·6631353, 2	7·3380718, 4
59	9·0878065, 4	8·5447893, 7 n	8·6436642, 4 n	8·1710688, 2 n	7·4017536, 9	7·6801341, 9	7·2705996, 5
60	9·0299632, 3	8·6197886, 9 n	8·6390938, 5 n	8·0953887, 1 n	7·5825530, 9	7·6851189, 5	7·1789309, 3
61	8·9646519, 2	8·6780936, 2 n	8·6299885, 8 n	8·0007268, 4 n	7·6978968, 5	7·6789688, 3	7·0515113, 9
62	8·8895617, 0	8·7244177, 7 n	8·6163637, 5 n	7·8775803, 8 n	7·7780303, 5	7·6619091, 8	6·8605835, 4
63	8·8010621, 1	8·7616225, 3 n	8·5981219, 2 n	7·7050525, 0 n	7·8354039, 1	7·6335532, 9	6·5060312, 4
64	8·6929536, 8	8·7915661, 6 n	8·5750440, 2 n	7·4198764, 8 n	7·8762952, 2	7·5928260, 5	5·8831502, 0 n
65	8·5532646, 3	8·8155147, 3 n	8·5467690, 1 n	6·3998121, 2 n	7·9042393, 5	7·5376919, 3	6·6575912, 2 n
66	8·3536770, 0	8·8343604, 8 n	8·5127569, 5 n	7·3088289, 6	7·9213383, 8	7·4645506, 5	6·9050183, 1 n
67	7·9918314, 5	8·8487460, 7 n	8·4722235, 6 n	7·6238682, 3	7·9288488, 6	7·3668374, 4	7·0470177, 2 n
68	7·4026121, 2 n	8·8591400, 4 n	8·4240339, 4 n	7·7947661, 0	7·9274617, 3	7·2312197, 5	7·1396398, 5 n
69	8·1592527, 9 n	8·8658832, 8 n	8·3665114, 9 n	7·9084051, 3	7·9174388, 2	7·0243707, 1	7·2019785, 2 n
70	8·4129535, 3 n	8·8692196, 9 n	8·2970867, 8 n	7·9903566, 2	7·8986571, 8	6·6097117, 7	7·2426681, 2 n
71	8·5665850, 2 n	8·8693152, 9 n	8·2115839, 7 n	8·0516633, 3	7·8705946, 7	6·3680822, 4 n	7·2661399, 3 n
72	8·6754634, 1 n	8·8662706, 1 n	8·1025918, 2 n	8·0980668, 2	7·8322284, 5	6·9289383, 8 n	7·2747073, 0 n
73	8·7587297, 4 n	8·8601279, 9 n	7·9550163, 1 n	8·1328962, 6	7·7818278, 7	7·1541591, 2 n	7·2693868, 5 n
74	8·8253039, 9 n	8·8508746, 6 n	7·7296547, 2 n	8·1582116, 7	7·7165141, 8	7·2904931, 6 n	7·2502125, 7 n
75	8·8800691, 7 n	8·8384424, 9 n	7·2425911, 2 n	8·1753273, 9	7·6313311, 3	7·3830519, 9 n	7·2162398, 2 n
76	8·9259869, 4 n	8·8227031, 6 n	7·2564675, 8	8·1850807, 0	7·5169569, 1	7·4483568, 9 n	7·1652799, 5 n
77	8·9649859, 6 n	8·8034604, 7 n	7·7211557, 9	8·1879714, 3	7·3528614, 1	7·4942047, 0 n	7·0931238, 9 n
78	8·9983895, 6 n	8·7804357, 0 n	7·9341508, 8	8·1842367, 4	7·0772552, 3	7·5247647, 0 n	6·9916070, 2 n
79	9·0271418, 8 n	8·7532459, 9 n	8·0707628, 1	8·1738858, 2	6·1191568, 7	7·5423809, 5 n	6·8429834, 7 n
80	9·0519372, 2 n	8·7213702, 5 n	8·1692278, 7	8·1566987, 4	6·9620329, 5 n	7·5483144, 7 n	6·5969378, 0 n
81	9·0732981, 5 n	8·6840963, 6 n	8·2443679, 3	8·1321991, 5	7·2855325, 5 n	7·5430792, 2 n	5·9355576, 5 n
82	9·0916247, 2 n	8·6404354, 2 n	8·3034811, 7	8·0995839, 7	7·4610631, 1 n	7·5265589, 5 n	6·3450795, 3
83	9·1072269, 4 n	8·5889783, 3 n	8·3506830, 7	8·0575923, 2	7·5778248, 6 n	7·4979823, 0 n	6·7145057, 9
84	9·1203466, 5 n	8·5276391, 8 n	8·3885038, 5	8·0042555, 9	7·6617795, 7 n	7·4557369, 2 n	6·9007002, 3
85	9·1311726, 6 n	8·4531632, 5 n	8·4185894, 5	7·9364121, 9	7·7240562, 1 n	7·3969243, 1 n	7·0193032, 7
86	9·1398512, 9 n	8·3600768, 3 n	8·4420490, 3	7·8486607, 3	7·7703843, 5 n	7·3163373, 6 n	7·1005325, 1
87	9·1464942, 1 n	8·2380953, 1 n	8·4596413, 9	7·7307750, 1	7·8040303, 3 n	7·2038919, 2 n	7·1566914, 8
88	9·1511833, 8 n	8·0641081, 7 n	8·4718805, 4	7·5596840, 4	7·8269438, 1 n	7·0366169, 3 n	7·1938033, 5
89	9·1539749, 9 n	7·7643375, 6 n	8·4790997, 3	7·2616387, 0	7·8402824, 2 n	6·7408362, 1 n	7·2150344, 7
90	9·1549019, 6 n	∞	8·4814860, 6	∞	7·8446639, 6 n	∞	7·2219521, 7

TABLE OF LOG G_4^3, G_5^3, G_6^3, ... G_{10}^3, FOR VALUES OF θ FROM 0° TO 90°.

$\mu = \cos\theta$. $m = 3$.

θ	Log G_4^3	Log G_5^3	Log G_6^3	Log G_7^3	Log G_8^3	Log G_9^3	Log G_{10}^3
0°	□	9·9488474, 8	9·8616973, 0	9·7477539, 5	9·6130553, 8	9·4617877, 0	9·2969774, 5
1		9·9486986, 3	9·8614492, 2	9·7473900, 9	9·6125591, 9	9·4611426, 5	9·2961669, 8
2		9·9482519, 9	9·8607047, 3	9·7462980, 2	9·6110697, 7	9·4592060, 7	9·2937333, 9
3		9·9475071, 6	9·8594629, 8	9·7444761, 6	9·6085844, 4	9·4559736, 8	9·2896701, 4
4		9·9464635, 2	9·8577225, 9	9·7419218, 7	9·6050986, 4	9·4514382, 1	9·2839662, 3
5		9·9451202, 1	9·8554816, 2	9·7386314, 9	9·6006060, 8	9·4455895, 6	9·2966062, 2
6		9·9434760, 6	9·8527375, 1	9·7346001, 6	9·5950984, 1	9·4384142, 7	9·2675695, 8
7		9·9415296, 9	9·8494871, 6	9·7298219, 6	9·5885653, 6	9·4298957, 4	9·2568307, 3
8		9·9392794, 5	9·8457268, 9	9·7242898, 4	9·5809946, 8	9·4200140, 3	9·2443588, 7
9		9·9367233, 7	9·8414522, 9	9·7179953, 5	9·5723716, 5	9·4087451, 5	9·2301367, 2
10		9·9338592, 7	9·8366584, 0	9·7109289, 1	9·5626794, 2	9·3960613, 6	9·2140607, 6
11		9·9306846, 2	9·8313395, 6	9·7030795, 0	9·5518985, 0	9·3819303, 7	9·1961400, 2
12		9·9271966, 2	9·8254893, 5	9·6944346, 5	9·5400067, 0	9·3663150, 7	9·1762951, 8
13		9·9233921, 6	9·8191006, 8	9·6849803, 5	9·5269788, 8	9·3491730, 5	9·1544577, 6
14		9·9192678, 3	9·8121656, 7	9·6747009, 7	9·5127866, 3	9·3304557, 4	9·1305483, 0
15		9·9148198, 2	9·8046755, 0	9·6635789, 0	9·4973977, 8	9·3101075, 6	9·1044744, 5
16		9·9100440, 4	9·7966205, 8	9·6515947, 5	9·4807763, 5	9·2880654, 6	9·0761297, 7
17		9·9049359, 8	9·7879903, 0	9·6387268, 6	9·4628816, 6	9·2642569, 6	9·0453894, 5
18		9·8994907, 6	9·7787730, 4	9·6249512, 7	9·4436680, 8	9·2385993, 6	9·0121080, 1
19		9·8937031, 2	9·7689561, 3	9·6102414, 3	9·4230841, 7	9·2109975, 5	8·9761140, 6
20		9·8875672, 9	9·7585255, 9	9·5945677, 4	9·4010717, 7	9·1813416, 9	8·9372040, 5
21		9·8810771, 2	9·7474662, 8	9·5778975, 5	9·3775652, 2	9·1495045, 7	8·8951345, 8
22		9·8742259, 1	9·7357615, 0	9·5601944, 1	9·3524899, 0	9·1153377, 0	8·8496110, 3
23		9·8670064, 8	9·7233931, 5	9·5414179, 2	9·3257609, 0	9·0786666, 7	8·8002725, 2
24		9·8594110, 4	9·7103413, 0	9·5215228, 8	9·2972808, 8	9·0392843, 6	8·7466692, 2
25		9·8514312, 4	9·6965841, 9	9·5004587, 6	9·2669378, 0	8·9969429, 9	8·6882319, 2
26		9·8430580, 4	9·6820979, 4	9·4781688, 8	9·2346019, 8	8·9513422, 8	8·6242224, 7
27		9·8342817, 6	9·6668564, 3	9·4545896, 2	9·2001223, 4	8·9021135, 2	8·5536596, 3
28		9·8250918, 9	9·6508308, 2	9·4296488, 0	9·1633208, 1	8·8487959, 7	8·4751946, 5
29		9·8154771, 5	9·6339894, 2	9·4032646, 4	9·1239862, 6	8·7908039, 6	8·3868963, 4
30		9·8054253, 3	9·6162972, 5	9·3753438, 0	9·0818653, 1	8·7273747, 4	8·2858490, 7
31		9·7949232, 8	9·5977155, 7	9·3457791, 5	9·0366500, 6	8·6574882, 1	8·1673276, 7
32		9·7839567, 7	9·5782013, 8	9·3144468, 3	8·9879608, 8	8·5797321, 7	8·0228583, 3
33		9·7725103, 6	9·5577068, 2	9·2812026, 6	8·9353218, 2	8·4920672, 0	7·8347003, 9
34		9·7605673, 7	9·5361784, 0	9·2458775, 7	8·8781245, 4	8·3913806, 5	7·5537204, 4
35		9·7481096, 6	9·5135560, 7	9·2082708, 8	8·8155717, 1	8·2725512, 3	6·8836955, 0
36		9·7351175, 5	9·4897722, 2	9·1681424, 6	8·7465892, 1	8·1262057, 9	7·2091636, 9 *n*
37		9·7215695, 6	9·4647502, 3	9·1252008, 8	8·6696773, 7	7·9320534, 8	7·5549541, 0 *n*
38		9·7074422, 9	9·4384029, 1	9·0790877, 6	8·5826464, 2	7·6299872, 6	7·7130888, 2 *n*
39		9·6927101, 3	9·4106303, 9	9·0293546, 8	8·4821070, 3	6·7047337, 6	7·8041912, 4 *n*
40		9·6773450, 0	9·3813174, 8	8·9754301, 5	8·3623804, 9	7·4372367, 1 *n*	7·8592061, 7 *n*

41	9·6613160, 7	9·3503304, 6	8·9165684, 3	8·2127872, 0	7·7387845, 2 *n*	7·8906618, 6 *n*
42	9·6445893, 5	9·3175127, 1	8·8517696, 0	8·0091456, 8	7·8888621, 5 *n*	7·9047204, 2 *n*
43	9·6271272, 7	9·2826789, 1	8·7796459, 7	7·6724940, 9	7·9797891, 2 *n*	7·9046944, 5 *n*
44	9·6088881, 2	9·2456079, 4	8·6981901, 8	6·3180842, 1 *n*	8·0380078, 3 *n*	7·8923695, 7 *n*
45	9·5898255, 4	9·2060323, 4	8·6043319, 1	7·6563598, 5 *n*	8·0747367, 4 *n*	7·8685562, 8 *n*
46	9·5698876, 7	9·1636243, 2	8·4930153, 1	7·9181331, 3 *n*	8·0956891, 4 *n*	7·8333205, 5 *n*
47	9·5490163, 3	0·1179759, 8	8·3549857, 6	8·0596050, 7 *n*	8·1040649, 1 *n*	7·7860099, 4 *n*
48	9·5271459, 3	9·0685700, 7	8·1702344, 2	8·1497956, 4 *n*	8·1017206, 1 *n*	7·7251018, 8 *n*
49	9·5042021, 0	9·0147365, 5	7·8798061, 2	8·2107547, 4 *n*	8·0896918, 0 *n*	7·6477828, 4 *n*
50	9·4801001, 2	8·9555850, 1	6·9985993, 2	8·2523184, 5 *n*	8·0684356, 0 *n*	7·5489738, 2 *n*
51	9·4547428, 1	8·8898942, 9	7·6954492, 8 *n*	8·2796747, 0 *n*	8·0379342, 2 *n*	7·4188529, 3 *n*
52	9·4280180, 0	8·8159233, 2	8·0124778, 3 *n*	8·2958337, 6 *n*	7·9976941, 0 *n*	7·2352354, 6 *n*
53	9·3997951, 8	8·7310630, 2	8·1762730, 4 *n*	8·3026345, 7 *n*	7·9466585, 6 *n*	6·9283241, 5 *n*
54	9·3699212, 7	8·6311366, 2	8·2813565, 8 *n*	8·3012110, 6 *n*	7·8829762, 4 *n*	5·4684950, 2 *n*
55	9·3382150, 2	8·5088116, 2	8·3546482, 8 *n*	8·2922257, 1 *n*	7·8035076, 4 *n*	6·8575495, 9
56	9·3044594, 9	8·3492895, 8	8·4076170, 9 *n*	8·2759918, 7 *n*	7·7027458, 1 *n*	7·1424113, 8
57	9·2683920, 7	8·1145866, 4	8·4461884, 8 *n*	8·2525256, 2 *n*	7·5700687, 0 *n*	7·2929734, 0
58	9·2296903, 4	7·6320055, 5	8·4737919, 6 *n*	8·2215514, 2 *n*	7·3810735, 4 *n*	7·3870871, 4
59	9·1879522, 9	7·5847562, 6 *n*	8·4925580, 0 *n*	8·1824648, 8 *n*	7·0550226, 5 *n*	7·4484728, 8
60	9·1426675, 2	8·0555170, 9 *n*	8·5038669, 2 *n*	8·1342345, 0 *n*	5·8494194, 1	7·4873915, 2
61	9·0931738, 6	8·2617615, 9 *n*	8·5086262, 6 *n*	8·0752074, 2 *n*	7·0765312, 2	7·5090398, 0
62	9·0385901, 3	8·3903079, 3 *n*	8·5074236, 6 *n*	8·0027309, 2 *n*	7·3451501, 5	7·5162047, 3
63	8·9777065, 1	8·4806507, 2 *n*	8·5006098, 0 *n*	7·9123575, 9 *n*	7·4941434, 5	7·5102969, 7
64	8·9087966, 8	8·5479507, 2 *n*	8·4883456, 5 *n*	7·7959842, 4 *n*	7·5910111, 5	7·4917761, 8
65	8·8292721, 8	8·5996350, 9 *n*	8·4706227, 4 *n*	7·6365691, 3 *n*	7·6574584, 0	7·4602737, 6
66	8·7349884, 7	8·6398829, 5 *n*	8·4472636, 4 *n*	7·3873257, 4 *n*	7·7031529, 0	7·4145037, 2
67	8·6186724, 1	8·6712635, 8 *n*	8·4179042, 8 *n*	6·7819563, 2 *n*	7·7331426, 8	7·3518943, 2
68	8·4656651, 9	8·6954540, 9 *n*	8·3819482, 0 *n*	7·0596088, 5	7·7502722, 9	7·2676855, 3
69	8·2384596, 2	8·7135945, 2 *n*	8·3384885, 6 *n*	7·4463305, 4	7·7561665, 7	7·1526277, 0
70	7·7683916, 6	8·7264782, 9 *n*	8·2861626, 6 *n*	7·6351689, 7	7·7516647, 2	6·9860727, 2
71	7·7089721, 3 *n*	8·7346626, 4 *n*	8·2228912, 0 *n*	7·7555516, 9	7·7370152, 3	6·7052134, 9
72	8·1936701, 8 *n*	8·7385337, 0 *n*	8·1453627, 0 *n*	7·8398562, 4	7·7119303, 3	5·7010496, 3
73	8·4087469, 1 *n*	8·7383480, 3 *n*	8·0479291, 3 *n*	7·9011846, 9	7·6755272, 2	6·5915543, 0 *n*
74	8·5457419, 4 *n*	8·7342557, 0 *n*	7·9198340, 5 *n*	7·9460949, 6	7·6261288, 2	6·9050021, 0 *n*
75	8·6446730, 0 *n*	8·7263148, 9 *n*	7·7364674, 2 *n*	7·9782796, 1	7·5608161, 3	7·0714696, 1 *n*
76	8·7208611, 4 *n*	8·7144968, 3 *n*	7·4162678, 2 *n*	7·9999504, 9	7·4744202, 5	7·1782803, 7 *n*
77	8·7818137, 3 *n*	8·6986841, 5 *n*	6·2659022, 0	8·0124639, 1	7·3570214, 9	7·2509540, 1 *n*
78	8·8317664, 0 *n*	8·6786615, 7 *n*	7·4618840, 3	8·0166184, 8	7·1862413, 4	7·3003367, 5 *n*
79	8·8733382, 7 *n*	8·6540978, 7 *n*	7·7404290, 2	8·0128043, 4	6·8913813, 3	7·3318725, 0 *n*
80	8·9082566, 6 *n*	8·6245137, 2 *n*	7·9015226, 2	8·0010705, 3	5·3324384, 6	7·3484742, 7 *n*
81	8·9377152, 8 *n*	8·5892313, 6 *n*	8·0121295, 5	7·9811365, 2	6·8573928, 1 *n*	7·3516030, 7 *n*
82	8·9625680, 1 *n*	8·5472900, 7 *n*	8·0939320, 3	7·9523433, 9	7·1555060, 8 *n*	7·3417391, 2 *n*
83	8·9834412, 1 *n*	8·4973040, 4 *n*	8·1566935, 9	7·9135367, 7	7·3212420, 7 *n*	7·3184911, 6 *n*
84	9·0008023, 2 *n*	8·4372060, 8 *n*	8·2056249, 2	7·8628305, 8	7·4314876, 0 *n*	7·2805123, 6 *n*
85	9·0150032, 5 *n*	8·3637569, 1 *n*	8·2438022, 4	7·7971255, 3	7·5098568, 3 *n*	7·2250843, 2 *n*
86	9·0263090, 8 *n*	8·2714944, 9 *n*	8·2731603, 7	7·7110678, 8	7·5666696, 1 *n*	7·1471257, 7 *n*
87	9·0349173, 3 *n*	8·1501452, 3 *n*	8·2949586, 6	7·5944665, 7	7·6072533, 8 *n*	7·0366488, 6 *n*
88	9·0409706, 8 *n*	7·9766039, 7 *n*	8·3100210, 7	7·4242754, 8	7·6346042, 2 *n*	6·8707374, 0 *n*
89	9·0445653, 2 *n*	7·6770984, 6 *n*	8·3188670, 8	7·1267643, 6	7·6504250, 7 *n*	6·5757649, 8 *n*
90	9·0457574, 9 *n*	∝	8·3217852, 2	∝	7·6556077, 3 *n*	∝

TABLE OF LOG G_5^4, G_6^4, G_7^4, ... G_{10}^4, FOR VALUES OF θ FROM 0° TO 90°.

$\mu = \cos\theta$. $m = 4$.

θ	Log G_5^4	Log G_6^4	Log G_7^4	Log G_8^4	Log G_9^4	Log G_{10}^4
0°	0	9·9586073, 1	9·8860566, 5	9·7891466, 3	9·6726410, 7	9·5400155, 0
1		9·9584617, 8	9·8858184, 9	9·7888026, 2	9·6721779, 5	9·5394200, 6
2		9·9580250, 8	9·8851038, 3	9·7877702, 2	9·6707880, 2	9·5376327, 6
3		9·9572968, 3	9·8839118, 8	9·7860480, 7	9·6684690, 6	9·5346502, 7
4		9·9562764, 8	9·8822414, 6	9·7836340, 1	9·6652175, 2	9·5304670, 4
5		9·9549632, 2	9·8800909, 3	9·7805250, 9	9·6610284, 4	9·5250753, 8
6		9·9533559, 9	9·8774580, 3	9·7767172, 4	9·6558952, 3	9·5184651, 2
7		9·9514535, 0	9·8743401, 1	9·7722056, 4	9·6498098, 6	9·5106237, 3
8		9·9492542, 1	9·8707339, 1	9·7669844, 7	9·6427626, 6	9·5015360, 7
9		9·9467563, 4	9·8666357, 0	9·7610468, 6	9·6347422, 2	9·4911842, 7
10		9·9439578, 6	9·8620412, 1	9·7543850, 2	9·6257354, 2	9·4795476, 1
11		9·9408564, 3	9·8569454, 9	9·7469899, 5	9·6157272, 3	9·4666022, 8
12		9·9374495, 1	9·8513431, 2	9·7388515, 6	9·6047004, 6	9·4523208, 1
13		9·9337341, 9	9·8452279, 4	9·7299585, 2	9·5926359, 1	9·4366723, 8
14		9·9297073, 6	9·8385932, 3	9·7202982, 7	9·5795119, 9	9·4196218, 7
15		9·9253655, 3	9·8314314, 9	9·7098567, 5	9·5653044, 1	9·4011296, 2
16		9·9207049, 4	9·8237345, 4	9·6986184, 7	9·5499861, 8	9·3811510, 0
17		9·9157214, 6	9·8154933, 8	9·6865663, 1	9·5335271, 3	9·3596355, 1
18		9·9104106, 2	9·8666981, 8	9·6736813, 8	9·5158936, 5	9·3365262, 3
19		9·9047676, 2	9·7973382, 6	9·6599429, 1	9·4970483, 3	9·3117589, 0
20		9·8987872, 2	9·7874019, 0	9·6453279, 1	9·4769493, 0	9·2852605, 2
21		9·8924637, 8	9·7768764, 1	9·6298112, 2	9·4555499, 6	9·2569482, 8
22		9·8857912, 4	9·7657479, 2	9·6133649, 9	9·4327980, 5	9·2267276, 0
23		9·8787630, 6	9·7540014, 0	9·5959586, 0	9·4086350, 3	9·1944902, 3
24		9·8713722, 1	9·7416204, 4	9·5775581, 1	9·3829948, 7	9·1601111, 9
25		9·8636111, 3	9·7285872, 3	9·5581261, 7	9·3558033, 1	9·1234459, 0
26		9·8554717, 0	9·7148823, 1	9·5376212, 0	9·3269758, 9	9·0843250, 3
27		9·8469451, 6	9·7004844, 5	9·5159970, 3	9·2964164, 3	9·0425493, 3
28		9·8380221, 1	9·6853705, 4	9·4932023, 2	9·2640149, 3	8·9978817, 0
29		9·8286924, 6	9·6695153, 4	9·4691796, 7	9·2296445, 3	8·9500367, 2
30		9·8189453, 2	9·6528910, 5	9·4438645, 1	9·1931578, 1	8·8986662, 3
31		9·8087689, 7	9·6354673, 0	9·4171842, 3	9·1543824, 6	8·8433390, 6
32		9·7981508, 0	9·6172106, 6	9·3890566, 3	9·1131149, 4	8·7835118, 3
33		9·7870771, 9	9·5980843, 0	9·3593882, 4	9·0691124, 4	8·7184840, 1
34		9·7755334, 2	9·5780474, 2	9·3280718, 8	9·0220813, 8	8·6473292, 7
35		9·7635036, 0	9·5570548, 6	9·2949842, 2	8·9716623, 7	8·5687837, 6
36		9·7509705, 3	9·5350563, 4	9·2599820, 9	8·9174080, 8	8·4810531, 7
37		9·7379155, 4	9·5119957, 9	9·2228979, 9	8·8587511, 7	8·3814587, 7
38		9·7243183, 6	9·4878101, 9	9·1835338, 7	8·7949549, 9	8·2657222, 9
39		9·7101569, 5	9·4624286, 5	9·1416533, 6	8·7250380, 1	8·1263421, 7
40		9·6954072, 5	9·4357709, 1	9·0969708, 2	8·6476475, 8	7·9481575, 9

41	9·6800429, 8	9·4077454, 6	9·0491359, 0	8·5608398, 9	7·6920048, 2
42	9·6640353, 6	9·3782476, 9	8·9977126, 4	8·4616700, 2	7·1752160, 0
43	9·6473527, 9	9·3471567, 5	8·9421472, 6	8·3453405, 9	7·1509865, 7 n
44	9·6299604, 3	9·3143322, 5	8·8817208, 8	8·2032002, 0	7·5844454, 4 n
45	9·6118198, 3	9·2796096, 9	8·8154755, 4	8·0169757, 1	7·7637992, 3 n
46	9·5928883, 3	9·2427942, 6	8·7420905, 1	7·7352650, 7	7·8654705, 7 n
47	9·5731184, 9	9·2036529, 8	8·6596705, 7	7·0271130, 3	7·9276505, 6 n
48	9·5524573, 2	9·1619035, 9	8·5653473, 9	7·4365045, 4 n	7·9649762, 6 n
49	9·5308453, 5	9·1171996, 6	8·4544634, 3	7·7762281, 4 n	7·9844922, 1 n
50	9·5082155, 9	9·0691082, 3	8·3186598, 8	7·9386192, 8 n	7·9899919, 0 n
51	9·4844921, 1	9·0170787, 3	8·1404426, 8	8·0367334, 7 n	7·9835969, 8 n
52	9·4595884, 2	8·9603947, 6	7·8715495, 9	8·1004598, 2 n	7·9664261, 7 n
53	9·4334053, 4	8·8980983, 5	7·2272154, 8	8·1421149, 7 n	7·9388938, 2 n
54	9·4058283, 9	8·8288663, 6	7·5375181, 3 n	8·1678942, 4 n	7·9008115, 7 n
55	9·3767244, 4	8·7507943, 4	7·9018385, 1 n	8·1812729, 2 n	7·8513586, 6 n
56	9·3459372, 8	8·6609912, 0	8·0760882, 2 n	8·1843040, 9 n	7·7888963, 0 n
57	9·3132819, 9	8·5547466, 3	8·1840579, 8 n	8·1782017, 2 n	7·7105336, 8 n
58	9·2785371, 7	8·4235722, 1	8·2573043, 5 n	8·1636163, 1 n	7·6111314, 6 n
59	9·2414347, 3	8·2495952, 8	8·3086918, 0 n	8·1407697, 4 n	7·4808287, 4 n
60	9·2016453, 6	7·9829664, 5	8·3446946, 0 n	8·1094989, 0 n	7·2973993, 6 n
61	9·1587582, 5	7·3161150, 8	8·3689849, 2 n	8·0692347, 0 n	6·9910133, 1 n
62	9·1122511, 9	7·6872069, 1 n	8·3837988, 6 n	8·0188980, 5 n	5·5280163, 4 n
63	9·0614460, 0	8·0490722, 0 n	8·3905479, 5 n	7·9566812, 1 n	6·9227929, 1 n
64	9·0054386, 5	8·2284288, 5 n	8·3901234, 9 n	7·8795918, 6 n	7·2092361, 2
65	8·9429852, 8	8·3433110, 9 n	8·3830554, 1 n	7·7824597, 2 n	7·3618146, 9
66	8·8723055, 4	8·4244559, 5 n	8·3695985, 7 n	7·6555110, 8 n	7·4583507, 4
67	8·7907193, 8	8·4845202, 0 n	8·3497694, 4 n	7·4769257, 8 n	7·5225889, 7
68	8·6939122, 3	8·5299314, 8 n	8·3233529, 8 n	7·1789742, 1 n	7·5648281, 5
69	8·5742454, 7	8·5643814, 9 n	8·2898737, 8 n	5·9131778, 3 n	7·5903245, 6
70	8·4161192, 1	8·5901612, 2 n	8·2485296, 4 n	7·1018844, 9	7·6019448, 9
71	8·1785605, 7	8·6087681, 4 n	8·1980624, 1 n	7·3996166, 3	7·6012220, 8
72	7·6610939, 5	8·6212087, 0 n	8·1365097, 1 n	7·5607839, 0	7·5887903, 1
73	7·7346302, 4 n	8·6281651, 8 n	8·0607295, 6 n	7·6658960, 5	7·5645560, 0
74	8·1741511, 4 n	8·6300901, 8 n	7·9653625, 9 n	7·7392811, 3	7·5276750, 0
75	8·3787936, 7 n	8·6272621, 0 n	7·8402949, 6 n	7·7915124, 0	7·4763402, 7
76	8·5103156, 1 n	8·6198168, 3 n	7·6628097, 6 n	7·8280583, 0	7·4072396, 0
77	8·6053708, 9 n	8·6077614, 9 n	7·3609567, 5 n	7·8520186, 1	7·3142844, 5
78	8·6783528, 3 n	8·5909774, 0 n	5·5763413, 5 n	7·8652141, 7	7·1852502, 7
79	8·7364046, 3 n	8·5692082, 3 n	7·3318199, 1	7·8686755, 8	6·9903965, 8
80	8·7835848, 9 n	8·5420347, 6 n	7·6265846, 5	7·8628710, 8	6·6176188, 2
81	8·8224123, 0 n	8·5088264, 2 n	7·7920258, 9	7·8478082, 5	6·1605585, 6 n
82	8·8545485, 8 n	8·4686605, 3 n	7·9037474, 4	7·8230292, 7	6·8365836, 0 n
83	8·8811389, 1 n	8·4201814, 3 n	7·9851726, 5	7·7875277, 7	7·0783371, 8 n
84	8·9029970, 9 n	8·3613473, 9 n	8·0466325, 0	7·7395240, 9	7·2218548, 4 n
85	8·9207126, 2 n	8·2889377, 0 n	8·0935572, 0	7·6759981, 8	7·3186475, 7 n
86	8·9347161, 5 n	8·1975068, 9 n	8·1291063, 6	7·5916544, 0	7·3867895, 9 n
87	8·9453210, 4 n	8·0767922, 2 n	8·1552294, 2	7·4763460, 7	7·4346120, 2 n
88	8·9527494, 5 n	7·9036986, 5 n	8·1731545, 0	7·3070572, 5	7·4664909, 7 n
89	8·9571497, 7 n	7·6044604, 2 n	8·1836362, 1	7·0100793, 7	7·4848149, 9 n
90	8·9586073, 1 n	∝	8·1870866, 4	∝	7·4907974, 8 n

TABLE OF LOG G_6^5, G_7^5, ... G_{10}^5, FOR VALUES OF θ FROM 0° TO 90°.

$\mu = \cos\theta. \qquad m = 5.$

θ	Log G_6^5	Log G_7^5	Log G_8^5	Log G_9^5	Log G_{10}^5
0°	0	9·9652378, 9	9·9030899, 9	9·8187691, 0	9·7161067, 6
1		9·9650945, 7	9·9028584, 6	9·8184383, 4	9·7156657, 4
2		9·9646644, 8	9·9021636, 4	9·8174456, 7	9·7143920, 6
3		9·9639472, 9	9·9010048, 9	9·8157899, 6	9·7121339, 3
4		9·9629424, 7	9·8993811, 3	9·8134693, 5	9·7090383, 7
5		9·9616492, 4	9·8972908, 2	9·8104811, 8	9·7050511, 6
6		9·9600666, 0	9·8947319, 7	9·8068219, 9	9·7001667, 5
7		9·9581933, 5	9·8917021, 6	9·8024875, 8	9·6943784, 5
8		9·9560280, 1	9·8881984, 9	9·7974729, 2	9·6876781, 5
9		9·9535689, 1	9·8842175, 4	9·7917720, 1	9·6800562, 7
10		9·9508141, 2	9·8797554, 8	9·7853781, 4	9·6715018, 9
11		9·9477614, 5	9·8748079, 1	9·7782835, 8	9·6620025, 2
12		9·9444084, 6	9·8693698, 5	9·7704795, 6	9·6515438, 8
13		9·9407524, 6	9·8634358, 9	9·7619564, 7	9·6401102, 1
14		9·9367904, 8	9·8569999, 3	9·7527034, 4	9·6276836, 2
15		9·9325192, 3	9·8500552, 8	9·7427085, 0	9·6142443, 1
16		9·9279351, 8	9·8425946, 3	9·7319584, 6	9·5997702, 6
17		9·9230344, 4	9·8346099, 7	9·7204388, 1	9·5842370, 7
18		9·9178128, 3	9·8260925, 7	9·7081335, 9	9·5676176, 6
19		9·9122658, 0	9·8170329, 0	9·6950252, 6	9·5498820, 7
20		9·9063884, 5	9·8074206, 4	9·6810946, 8	9·5309972, 0
21		9·9001755, 2	9·7972445, 7	9·6663207, 9	9·5109262, 5
22		9·8936213, 1	9·7864925, 4	9·6506806, 0	9·4896285, 3
23		9·8867197, 1	9·7751512, 6	9·6341487, 4	9·4670586, 7
24		9·8794641, 7	9·7632064, 8	9·6166975, 7	9·4431663, 2
25		9·8718476, 2	9·7506426, 2	9·5982966, 1	9·4178952, 5
26		9·8638625, 1	9·7374428, 4	9·5789123, 9	9·3911825, 4
27		9·8555007, 2	9·7235888, 3	9·5585079, 5	9·3629574, 3
28		9·8467534, 9	9·7090606, 4	9·5370425, 1	9·3331402, 0
29		9·8376115, 1	9·6938366, 8	9·5144709, 7	9·3016405, 9
30		9·8280647, 0	9·6778933, 1	9·4907431, 7	9·2683557, 3
31		9·8181022, 5	9·6612048, 4	9·4658033, 7	9·2331680, 0
32		9·8077125, 6	9·6437430, 9	9·4395890, 7	9·1959413, 4
33		9·7968831, 1	9·6254772, 7	9·4120303, 2	9·1565183, 2
34		9·7856004, 5	9·6063735, 7	9·3830482, 1	9·1147141, 1
35		9·7738500, 4	9·5863947, 1	9·3525532, 3	9·0703100, 8
36		9·7616162, 3	9·5654996, 5	9·3204434, 8	9·0230448, 2
37		9·7488820, 4	9·5436429, 2	9·2866021, 8	8·9726017, 7
38		9·7356291, 3	9·5207739, 5	9·2508943, 5	8·9185915, 7
39		9·7218376, 2	9·4968365, 0	9·2131629, 5	8·8605275, 6
40		9·7074858, 7	9·4717674, 8	9·1732236, 9	8·7977889, 3

41	9·6925503, 6	9·4454960, 1	9·1308577, 9	8·7295643, 0
42	9·6770054, 1	9·4179419, 9	9·0858030, 8	8·6547629, 7
43	9·6608229, 5	9·3890143, 9	9·0377406, 7	8·5718635, 5
44	9·6439722, 1	9·3586090, 9	8·9862772, 9	8·4786493, 1
45	9·6264193, 3	9·3266062, 5	8·9309190, 9	8·3716909, 5
46	9·6081270, 4	9·2928669, 8	8·8710332, 4	8·2452446, 1
47	9·5890540, 5	9·2572287, 0	8·8057885, 3	8·0884996, 8
48	9·5691545, 9	9·2194996, 2	8·7340605, 0	7·8769160, 8
49	9·5483775, 8	9·1794507, 0	8·6542710, 0	7·5297587, 8
50	9·5266659, 5	9·1368056, 4	8·5641006, 4	6·2952591, 4 *n*
51	9·5039554, 8	9·0912256, 9	8·4599258, 4	7·5109898, 0 *n*
52	9·4801736, 8	9·0422897, 3	8·3355946, 9	7·7617301, 8 *n*
53	9·4552382, 0	8·9894638, 4	8·1793021, 2	7·8936341, 7 *n*
54	9·4290549, 5	8·9320563, 2	7·9633585, 1	7·9742504, 8 *n*
55	9·4015157, 3	8·8691471, 4	7·5893899, 0	8·0252599, 9 *n*
56	9·3724952, 1	8·7994728, 9	6·9136019, 5 *n*	8·0563040, 5 *n*
57	9·3418470, 7	8·7212282, 6	7·6935933, 0 *n*	8·0724076, 3 *n*
58	9·3093988, 1	8·6316986, 9	7·9288831, 3 *n*	8·0764055, 8 *n*
59	9·2749452, 0	8·5265145, 2	8·0604281, 2 *n*	8·0699269, 7 *n*
60	9·2382391, 6	8·3979400, 1	8·1452147, 5 *n*	8·0538333, 9 *n*
61	9·1989793, 7	8·2301399, 6	8·2026719, 1 *n*	8·0284248, 5 *n*
62	9·1567927, 9	7·9813147, 4	8·2417222, 5 *n*	7·9935080, 9 *n*
63	9·1112097, 0	7·4429013, 2	8·2671442, 7 *n*	7·9483640, 6 *n*
64	9·0616265, 4	7·5356447, 1 *n*	8·2817313, 2 *n*	7·8916043, 5 *n*
65	9·0072488, 7	7·9562366, 4 *n*	8·2871881, 3 *n*	7·8208363, 4 *n*
66	8·9470002, 9	8·1479535, 1 *n*	8·2845518, 7 *n*	7·7319423, 5 *n*
67	8·8793696, 2	8·2670069, 0 *n*	8·2744056, 5 *n*	7·6173914, 5 *n*
68	8·8021373, 5	8·3493307, 7 *n*	8·2569826, 7 *n*	7·4615165, 3 *n*
69	8·7118452, 6	8·4090746, 7 *n*	8·2322088, 4 *n*	7·2227320, 8 *n*
70	8·6026534, 8	8·4532461, 5 *n*	8·1996940, 4 *n*	6·6935686, 8 *n*
71	8·4634680, 4	8·4857946, 5 *n*	8·1586733, 2 *n*	6·7886066, 9
72	8·2687749, 5	8·5091339, 4 *n*	8·1078727, 9 *n*	7·2142237, 4
73	7·9323788, 8	8·5248120, 4 *n*	8·0452482, 4 *n*	7·4082688, 0
74	6·9764095, 9 *n*	8·5338439, 7 *n*	7·9674539, 0 *n*	7·5275744, 8
75	7·9972019, 7 *n*	8·5368893, 4 *n*	7·8686779, 3 *n*	7·6080776, 0
76	8·2647441, 8 *n*	8·5343512, 0 *n*	7·7376918, 8 *n*	7·6638365, 7
77	8·4202875, 3 *n*	8·5264298, 8 *n*	7·5483279, 9 *n*	7·7016983, 4
78	8·5275759, 0 *n*	8·5131493, 9 *n*	7·2104202, 0 *n*	7·7253888, 6
79	8·6076159, 5 *n*	8·4943607, 1 *n*	6·3808260, 6	7·7370112, 5
80	8·6699617, 0 *n*	8·4697263, 1 *n*	7·3078125, 6	7·7376839, 3
81	8·7197565, 9 *n*	8·4386795, 0 *n*	7·5712999, 6	7·7278331, 3
82	8·7600749, 9 *n*	8·4003473, 1 *n*	7·7245789, 5	7·7072797, 0
83	8·7928883, 3 *n*	8·3534130, 1 *n*	7·8288763, 4	7·6752108, 3
84	8·8195234, 0 *n*	8·2958658, 3 *n*	7·9045954, 6	7·6299831, 8
85	8·8409021, 5 *n*	8·2245097, 5 *n*	7·9610141, 4	7·5686791, 9
86	8·8576768, 4 *n*	8·1339174, 7 *n*	8·0030717, 4	7·4860704, 3
87	8·8703104, 3 *n*	8·0138412, 7 *n*	8·0336440, 6	7·3720665, 1
88	8·8791251, 9 *n*	7·8411961, 1 *n*	8·0544721, 0	7·2036854, 7
89	8·8843335, 9 *n*	7·5422239, 6 *n*	8·0665979, 8	6·9072447, 1
90	8·8860566, 5 *n*	∞	8·0705810, 7	∞

TABLE OF LOG G_7^6, G_8^6, ... G_{10}^6, FOR VALUES OF θ FROM 0° TO 90°.

$\mu = \cos\theta$. $m = 6$.

θ	Log G_7^6	Log G_8^6	Log G_9^6	Log G_{10}^6	θ	Log G_7^6	Log G_8^6	Log G_9^6	Log G_{10}^6
0°	0	9·9700367, 8	9·9156791, 1	9·8410455, 0	46°	0	9·6189717, 8	9·3276057, 4	8·9530749, 6
1		9·9698950, 2	9·9154523, 1	9·8407241, 9	47		9·6003793, 8	9·2941566, 7	8·8962264, 6
2		9·9694696, 7	9·9147716, 9	9·8397599, 1	48		9·5810037, 3	9·2589049, 2	8·8348338, 2
3		9·9687603, 8	9·9136366, 6	9·8381516, 6	49		9·5607991, 6	9·2216789, 4	8·7680727, 1
4		9·9677666, 4	9·9120461, 8	9·8358977, 3	50		9·5397148, 2	9·1822755, 3	8·6948302, 7
5		9·9664877, 2	9·9099988, 7	9·8329957, 3					
6		9·9649226, 5	9·9074928, 8	9·8294426, 0	51		9·5176939, 0	9·1404518, 2	8·6135492, 3
7		9·9630702, 6	9·9045259, 4	9·8252345, 1	52		9·4946726, 3	9·0959137, 8	8·5219481, 0
8		9·9609291, 8	9·9010954, 0	9·8203669, 5	53		9·4705790, 1	9·0482999, 9	8·4164770, 8
9		9·9584977, 5	9·8971980, 6	9·8148345, 4	54		9·4453314, 4	8·9971590, 9	8·2911407, 7
10		9·9557741, 4	9·8928303, 6	9·8086312, 4	55		9·4188368, 0	8·9419162, 2	8·1345197, 6
					56		9·3909881, 2	8·8818220, 4	7·9201666, 5
11		9·9527562, 6	9·8879882, 6	9·8017501, 4	57		9·3616617, 6	8·8158726, 3	7·5569197, 3
12		9·9494417, 6	9·8826671, 9	9·7941834, 6	58		9·3307135, 4	8·7426765, 1	6·7271914, 0 n
13		9·9458280, 6	9·8768620, 8	9·7859225, 0	59		9·2979738, 9	8·6602211, 4	7·6143297, 3 n
14		9·9419123, 1	9·8705673, 7	9·7769577, 0	60		9·2632414, 3	8·5654310, 3	7·8548792, 0 n
15		9·9376913, 9	9·8637768, 9	9·7672783, 8					
16		9·9331618, 9	9·8564839, 0	9·7568728, 4	61		9·2262742, 5	8·4532447, 6	7·9863856, 3 n
17		9·9283201, 1	9·8486810, 4	9·7457282, 0	62		9·1867780, 7	8·3143996, 5	8·0695004, 8 n
18		9·9231620, 4	9·8403603, 2	9·7338304, 3	63		9·1443895, 5	8·1288775, 7	8·1244507, 4 n
19		9·9176833, 4	9·8315130, 3	9·7211641, 1	64		9·0986526, 9	7·8377045, 7	8·1604109, 0 n
20		9·9118793, 5	9·8221297, 1	9·7077123, 9	65		9·0489834, 8	6·9554712, 7	8·1822394, 2 n
					66		8·9946163, 9	7·6521205, 8 n	8·1927338, 4 n
21		9·9057450, 3	9·8122001, 7	9·6934569, 8	67		8·9345193, 6	7·9684511, 9 n	8·1935567, 3 n
22		9·8992749, 6	9·8017132, 9	9·6783777, 5	68		8·8672519, 4	8·1315692, 7 n	8·1856642, 4 n
23		9·8924633, 3	9·7906570, 8	9·6624528, 2	69		8·7907137, 6	8·2359595, 0 n	8·1695130, 8 n
24		9·8853038, 9	9·7790185, 6	9·6456582, 5	70		8·7016639, 0	8·3085161, 8 n	8·1451538, 2 n
25		9·8777899, 6	9·7667837, 2	9·6279679, 1					
26		9·8699143, 6	9·7539373, 6	9·6093531, 9	71		8·5947013, 1	8·3606829, 5 n	8·1122429, 4 n
27		9·8616693, 7	9·7404630, 0	9·5897826, 2	72		8·4597667, 6	8·3983553, 5 n	8·0699883, 4 n
28		9·8530467, 6	9·7263428, 8	9·5692219, 5	73		8·2744936, 1	8·4249260, 4 n	8·0169957, 6 n
29		9·8440376, 6	9·7115576, 5	9·5476331, 6	74		7·9689160, 9	8·4424808, 7 n	7·9509550, 4 n
30		9·8346326, 1	9·6960863, 3	9·5249745, 7	75		6·5060041, 6	8·4523440, 6 n	7·8679775, 0 n
					76		7·9106494, 9 n	8·4553524, 6 n	7·7610794, 6 n
31		9·8248213, 9	9·6799061, 4	9·5012000, 5	77		8·2058453, 2 n	8·4520020, 0 n	7·6160867, 3 n
32		9·8145930, 9	9·6629922, 8	9·4762584, 1	78		8·3699464, 9 n	8·4425242, 9 n	7·3969295, 0 n
33		9·8039359, 8	9·6453177, 3	9·4500929, 2	79		8·4808488, 3 n	8·4269208, 0 n	6·9463638, 3 n
34		9·7928374, 0	9·6268529, 0	9·4226400, 5	80		8·5624472, 8 n	8·4049646, 7 n	6·8292780, 4
35		9·7812837, 8	9·6075654, 8	9·3938287, 3					
36		9·7692604, 8	9·5874199, 7	9·3635788, 7	81		8·6252602, 7 n	8·3761727, 7 n	7·3342265, 0
37		9·7567517, 1	9·5663773, 4	9·3318000, 3	82		8·6748383, 7 n	8·3397353, 8 n	7·5498024, 1
38		9·7437403, 9	9·5443944, 8	9·2983890, 3	83		8·7144515, 6 n	8·2943853, 2 n	7·6828992, 5
39		9·7302080, 9	9·5214237, 4	9·2632280, 7	84		8·7461706, 2 n	8·2381492, 7 n	7·7750225, 0
40		9·7161348, 0	9·4974121, 3	9·2261811, 0	85		8·7713710, 1 n	8·1678600, 2 n	7·8417879, 3
					86		8·7909934, 0 n	8·0781133, 8 n	7·8906992, 8
41		9·7014988, 0	9·4723005, 2	9·1870899, 3	87		8·8056885, 3 n	7·9586785, 3 n	7·9258512, 7
42		9·6862764, 9	9·4460228, 7	9·1457694, 9	88		8·8159010, 3 n	7·7864852, 3 n	7·9496236, 8
43		9·6704421, 5	9·4185047, 5	9·1020003, 6	89		8·8219199, 9 n	7·4877773, 3 n	7·9634019, 9
44		9·6539676, 1	9·3896619, 3	9·0555197, 3	90		8·8239087, 4 n	∞	7·9679187, 3
45		9·6368221, 0	9·3593987, 7	9·0060087, 4					

TABLE OF LOG G_8^7, G_9^7, G_{10}^7 AND G_{10}^8, FOR VALUES OF θ FROM 0° TO 90°. $\mu = \cos\theta$.

		$m=7$.		$m=8$.			$m=7$.		$m=8$.
θ	G_8^7 Log	Log G_9^7	Log G_{10}^7	Log G_{10}^8	θ	G_8^7 Log	Log G_9^7	Log G_{10}^7	Log G_{10}^8
0°	0	9·9736710, 6	9·9253663, 8	9·9765189, 0	46°	0	9·6270858, 5	9·3531940, 5	9·6333863, 1
1		9·9735304, 9	9·9251431, 2	9·9763792, 5	47		9·6088449, 3	9·3212427, 6	9·6154136, 1
2		9·9731086, 8	9·9244731, 4	9·9759602, 0	48		9·5898516, 1	9·2876707, 7	9·5967116, 4
3		9·9724053, 1	9·9233558, 9	9·9752614, 5	49		9·5700639, 2	9·2523392, 4	9·5772411, 1
4		9·9714198, 9	9·9217903, 9	9·9742824, 9	50		9·5494353, 6	9·2150863, 0	9·5569586, 7
5		9·9701517, 0	9·9197753, 3	9·9730226, 4					
6		9·9685998, 1	9·9173089, 5	9·9714809, 9	51		9·5279141, 2	9·1757213, 2	9·5358161, 9
7		9·9667630, 7	9·9143891, 2	9·9696564, 2	52		9·5054423, 8	9·1340177, 5	9·5137600, 9
8		9·9646401, 5	9·9110133, 5	9·9675476, 4	53		9·4819551, 6	9·0897028, 9	9·4907304, 5
9		9·9622294, 7	9·9071786, 1	9·9651530, 8	54		9·4573791, 8	9·0424440, 3	9·4666599, 4
10		9·9595292, 4	9·9028816, 0	9·9624710, 2	55		9·4316313, 4	8·9918291, 0	9·4414725, 9
					56		9·4046168, 0	8·9373380, 7	9·4150821, 1
11		9·9565374, 2	9·8981184, 2	9·9594994, 8	57		9·3762267, 0	8·8783008, 6	9·3873900, 1
12		9·9532517, 7	9·8928848, 5	9·9562362, 3	58		9·3463351, 7	8·8138311, 5	9·3582830, 0
13		9·9496697, 6	9·8871760, 9	9·9526788, 6	59		9·3147953, 0	8·7427201, 9	9·3276299, 3
14		9·9457886, 6	9·8809869, 4	9·9488246, 8	60		9·2814344, 4	8·6632545, 4	9·2952776, 7
15		9·9416054, 3	9·8743115, 9	9·9446707, 2					
16		9·9371167, 7	9·8671437, 2	9·9402138, 0	61		9·2460474, 1	8·5728826, 3	9·2610457, 7
17		9·9323191, 3	9·8594764, 6	9·9354504, 3	62		9·2083876, 5	8·4675505, 4	9·2247194, 0
18		9·9272086, 1	9·8513023, 1	9·9303768, 5	63		9·1681551, 0	8·3402074, 7	9·1860398, 8
19		9·9217810, 5	9·8426131, 7	9·9249889, 8	64		9·1249791, 2	8·1768133, 2	9·1446915, 0
20		9·9160319, 2	9·8334002, 0	9·9192824, 4	65		9·0783939, 9	7·9421589, 1	9·1002830, 6
					66		9·0278027, 1	7·4866826, 2	9·0523211, 2
21		9·9099563, 7	9·8236538, 4	9·9132525, 2	67		8·9724216, 9	7·3098749, 2 *n*	9·0001703, 4
22		9·9035492, 1	9·8133638, 6	9·9068941, 6	68		8·9111926, 2	7·8182146, 7 *n*	8·9429922, 7
23		9·8968048, 2	9·8025190, 6	9·9002019, 1	69		8·8426345, 7	8·0236676, 2 *n*	8·8796463, 5
24		9·8897171, 8	9·7911074, 5	9·8931699, 5	70		8·7645814, 6	8·1459552, 2 *n*	8·8085229, 0
25		9·8822798, 7	9·7791160, 2	9·8857920, 3					
26		9·8744860, 0	9·7665307, 8	9·8780614, 9	71		8·6736759, 1	8·2278103, 7 *n*	8·7272405, 7
27		9·8663281, 7	9·7533365, 6	9·8699711, 9	72		8·5642868, 7	8·2851894, 3 *n*	8·6320513, 5
28		9·8577984, 6	9·7395170, 0	9·8615134, 3	73		8·4258223, 5	8·3257551, 2 *n*	8·5165304, 8
29		9·8488884, 1	9·7250543, 7	9·8526800, 5	74		8·2343254, 0	8·3537216, 3 *n*	8·3681821, 5
30		9·8395889, 5	9·7099295, 6	9·8434622, 7	75		7·9118907, 6	8·3715957, 2 *n*	8·1570249, 8
					76		6·4732387, 3 *n*	8·3809239, 9 *n*	7·7704558, 1
31		9·8298903, 2	9·6941217, 5	9·8338507, 2	77		7·9149008, 7 *n*	8·3826543, 2 *n*	7·3071964, 2 *n*
32		9·8197821, 0	9·6776084, 3	9·8238353, 3	78		8·1930204, 9 *n*	8·3773190, 0 *n*	7·9733269, 3 *n*
33		9·8092531, 0	9·6603651, 1	9·8134053, 3	79		8·3505476, 5 *n*	8·3651274, 9 *n*	8·2101448, 3 *n*
34		9·7982913, 0	9·6423651, 7	9·8025491, 5	80		8·4574252, 7 *n*	8·3460007, 0 *n*	8·3517555, 5 *n*
35		9·7868837, 8	9·6235795, 1	9·7912543, 7					
36		9·7750166, 4	9·6039764, 0	9·7795076, 9	81		8·5359493, 7 *n*	8·3195630, 3 *n*	8·4496301, 9 *n*
37		9·7626749, 2	9·5835210, 2	9·7672947, 4	82		8·5960952, 2 *n*	8·2850854, 0 *n*	8·5219539, 8 *n*
38		9·7498424, 9	9·5621752, 3	9·7546001, 0	83		8·6431701, 9 *n*	8·2413596, 8 *n*	8·5772555, 2 *n*
39		9·7365019, 3	9·5398969, 4	9·7414071, 1	84		8·6803113, 0 *n*	8·1864587, 9 *n*	8·6201920, 9 *n*
40		9·7226343, 8	9·5166397, 3	9·7276977, 9	85		8·7095024, 0 *n*	8·1172507, 4 *n*	8·6535546, 1 *n*
					86		8·7320523, 1 *n*	8·0283577, 2 *n*	8·6791153, 3 *n*
41		9·7082194, 5	9·4923521, 0	9·7134527, 2	87		8·7488427, 0 *n*	7·9095678, 0 *n*	8·6980355, 2 *n*
42		9·6932349, 7	9·4669767, 5	9·6986508, 1	88		8·7604644, 0 *n*	7·7378237, 4 *n*	8·7110781, 0 *n*
43		9·6776568, 7	9·4404498, 2	9·6832692, 3	89		8·7672964, 5 *n*	7·4393848, 1 *n*	8·7187257, 6 *n*
44		9·6614588, 8	9·4126994, 3	9·6672830, 9	90		8·7695510, 8 *n*	$\propto$	8·7212464, 0 *n*
45		9·6446123, 4	9·3836447, 6	9·6506653, 2					

[The following is a specimen of the mode of testing the correctness of the results of the calculations.]

Take $G_{10}{}^4$ for $\mu = \cdot 80$.

$$\begin{array}{r} \cdot 0735623529 \\ \cdot 80 \\ \hline \cdot 0588498823{,}2 \\ \hline \end{array} \qquad \begin{array}{rr} & \cdot 1689846154 \\ & 65 \\ \hline & 8449230770 \\ & 10\ 139076924 \\ \hline 323\,) & 10 \cdot 9840000010 \\ \hline & -\cdot 0340061919{,}5 \\ & \cdot 0588498823{,}2 \\ \hline G_{10}{}^4 & \cdot 0248436903{,}7 \\ \hline \end{array}$$

Next for $\mu = \cdot 81$.

$$\begin{array}{r} \cdot 0837131460 \\ 18 \\ \hline 6697051680 \\ 83713146 \\ \hline \cdot 0678076482{,}6 \\ -\cdot 0369095004{,}7 \\ \hline \cdot 0308981477{,}9 \\ \hline \end{array} \qquad \begin{array}{rr} & \cdot 1834118254 \\ & 65 \\ \hline & 9170591270 \\ & 11004709524 \\ \hline 323\,) & 11 \cdot 921768651{,}0 \\ \hline & \cdot 036909500{,}47 \\ \hline \end{array}$$

Since $G_{10}{}^4$, $G_{10}{}^5$, $G_{10}{}^6$, &c. are the respective successive differential coefficients of a function, each multiplied by a constant, and since $G_{10}{}^4$ is of six dimensions in μ and the coefficient of the highest power of μ in each of them is 1; it is clear that the coefficients of the successive terms in the expansion of $G_{10}{}^4 + \delta G_{10}{}^4$ in powers of $\delta\mu$ will be the same as those of a binomial raised to the sixth power.

Thus
$$G_{10}{}^4 + \delta G_{10}{}^4 = G_{10}{}^4 + 6G_{10}{}^5 \,.\, \delta\mu + 15G_{10}{}^6 \,.\, \delta\mu^2$$
$$+ 20G_{10}{}^7 \,.\, \delta\mu^3 + 15G_{10}{}^8 \,.\, \delta\mu^4 + 6G_{10}{}^9 \,.\, \delta\mu^5 + G_{10}^{10} \,.\, \delta\mu^6,$$
where
$$G_{10}{}^9 = \mu \quad \text{and} \quad G_{10}^{10} = 1.$$

Now take $\delta\mu$ successively $= \cdot 01$ and $-\cdot 01$ and $\mu = \cdot 80$.

$$\begin{array}{rr} G_{10}{}^4 & \cdot 0248436903{,}7 \\ 15G_{10}{}^6 \,.\, \delta\mu^2 & 3251739{,}9 \\ 15G_{10}{}^8 \,.\, \delta\mu^4 & 881{,}0 \\ \hline & \cdot 0251689524{,}6 \\ & \pm \quad 57291952{,}7 \\ \hline \mu = \cdot 81,\ G_{10}{}^4 = & \cdot 0308981477{,}3 \\ \hline \mu = \cdot 79,\ G_{10}{}^4 = & \cdot 0194397571{,}9 \\ \hline \end{array} \qquad \begin{array}{rr} 6G_{10}{}^5 \,.\, \delta\mu & 57214811{,}1 \\ 20G_{10}{}^7 \,.\, \delta\mu^3 & 77136{,}8 \\ 6G_{10}{}^9 \,.\, \delta\mu^5 & 4{,}8 \\ \hline & 57291952{,}7 \\ \hline \end{array}$$

SECTION II.

COMPUTATION OF THE VALUES OF LOG $G_n^{\prime m}$ FOR THE EARTH'S SURFACE.

1. TAKING into account the spheroidal figure of the Earth, let r, θ', λ be polar coordinates of a point referred to the Earth's centre as origin and axis of figure as initial line.

Also let a be the semi-axis major of the terrestrial meridian,

θ' the angle from the axis of rotation,

and θ the geographical colatitude of the point.

Then
$$r^2 = \frac{a^4 \sin^2\theta + b^4 \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta},$$

$$r \sin\theta' = \frac{a^2 \sin\theta}{\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}},$$

$$r \cos\theta' = \frac{b^2 \cos\theta}{\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}.$$

Also
$$\tan\theta' = \frac{a^2}{b^2}\tan\theta,$$

or
$$\tan\theta = \frac{b^2}{a^2}\tan\theta' = \frac{b^2 \sin\theta'}{a^2 \cos\theta'};$$

$$\therefore \quad \sin\theta = \frac{b^2 \sin\theta'}{\sqrt{a^4 \cos^2\theta' + b^4 \sin^2\theta'}},$$

$$\cos\theta = \frac{a^2 \cos\theta'}{\sqrt{a^4 \cos^2\theta' + b^4 \sin^2\theta'}},$$

and
$$r^2 = \frac{a^2 b^2}{b^2 \sin^2\theta' + a^2 \cos^2\theta'}.$$

If e be the eccentricity of the terrestrial meridian,

$$b^2 = a^2(1 - e^2),$$

and we have

$$r^2 = \frac{a^2(1-e^2)}{1 - e^2 \sin^2\theta'},$$

also
$$\frac{r^2}{a^2}=\frac{\sin^2\theta+(1-e^2)^2\cos^2\theta}{\sin^2\theta+(1-e^2)\cos^2\theta}=\frac{1-2e^2\cos^2\theta+e^4\cos^2\theta}{1-e^2\cos^2\theta},$$

$$\frac{r}{a}\sin\theta'=\frac{\sin\theta}{\sqrt{\sin^2\theta+(1-e^2)\cos^2\theta}}=\frac{\sin\theta}{\sqrt{1-e^2\cos^2\theta}},$$

$$\frac{r}{a}\cos\theta'=\frac{(1-e^2)\cos\theta}{\sqrt{\sin^2\theta+(1-e^2)\cos^2\theta}}=\frac{(1-e^2)\cos\theta}{\sqrt{1-e^2\cos^2\theta}},$$

and
$$\tan\theta'=\frac{1}{1-e^2}\tan\theta.$$

These formulae give the radius vector and the geocentric colatitude in terms of the geographical colatitude.

2. Now
$$\frac{r}{a}\cos(\theta'-\theta)=\frac{(1-e^2)\cos^2\theta+\sin^2\theta}{\sqrt{1-e^2\cos^2\theta}}=\sqrt{1-e^2\cos^2\theta},$$

and
$$\frac{r}{a}\sin(\theta'-\theta)=\frac{\sin\theta\cos\theta}{\sqrt{1-e^2\cos^2\theta}}\{1-(1-e^2)\}=\frac{e^2\sin\theta\cos\theta}{\sqrt{1-e^2\cos^2\theta}}.$$

Also
$$\cos(\theta'-\theta)=\frac{\cos^2\theta'+(1-e^2)\sin^2\theta'}{\sqrt{\cos^2\theta'+(1-e^2)^2\sin^2\theta'}}$$

$$=\frac{1-e^2\sin^2\theta'}{\sqrt{1-2e^2\sin^2\theta'+e^4\sin^2\theta'}};$$

and
$$\sin(\theta'-\theta)=\frac{e^2\sin\theta'\cos\theta'}{\sqrt{1-2e^2\sin^2\theta'+e^4\sin^2\theta'}}.$$

Also
$$\frac{a}{r}=\frac{\sqrt{1-e^2\cos^2\theta}}{\sqrt{1-2e^2\cos^2\theta+e^4\cos^2\theta}},$$

$$\sin\theta'=\frac{\sin\theta}{\sqrt{1-2e^2\cos^2\theta+e^4\cos^2\theta}},$$

$$\cos\theta'=\frac{(1-e^2)\cos\theta}{\sqrt{1-2e^2\cos^2\theta+e^4\cos^2\theta}}.$$

Also we get, taking ψ as the angle of the vertical, i.e. $(\theta'-\theta)$,

$$\cos\psi=\cos(\theta'-\theta)=\frac{1-e^2\cos^2\theta}{\sqrt{1-e^2(2-e^2)\cos^2\theta}},$$

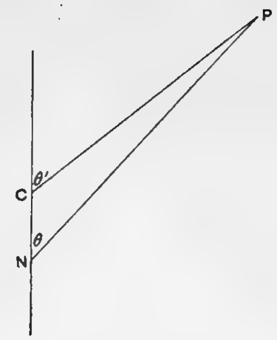
$$\sin\psi=\sin(\theta'-\theta)=\frac{e^2\sin\theta\cos\theta}{\sqrt{1-e^2(2-e^2)\cos^2\theta}}.$$

If N be the length of the normal PN terminated by the axis of rotation,

$$\sin\theta' = \frac{N}{r}\sin\theta;$$

hence
$$\frac{N}{a} = \frac{r}{a}\frac{\sin\theta'}{\sin\theta} = \frac{1}{\sqrt{1-e^2\cos^2\theta}},$$

$$\frac{N}{r} = \frac{1}{\sqrt{1-e^2(2-e^2)\cos^2\theta}},$$



and
$$\cos\theta' = \frac{N}{r}(1-e^2)\cos\theta;$$

also
$$\cos(\theta'-\theta) = \frac{a^2}{Nr},$$

and
$$\sin(\theta'-\theta) = \frac{N}{r}\,.\,e^2\sin\theta\cos\theta.$$

By these formulae the values of $\cos\theta'$, $\sin\theta'$, $\cos(\theta'-\theta)$, $\sin(\theta'-\theta)$ for values of θ differing by 1° from 0° to 90° have been computed, employing Bessel's dimensions of the Earth, as given in Encke's paper and tables in the *Berliner Jahrbuch*, 1852.

Encke gives
$$\log_{10} e = 8{\cdot}9122052075,$$
$$\log_{10}\sqrt{1-e^2} = 9{\cdot}9985458202,$$

also, n being $= \dfrac{a-b}{a+b}$,
$$\log_{10} n = 7{\cdot}2238033861,$$
$$\log_{10}(1+n^2) = 0{\cdot}0000012173.$$

Now
$$n = \frac{e^2}{(1+\sqrt{1-e^2})^2},$$

also
$$e^2 = \frac{4n}{(1+n)^2} \text{ and } \frac{e^2}{2-e^2} = \frac{2n}{1+n^2}.$$

Hence
$$2-e^2 = \frac{e^2}{2n}(1+n^2),$$

also
$$\log_{10} 2 = 0{\cdot}3010299957;$$

therefore
$$\log_{10}(2-e^2) = 0{\cdot}2995782505,$$

and
$$\log_{10} e^2(2-e^2) = 8{\cdot}1239886655.$$

These agree with the results obtained by Bessel, whose tables extend only to the seventh place of decimals.

3. The value of ψ the angle of the vertical has also been determined by another approximate method depending on the development of

$$\log_\epsilon \frac{1}{2}(1+\epsilon^x)$$

in powers of x.

To develope $\log_\epsilon \frac{1}{2}(1+\epsilon^x)$ in powers of x.

$$\frac{1}{2}(1+\epsilon^x) = 1 + \frac{1}{2}(\epsilon^x - 1);$$

$$\therefore \log_\epsilon \frac{1}{2}(1+\epsilon^x) = \frac{1}{2}(\epsilon^x - 1)$$

$$-\frac{1}{2}\cdot\frac{1}{2^2}(\epsilon^x-1)^2 + \frac{1}{3}\cdot\frac{1}{2^3}(\epsilon^x-1)^3 - \&c.$$

$$-\frac{(-1)^n}{n}\cdot\frac{1}{2^n}(\epsilon^x-1)^n + \&c.$$

Or

$$\log_\epsilon \frac{1}{2}(1+\epsilon^x) = \frac{1}{2}(\epsilon^x-1) - \frac{1}{2}\cdot\frac{1}{2^2}(\epsilon^{2x} - 2\epsilon^x + 1)$$

$$+\frac{1}{3}\cdot\frac{1}{2^3}(\epsilon^{3x} - 3\epsilon^{2x} + 3\epsilon^x - 1)$$

$$-\frac{1}{4}\cdot\frac{1}{2^4}(\epsilon^{4x} - 4\epsilon^{3x} + 6\epsilon^{2x} - 4\epsilon^x + 1) + \&c.$$

$$-\frac{(-1)^n}{n}\cdot\frac{1}{2^n}\left(\epsilon^{nx} - n\epsilon^{(n-1)x} + \frac{n(n-1)}{1.2}\epsilon^{(n-2)x} - \&c.\right) + \&c.$$

Now substitute for ϵ^x, ϵ^{2x}, ϵ^{3x}, &c. their developments in powers of x and we have

$$\log_\epsilon \frac{1}{2}(1+\epsilon^x) = \frac{1}{2}\left\{x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.\right\}$$

$$-\frac{1}{2}\cdot\frac{1}{2^2}\left\{\frac{1}{1.2}[2^2 - 2.1^2 + 1.0^2]x^2\right.$$

$$\left.+\frac{x^3}{1.2.3}[2^3 - 2.1^3 + 1.0^3] + \&c.\right\}$$

$$+\frac{1}{3}\cdot\frac{1}{2^3}\left\{\frac{x^3}{1.2.3}[3^3-3.2^3+3.1^3-1.0^3]\right.$$

$$\left.+\frac{x^4}{1.2.3.4}[3^4-3.2^4+3.1^4-1.0^4]+\&c.\right\}$$

$$-\frac{1}{4}\cdot\frac{1}{2^4}\left\{\frac{x^4}{1.2.3.4}[4^4-4.3^4+6.2^4-4.1^4+1.0^4]\right.$$

$$\left.+\frac{x^5}{1.2.3.4.5}[4^5-4.3^5+6.2^5-4.1^5+1.0^5]+\&c.\right\}$$

$$+\&c.$$

$$-\frac{(-1)^n}{n}\cdot\frac{1}{2^n}\left\{\frac{x^n}{n\,!}\left[n^n-n(n-1)^n+\frac{n(n-1)}{1.2}(n-2)^n-\&c.\right]\right.$$

$$\left.+\frac{x^{n+1}}{(n+1)\,!}\left[n^{n+1}-n(n-1)^{n+1}+\&c.\right]\right\}+\&c.$$

Or, employing the symbol of operation $\Delta^n 0^m$ in the usual sense,

$$\log_\epsilon\frac{1}{2}(1+\epsilon^x)=\frac{1}{2}\left\{x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\&c.\right\}$$

$$-\frac{1}{2}\cdot\frac{1}{2^2}\left\{\frac{x^2}{1.2}\Delta^2 0^2+\frac{x^3}{1.2.3}\Delta^2 0^3+\frac{x^4}{1.2.3.4}\Delta^2 0^4+\&c.\right\}$$

$$+\frac{1}{3}\cdot\frac{1}{2^3}\left\{\frac{x^3}{1.2.3}\Delta^3 0^3+\frac{x^4}{1.2.3.4}\Delta^3 0^4+\&c.\right\}$$

$$-\frac{1}{4}\cdot\frac{1}{2^4}\left\{\frac{x^4}{1.2.3.4}\Delta^4 0^4+\&c.\right\}+\&c.$$

$$-\frac{(-1)^n}{n}\cdot\frac{1}{2^n}\left\{\frac{x^n}{n\,!}\Delta^n 0^n+\frac{x^{n+1}}{(n+1)\,!}\Delta^n 0^{n+1}+\&c.\right\}$$

$$+\&c.$$

where the first line may be expressed similarly to the rest, viz.

$$\frac{1}{2}\left\{x\Delta 0^1+\frac{x^2}{1.2}\Delta 0^2+\frac{x^3}{1.2.3}\Delta 0^3+\&c.\right\},$$

since $\Delta 0^1$, $\Delta 0^2$, $\Delta 0^3$, &c. are all $=1$.

Hence, we have

$$\log_\epsilon \frac{1}{2}(1+\epsilon^x) = \Delta 0^1\left(\frac{1}{2}x\right) + \frac{x^2}{1.2}\left\{\frac{1}{2}\Delta 0^2 - \frac{1}{2}\cdot\frac{1}{2^2}\Delta^2 0^2\right\}$$

$$+\frac{x^3}{1.2.3}\left\{\frac{1}{2}\Delta 0^3 - \frac{1}{2}\cdot\frac{1}{2^2}\Delta^2 0^3 + \frac{1}{3}\cdot\frac{1}{2^3}\Delta^3 0^3\right\}$$

$$+\frac{x^4}{1.2.3.4}\left\{\frac{1}{2}\Delta 0^4 - \frac{1}{2}\cdot\frac{1}{2^2}\Delta^2 0^4 + \frac{1}{3}\cdot\frac{1}{2^3}\Delta^3 0^4 - \frac{1}{4}\cdot\frac{1}{2^4}\Delta^4 0^4\right\} + \&c.$$

$$+\frac{x^n}{n\,!}\left\{\frac{1}{2}\Delta 0^n - \frac{1}{2}\cdot\frac{1}{2^2}\Delta^2 0^n + \frac{1}{3}\cdot\frac{1}{2^3}\Delta^3 0^n - \&c. - \frac{(-1)^n}{n}\frac{1}{2^n}\Delta^n 0^n\right\}$$

$$+ \&c.$$

observing that $\Delta^n 0^m = 0$ when n is greater than m.

The above may be symbolically expressed thus

$$\log\frac{1}{2}(1+\epsilon^x) = x\log\left(1+\frac{1}{2}\Delta\right)0^1 + \frac{x^2}{1.2}\log\left(1+\frac{1}{2}\Delta\right)0^2 + \frac{x^3}{1.2.3}\log\left(1+\frac{1}{2}\Delta\right)0^3 + \&c.$$

$$+\frac{x^n}{n!}\log\left(1+\frac{1}{2}\Delta\right)0^n + \&c.;$$

or

$$= \log\left(1+\frac{1}{2}\Delta\right)\left\{0^1 x + 0^2\frac{x^2}{1.2} + 0^3\frac{x^3}{1.2.3} + \&c. + 0^n\frac{x^n}{n\,!} + \&c.\right\},$$

where the symbols of operation Δ, Δ^2, &c. act only on the quantities denoted by 0^1, 0^2, 0^3, &c.

This again may be symbolically represented thus

$$\log\frac{1}{2}(1+\epsilon^x) = \log\left(1+\frac{1}{2}\Delta\right)[\epsilon^{0^1 x} - 1], \text{ or by } \log\left(1+\frac{1}{2}\Delta\right)\epsilon^{0^1 x} \text{ simply,}$$

since Δ, Δ^2, &c. performed on the constant 1 produce zero.

If we substitute the known values of $\Delta^n 0^m$ as given in Herschel's "Examples of Finite Differences," p. 9, we have

$$\log_\epsilon \frac{1}{2}(1+\epsilon^x) = x\left[\frac{1}{2}\right] + \frac{x^2}{1.2}\left[\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2}2\right] + \frac{x^3}{1.2.3}\left[\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2}6 + \frac{1}{3}\cdot\frac{1}{2^3}6\right]$$

$$+\frac{x^4}{1.2.3.4}\left[\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2}14 + \frac{1}{3}\cdot\frac{1}{2^3}36 - \frac{1}{4}\cdot\frac{1}{2^4}24\right] + \&c.$$

The coefficient of $\frac{x^2}{1\,.\,2}$ is $\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$,

the coefficient of $\frac{x^3}{1\,.\,2\,.\,3}$ is $\frac{1}{2}-\frac{3}{4}+\frac{1}{4}=0$,

the coefficient of $\frac{x^4}{1\,.\,2\,.\,3\,.\,4}$ is $\frac{1}{2}-\frac{7}{4}+\frac{3}{2}-\frac{3}{8}=-\frac{1}{8}$, &c., &c.

Hence to the 4th power of x we have

$$\log_\epsilon \frac{1}{2}(1+\epsilon^x)=\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4.$$

If $x=-\log\cos\psi=\log\sec\psi$, where ψ is the angle of the vertical, we have

$$\log_\epsilon \frac{1}{2}(1+\cos\psi)=\log_\epsilon \frac{1}{2}(1+\epsilon^{-x})=-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\&c.$$

4. Also $1-\cos\psi=\frac{\sin^2\psi}{1+\cos\psi}$,

hence $\log(1-\cos\psi)=2\log\sin\psi-\log 2-\log\frac{1}{2}(1+\cos\psi)$

$$=2\log\sin\psi-\log 2+\frac{1}{2}\log(\sec\psi) \text{ nearly,}$$

neglecting the square of $\log(\sec\psi)$.

Or if a is the modulus of common logarithms, we have

$$\log_{10}\frac{1}{2}(1+\cos\psi)=a\log_\epsilon\frac{1}{2}(1+\cos\psi)=-\frac{1}{2}ax+\frac{1}{8}ax^2-\frac{1}{192}ax^4 \text{ nearly}$$

$$=-\frac{1}{2}ax+\frac{1}{8}\frac{(ax)^2}{a}-\frac{1}{192}\frac{(ax)^4}{a^3},$$

or $$-\log_{10}\frac{1}{2}(1+\cos\psi)=\frac{1}{2}\log_{10}\sec\psi-\frac{1}{8}\frac{(\log_{10}\sec\psi)^2}{a}+\frac{1}{192}\frac{(\log_{10}\sec\psi)^4}{a^3}.$$

By these formulae the values of $\log(1-\cos\psi)$ and $\log(1+\cos\psi)$ for every 5th degree of geographical colatitude have been determined.

5. Taking $\mu=\cos\theta$ and $\mu'=\cos\theta'$, the following formula is convenient as a test of accuracy in the determination of $\mu-\mu'$.

$$\begin{aligned}\mu-\mu' &= \cos\theta\{1-\cos(\theta'-\theta)\}+\sin\theta\sin(\theta'-\theta)\\ &= \sin\theta\sin(\theta'-\theta)+\cos\theta\,\frac{\sin^2(\theta'-\theta)}{1+\cos(\theta'-\theta)}\\ &= e^2\sin\theta'\sin\theta\cos\theta+\frac{e^4\sin^2\theta'\cos^3\theta}{1+\cos(\theta'-\theta)}\\ &= e^2\sin\theta'\sin\theta\cos\theta\left[1+\frac{e^2\sin\theta'\cos^2\theta}{\sin\theta\{1+\cos(\theta'-\theta)\}}\right].\end{aligned}$$

Also $\qquad \sin\psi=e^2\sin\theta'\cos\theta,$

hence $$\mu-\mu'=\sin\psi\sin\theta\left[1+e^2\frac{N}{r}\cos^2\theta\left(\frac{1}{1+\cos\psi}\right)\right].$$

6. The following theorem is also useful for testing the accuracy of numerical determinations of similar functions of μ and μ'.

Let G be any function of μ, and G' the same function of μ'.

Since $\mu=\mu'+(\mu-\mu')$ and $\mu'=\mu-(\mu-\mu')$, we get from Taylor's Theorem,

$$\begin{aligned}2(G-G') &= (\mu-\mu')\left(\frac{dG}{d\mu}+\frac{dG'}{d\mu'}\right)-\frac{1}{2}(\mu-\mu')^2\left(\frac{d^2G}{d\mu^2}-\frac{d^2G'}{d\mu'^2}\right)\\ &+\frac{1}{6}(\mu-\mu')^3\left(\frac{d^3G}{d\mu^3}+\frac{d^3G'}{d\mu'^3}\right)-\frac{1}{24}(\mu-\mu')^4\left(\frac{d^4G}{d\mu^4}-\frac{d^4G'}{d\mu'^4}\right)\\ &+\frac{1}{120}(\mu-\mu')^5\left(\frac{d^5G}{d\mu^5}+\frac{d^5G'}{d\mu'^5}\right)+\&c.\end{aligned}$$

But by the same formula

$$\begin{aligned}2\left(\frac{d^2G}{d\mu^2}-\frac{d^2G'}{d\mu'^2}\right) &= (\mu-\mu')\left(\frac{d^3G}{d\mu^3}+\frac{d^3G'}{d\mu'^3}\right)-\frac{1}{2}(\mu-\mu')^2\left(\frac{d^4G}{d\mu^4}-\frac{d^4G'}{d\mu'^4}\right)\\ &+\frac{1}{6}(\mu-\mu')^3\left(\frac{d^5G}{d\mu^5}+\frac{d^5G'}{d\mu'^5}\right)+\&c.,\end{aligned}$$

and $$2\left(\frac{d^4G}{d\mu^4}-\frac{d^4G'}{d\mu'^4}\right)=(\mu-\mu')\left(\frac{d^5G}{d\mu^5}+\frac{d^5G'}{d\mu'^5}\right)+\&c.$$

Combining these formulae we get

$$G - G' = \left(\frac{dG}{d\mu}\right)(\mu - \mu') - \frac{1}{12}\left(\frac{d^3G}{d\mu^3}\right)(\mu - \mu')^3 + \frac{1}{120}\left(\frac{d^5G}{d\mu^5}\right)(\mu - \mu')^5 - \&c.,$$

where for $\left(\dfrac{dG}{d\mu}\right)$ we must put $\dfrac{1}{2}\left[\dfrac{dG}{d\mu} + \dfrac{dG'}{d\mu'}\right]$,

for $\left(\dfrac{d^3G}{d\mu^3}\right)$,, ,, ,, $\dfrac{1}{2}\left[\dfrac{d^3G}{d\mu^3} + \dfrac{d^3G'}{d\mu'^3}\right]$, &c.

Now if G_n^m, G_n^{m+1}, &c., have the signification given them in Section I. (see p. 259), each of these quantities is proportional to the differential coefficient of the one immediately preceding taken with respect to μ, also the highest powers of μ in these quantities are respectively $n-m$, $n-m-1$, &c., and the coefficients of these highest powers are in every case unity.

Hence $$\frac{d}{d\mu}(G_n^m) = (n-m)\, G_n^{m+1},$$

$$\frac{d^2}{d\mu^2}(G_n^m) = (n-m)(n-m-1)\, G_n^{m+2}, \ \&c.$$

Hence if as above we denote by (G_n^{m+1}), (G_n^{m+3}), &c., the means of the several quantities obtained by substituting μ and μ' for μ in the respective functions, we have

$$G_n^m - G_n'^m = (n-m)(G_n^{m+1})(\mu - \mu') - \frac{(n-m)(n-m-1)(n-m-2)}{12}(G_n^{m+3})(\mu-\mu')^3$$

$$+ \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)(n-m-4)}{120}(G_n^{m+5})(\mu-\mu')^5$$

$$- \&c., \&c.,$$

which is a convenient test formula for examining the values of G_n^m and G'^m_n by taking their difference.

TABLE OF THE VALUES OF LOG COS ψ AND LOG SIN ψ; ψ BEING THE ANGLE OF THE VERTICAL, AND θ THE GEOGRAPHICAL COLATITUDE.

θ	Log cos ψ	Log sin ψ	θ	Log cos ψ	Log sin ψ	θ	Log cos ψ	Log sin ψ
0°	0·0000000, 0	$-\infty$						
1	0·0000000, 0	6·0691071	31°	9·9999981, 0	7·4714484	61°	9·9999982, 6	7·4524810
2	9·9999999, 9	6·3698697	32	9·9999980, 3	7·4791283	62	9·9999983, 3	7·4425923
3	9·9999999, 7	6·5455153	33	9·9999979, 6	7·4861522	63	9·9999984, 1	7·4319343
4	9·9999999, 5	6·6698298	34	9·9999979, 0	7·4925410	64	9·9999984, 9	7·4204684
5	9·9999999, 3	6·7659368	35	9·9999978, 5	7·4983135	65	9·9999985, 8	7·4081510
6	9·9999998, 9	6·8441357	36	9·9999977, 9	7·5034859	66	9·9999986, 6	7·3949323
7	9·9999998, 6	6·9099205	37	9·9999977, 5	7·5080726	67	9·9999987, 5	7·3807560
8	9·9999998, 1	6·9665701	38	9·9999977, 0	7·5120859	68	9·9999988, 3	7·3655575
9	9·9999997, 7	7·0161995	39	9·9999976, 7	7·5155367	69	9·9999989, 2	7·3492627
10	9·9999997, 1	7·0602522	40	9·9999976, 4	7·5184338	70	9·9999990, 0	7·3317861
11	9·9999996, 6	7·0997576	41	9·9999976, 1	7·5207850	71	9·9999990, 8	7·3130288
12	9·9999995, 9	7·1354755	42	9·9999975, 9	7·5225961	72	9·9999991, 6	7·2928752
13	9·9999995, 3	7·1679826	43	9·9999975, 8	7·5238720	73	9·9999992, 4	7·2711892
14	9·9999994, 6	7·1977268	44	9·9999975, 7	7·5246158	74	9·9999993, 2	7·2478097
15	9·9999993, 9	7·2250627	45	9·9999975, 7	7·5248297	75	9·9999994, 0	7·2225440
16	9·9999993, 1	7·2502761	46	9·9999975, 7	7·5245143	76	9·9999994, 7	7·1951589
17	9·9999992, 3	7·2736003	47	9·9999975, 8	7·5236691	77	9·9999995, 3	7·1653686
18	9·9999991, 5	7·2952281	48	9·9999975, 9	7·5222921	78	9·9999996, 0	7·1328186
19	9·9999990, 7	7·3153206	49	9·9999976, 1	7·5203802	79	9·9999996, 6	7·0970610
20	9·9999989, 9	7·3340140	50	9·9999976, 4	7·5179288	80	9·9999997, 2	7·0575192
21	9·9999989, 0	7·3514240	51	9·9999976, 7	7·5149320	81	9·9999997, 7	7·0134335
22	9·9999988, 2	7·3676496	52	9·9999977, 1	7·5113823	82	9·9999998, 2	6·9637745
23	9·9999987, 3	7·3827763	53	9·9999977, 5	7·5072709	83	9·9999998, 6	6·9070985
24	9·9999986, 5	7·3968784	54	9·9999978, 0	7·5025872	84	9·9999999, 0	6·8412909
25	9·9999985, 7	7·4100204	55	9·9999978, 5	7·4973188	85	9·9999999, 3	6·7630726
26	9·9999984, 8	7·4222590	56	9·9999979, 1	7·4914515	86	9·9999999, 5	6·6669498
27	9·9999984, 0	7·4336438	57	9·9999979, 7	7·4849692	87	9·9999999, 7	6·5426229
28	9·9999983, 2	7·4442186	58	9·9999980, 4	7·4778534	88	9·9999999, 9	6·3669685
29	9·9999982, 4	7·4540222	59	9·9999981, 1	7·4700830	89	0·0000000, 0	6·0662005
30	9·9999981, 7	7·4630887	60	9·9999981, 8	7·4616345	90	0·0000000, 0	$-\infty$

TABLE OF THE VALUES OF LOG μ', OR LOG COS θ'; θ' BEING THE GEOCENTRIC, AND θ THE GEOGRAPHICAL COLATITUDE.

θ	Log μ'	θ	Log μ'	θ	Log μ'
0°	0·0000000, 0				
1	9·9999329, 6	31°	9·9322903, 1	61°	9·6833429, 6
2	9·9997317, 9	32	9·9275998, 3	62	9·6693386, 1
3	9·9993963, 9	33	9·9227246, 2	63	9·6547346, 6
4	9·9989265, 4	34	9·9176605, 9	64	9·6394894, 8
5	9·9983219, 9	35	9·9124033, 9	65	9·6235565, 0
6	9·9975823, 6	36	9·9069484, 2	66	9·6068834, 0
7	9·9967072, 3	37	9·9012907, 6	67	9·5894111, 6
8	9·9956960, 7	38	9·8954251, 7	68	9·5710728, 5
9	9·9945482, 9	39	9·8893460, 9	69	9·5517921, 4
10	9·9932631, 9	40	9·8830475, 8	70	9·5314815, 3
11	9·9918400, 0	41	9·8765232, 9	71	9·5100399, 9
12	9·9902778, 6	42	9·8697664, 6	72	9·4873500, 5
13	9·9885758, 2	43	9·8627698, 8	73	9·4632740, 7
14	9·9867328, 3	44	9·8555258, 0	74	9·4376493, 2
15	9·9847477, 3	45	9·8480259, 5	75	9·4102814, 8
16	9·9826193, 0	46	9·8402614, 8	76	9·3809359, 6
17	9·9803461, 8	47	9·8322228, 7	77	9·3493259, 1
18	9·9779269, 1	48	9·8238999, 0	78	9·3150954, 7
19	9·9753599, 4	49	9·8152815, 8	79	9·2777956, 9
20	9·9726435, 8	50	9·8063560, 8	80	9·2368490, 0
21	9·9697760, 4	51	9·7971106, 3	81	9·1914947, 9
22	9·9667553, 7	52	9·7875314, 2	82	9·1407029, 1
23	9·9635795, 3	53	9·7776035, 3	83	9·0830290, 2
24	9·9602463, 2	54	9·7673107, 4	84	9·0163577, 7
25	9·9567533, 9	55	9·7566354, 7	85	8·9374095, 9
26	9·9530982, 6	56	9·7455585, 5	86	8·8406902, 2
27	9·9492782, 5	57	9·7340590, 6	87	8·7158997, 2
28	9·9452905, 6	58	9·7221141, 3	88	8·5399143, 2
29	9·9411321, 7	59	9·7096986, 7	89	8·2389478, 4
30	9·9367998, 8	60	9·6967850, 9	90	inf. ∞

Table of the Values of Log H'^{1}_{1} or Log sin θ', θ = Geographical Colatitude, θ' = Geocentric Colatitude.

θ	Log H'^{1}_{1}	θ	Log H'^{1}_{1}	θ	Log H'^{1}_{1}
1°	8·2447627, 9	31°	9·7139724, 1	61°	9·9424993, 5
[illegible]	8·5457239, 6	32	9·7262974, 2	62	9·9465726, 0
3	8·7217005, 0	33	9·7381503, 4	63	9·9504771, 4
4	8·8464786, 3	34	9·7495563, 8	64	9·9542160, 7
5	8·9431821. 3	35	9·7605385, 3	65	9·9577923, 1
6	9·0221109, 4	36	9·7711178, 2	66	9·9612086, 3
7	9·0887593, 5	37	9·7813135, 3	67	9·9644675, 9
8	9·1464069, 6	38	9·7911433, 7	68	9·9675716, 5
9	9·1971691, 6	39	9·8006236, 6	69	9·9705230, 8
10	9·2424903, 2	40	9·8097694, 7	70	9·9733240, 2
11	9·2834006, 3	41	9·8185947, 0	71	9·9759765, 0
12	9·3206607, 4	42	9·8271122, 6	72	9·9784823, 7
13	9·3548482, 8	43	9·8353341, 0	73	9·9808434, 1
14	9·3864122, 4	44	9·8432713, 4	74	9·9830612, 4
15	9·4157085, 4	45	9·8509343, 1	75	9·9851373, 9
16	9·4430241, 0	46	9·8583326, 6	76	9·9870732, 6
17	9·4685935, 6	47	9·8654753, 6	77	9·9888701, 7
18	9·4926113, 1	48	9·8723708, 2	78	9·9905293, 1
19	9·5152401, 5	49	9·8790268, 9	79	9·9920517, 8
20	9·5366178, 1	50	9·8854509, 1	80	9·9934385, 9
21	9·5568618, 2	51	9·8916497, 8	81	9·9946906, 4
22	9·5760732, 9	52	9·8976299, 5	82	9·9958087, 4
23	9·5943398, 2	53	9·9033974, 8	83	9·9967936, 2
24	9·6117378, 2	54	9·9089580, 6	84	9·9976459, 2
25	9·6283343, 0	55	9·9143170, 5	85	9·9983661, 7
26	9·6441883, 9	56	9·9194794, 7	86	9·9989548, 5
27	9·6593524, 9	57	9·9244500, 6	87	9·9994123, 2
28	9·6738732, 8	58	9·9292332, 7	88	9·9997388, 8
29	9·6877925, 0	59	9·9338332, 9	89	9·9999347, 3
30	9·7011476, 1	60	9·9382540, 8	90	0·0000000, 0

Table of the Values of Log H'^{2}_{2} = Log ($\sin^2 \theta'$), θ = Geographical Colatitude, θ' = Geocentric Colatitude.

θ	Log H'^{2}_{2}	θ	Log H'^{2}_{2}	θ	Log H'^{2}_{2}
1°	6·4895255, 7	31°	9·4279448, 2	61°	9·8849986, 9
2	7·0914479, 1	32	9·4525948, 3	62	9·8931452, 0
3	7·4434010, 1	33	9·4763006, 8	63	9·9009542, 8
4	7·6929572, 6	34	9·4991127, 7	64	9·9084321, 4
5	7·8863642, 6	35	9·5210770, 7	65	9·9155846, 3
6	8·0442218, 7	36	9·5422356, 4	66	9·9224172, 5
7	8·1775187, 0	37	9·5626270, 5	67	9·9289351, 9
8	8·2928139, 2	38	9·5822867, 3	68	9·9351433, 0
9	8·3943383, 2	39	9·6012473, 2	69	9·9410461, 6
10	8·4849806, 4	40	9·6195389, 3	70	9·9466480, 5
11	8·5668012, 6	41	9·6371894, 0	71	9·9519529, 9
12	8·6413214, 8	42	9·6542245, 2	72	9·9569647, 5
13	8·7096965, 6	43	9·6706682, 0	73	9·9616868, 2
14	8·7728244, 9	44	9·6865426, 8	74	9·9661224, 8
15	8·8314170, 9	45	9·7018686, 3	75	9·9702747, 8
16	8·8860481, 9	46	9·7166653, 1	76	9·9741465, 3
17	8·9371871, 2	47	9·7309507, 2	77	9·9777403, 5
18	8·9852226, 2	48	9·7447416, 4	78	9·9810586, 2
19	9·0304803, 1	49	9·7580537, 8	79	9·9841035, 7
20	9·0732356, 2	50	9·7709018, 2	80	9·9868771, 8
21	9·1137236, 4	51	9·7832995, 6	81	9·9893812, 7
22	9·1521465, 8	52	9·7952598, 9	82	9·9916174, 8
23	9·1886796, 4	53	9·8067949, 5	83	9·9935872, 4
24	9·2234756, 4	54	9·8179161, 2	84	9·9952918, 3
25	9·2566686, 0	55	9·8286340, 9	85	9·9967323, 4
26	9·2883767, 8	56	9·8389589, 4	86	9·9979097, 0
27	9·3187049, 9	57	9·8489001, 2	87	9·9988246, 4
28	9·3477465, 7	58	9·8584665, 3	88	9·9994777, 6
29	9·3755849, 9	59	9·8676665, 8	89	9·9998694, 6
30	9·4022952, 3	60	9·8765081, 6	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^3_3 = LOG ($\sin^3 \theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^3_3	θ	Log H'^3_3	θ	Log H'^3_3
1°	4·7342883, 6	31°	9·1419172, 2	61°	9·8274980, 4
2	5·6371718, 7	32	9·1788922, 5	62	9·8397178, 0
3	6·1651015, 1	33	9·2144510, 2	63	9·8514314, 2
4	6·5394358, 9	34	9·2486691, 5	64	9·8626482, 1
5	6·8295463, 8	35	9·2816156, 0	65	9·8733769, 4
6	7·0663328, 1	36	9·3133534, 6	66	9·8836258, 8
7	7·2662780, 5	37	9·3439405, 8	67	9·8934027, 8
8	7·4392208, 8	38	9·3734301, 0	68	9·9027149, 5
9	7·5915074, 8	39	9·4018709, 8	69	9·9115692, 4
10	7·7274709, 6	40	9·4293084, 0	70	9·9199720, 7
11	7·8502018, 9	41	9·4557841, 0	71	9·9279294, 9
12	7·9619822, 2	42	9·4813367, 8	72	9·9354471, 2
13	8·0645448, 4	43	9·5060023, 1	73	9·9425302, 3
14	8·1592367, 3	44	9·5298140, 2	74	9·9491837, 2
15	8·2471256, 3	45	9·5528029, 4	75	9·9554121, 7
16	8·3290722, 9	46	9·5749979, 7	76	9·9612197, 9
17	8·4057806, 8	47	9·5964260, 8	77	9·9666105, 2
18	8·4778339, 4	48	9·6171124, 6	78	9·9715879, 4
19	8·5457204, 6	49	9·6370806, 6	79	9·9761553, 5
20	8·6098534, 4	50	9·6563527, 4	80	9·9803157, 7
21	8·6705854, 6	51	9·6749493, 4	81	9·9840719, 1
22	8·7282198, 7	52	9·6928898, 4	82	9·9874262, 2
23	8·7830194, 7	53	9·7101924, 3	83	9·9903808, 6
24	8·8352134, 6	54	9·7268741, 8	84	9·9929377, 5
25	8·8850029, 1	55	9·7429511, 4	85	9·9950985, 2
26	8·9325651, 8	56	9·7584384, 1	86	9·9968645, 4
27	8·9780574, 8	57	9·7733501, 7	87	9·9982369, 6
28	9·0216198, 5	58	9·7876998, 0	88	9·9992166, 3
29	9·0633774, 9	59	9·8014998, 7	89	9·9998041, 9
30	9·1034428, 4	60	9·8147622, 3	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^4_4 = LOG ($\sin^4 \theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^4_4	θ	Log H'^4_4	θ	Log H'^4_4
1°	2·9790511, 4	31°	8·8558896, 3	61°	9·7699973, 9
2	4·1828958, 3	32	8·9051896, 7	62	9·7862904, 1
3	4·8868020, 1	33	8·9526013, 7	63	9·8019085, 6
4	5·3859145, 2	34	8·9982255, 3	64	9·8168642, 8
5	5·7727285, 1	35	9·0421541, 4	65	9·8311692, 5
6	6·0884437, 5	36	9·0844712, 8	66	9·8448345, 0
7	6·3550374, 0	37	9·1252541, 0	67	9·8578703, 7
8	6·5856278, 4	38	9·1645734, 7	68	9·8702866, 0
9	6·7886766, 5	39	9·2024946, 5	69	9·8820923, 1
10	6·9699612, 8	40	9·2390778, 6	70	9·8932961, 0
11	7·1336025, 2	41	9·2743788, 1	71	9·9039059, 9
12	7·2826429, 6	42	9·3084490, 4	72	9·9139294, 9
13	7·4193931, 3	43	9·3413364, 1	73	9·9233736, 4
14	7·5456489, 7	44	9·3730853, 6	74	9·9322449, 6
15	7·6628341, 8	45	9·4037372, 5	75	9·9405495, 5
16	7·7720963, 9	46	9·4333306, 3	76	9·9482930, 6
17	7·8743742, 4	47	9·4619014, 4	77	9·9554806, 9
18	7·9704452, 5	48	9·4894832, 8	78	9·9621172, 5
19	8·0609606, 2	49	9·5161075, 5	79	9·9682071, 4
20	8·1464712, 5	50	9·5418036, 5	80	9·9737543, 6
21	8·2274472, 8	51	9·5665991, 2	81	9·9787625, 5
22	8·3042931, 6	52	9·5905197, 9	82	9·9832349, 6
23	8·3773592, 9	53	9·6135899, 1	83	9·9871744, 9
24	8·4469512, 8	54	9·6358322, 4	84	9·9905836, 7
25	8·5133372, 1	55	9·6572681, 9	85	9·9934646, 9
26	8·5767535, 7	56	9·6779178, 8	86	9·9958193, 9
27	8·6374099, 8	57	9·6978002, 3	87	9·9976492, 8
28	8·6954931, 3	58	9·7169330, 7	88	9·9989555, 1
29	8·7511699, 9	59	9·7353331, 6	89	9·9997389, 2
30	8·8045904, 6	60	9·7530163, 1	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^{5}_{5} = LOG ($\sin^5 \theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^{5}_{5}	θ	Log H'^{5}_{5}	θ	Log H'^{5}_{5}
1°	1·2238139, 3	31°	8·5698620, 4	61°	9·7124967, 4
2	2·7286197, 9	32	8·6314870, 8	62	9·7328630, 1
3	3·6085025, 2	33	8·6907517, 1	63	9·7523857, 0
4	4·2323931, 6	34	8·7477819, 2	64	9·7710803, 5
5	4·7159106, 4	35	8·8026926, 7	65	9·7889615, 7
6	5·1105546, 8	36	8·8555891, 1	66	9·8060431, 3
7	5·4437967, 5	37	8·9065676, 3	67	9·8223379, 6
8	5·7320348, 0	38	8·9557168, 4	68	9·8378582, 4
9	5·9858458, 1	39	9·0031183, 1	69	9·8526153, 9
10	6·2124516, 0	40	9·0488473, 3	70	9·8666201, 2
11	6·4170031, 5	41	9·0929735, 1	71	9·8798824, 8
12	6·6033037, 0	42	9·1355613, 0	72	9·8924118, 7
13	6·7742414, 1	43	9·1766705, 1	73	9·9042170, 4
14	6·9320612, 2	44	9·2163567, 0	74	9·9153062, 0
15	7·0785427, 2	45	9·2546715, 6	75	9·9256869, 4
16	7·2151204, 8	46	9·2916632, 9	76	9·9353663, 2
17	7·3429678, 1	47	9·3273768, 1	77	9·9443508, 6
18	7·4630565, 6	48	9·3618541, 1	78	9·9526465, 6
19	7·5762007, 7	49	9·3951344, 4	79	9·9602589, 2
20	7·6830890, 6	50	9·4272545, 6	80	9·9671929, 5
21	7·7843091, 0	51	9·4582488, 9	81	9·9734531, 8
22	7·8803664, 5	52	9·4881497, 3	82	9·9790437, 0
23	7·9716991, 1	53	9·5169873, 8	83	9·9839681, 1
24	8·0586890, 9	54	9·5447903, 0	84	9·9882295, 8
25	8·1416715, 1	55	9·5715852, 4	85	9·9918308, 6
26	8·2209419, 6	56	9·5973973, 5	86	9·9947742, 4
27	8·2967624, 7	57	9·6222502, 9	87	9·9970616, 0
28	8·3693664, 1	58	9·6461663, 4	88	9·9986943, 9
29	8·4389624, 8	59	9·6691664, 5	89	9·9996736, 5
30	8·5057380, 7	60	9·6912703, 9	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^{6}_{6} = LOG ($\sin^6 \theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^{6}_{6}	θ	Log H'^{6}_{6}	θ	Log H'^{6}_{6}
1°	89·4685767, 2	31°	8·2838344, 5	61°	9·6549960, 8
2	1·2743437, 4	32	8·3577845, 0	62	9·6794356, 1
3	2·3302030, 2	33	8·4289020, 5	63	9·7028628, 4
4	3·0788717, 9	34	8·4973383, 0	64	9·7252964, 1
5	3·6590927, 7	35	8·5632312, 0	65	9·7467538, 8
6	4·1326656, 2	36	8·6267069, 3	66	9·7672517, 5
7	4·5325561, 0	37	8·6878811, 6	67	9·7868055, 6
8	4·8784417, 6	38	8·7468602, 0	68	9·8054298, 9
9	5·1830149, 7	39	8·8037419, 7	69	9·8231384, 7
10	5·4549419, 2	40	8·8586168, 0	70	9·8399441, 5
11	5·7004037, 8	41	8·9115682, 1	71	9·8558589, 8
12	5·9239644, 4	42	8·9626735, 6	72	9·8708942, 4
13	6·1290896, 9	43	9·0120046, 1	73	9·8850604, 5
14	6·3184734, 6	44	9·0596280, 4	74	9·8983674, 4
15	6·4942512, 7	45	9·1056058, 8	75	9·9108243, 3
16	6·6581445, 8	46	9·1499959, 4	76	9·9224395, 9
17	6·8115613, 7	47	9·1928521, 7	77	9·9332210, 4
18	6·9556678, 7	48	9·2342249, 3	78	9·9431758, 7
19	7·0914409, 3	49	9·2741613, 3	79	9·9523107, 0
20	7·2197068, 7	50	9·3127054, 7	80	9·9606315, 4
21	7·3411709, 2	51	9·3498986, 7	81	9·9681438, 2
22	7·4564397, 3	52	9·3857796, 8	82	9·9748524, 4
23	7·5660389, 3	53	9·4203848, 6	83	9·9807617, 3
24	7·6704269, 1	54	9·4537483, 6	84	9·9858755, 0
25	7·7700058, 1	55	9·4859022, 8	85	9·9901970, 3
26	7·8651303, 5	56	9·5168768, 1	86	9·9937290, 9
27	7·9561149, 7	57	9·5467003, 5	87	9·9964739, 2
28	8·0432397, 0	58	9·5753996, 0	88	9·9984332, 7
29	8·1267549, 8	59	9·6029997, 5	89	9·9996083, 8
30	8·2068856, 8	60	9·6295244, 7	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^{7}_{7} = LOG ($\sin^7 \theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^{7}_{7}	θ	Log H'^{7}_{7}	θ	Log H'^{7}_{7}
1°	87·7133395, 0	31°	7·9978068, 6	61°	9·5974954, 3
2	89·8200677, 0	32	8·0840819, 2	62	9·6260082, 1
3	1·0519035, 3	33	8·1670523, 9	63	9·6533399, 8
4	1·9253504, 2	34	8·2468946, 8	64	9·6795124, 8
5	2·6022749, 0	35	8·3237697, 4	65	9·7045461, 9
6	3·1547765, 5	36	8·3978247, 5	66	9·7284603, 8
7	3·6213154, 5	37	8·4691946, 8	67	9·7512731, 5
8	4·0248487, 2	38	8·5380035, 7	68	9·7730015, 4
9	4·3801841, 3	39	8 6043656, 3	69	9·7936615, 5
10	4·6974322, 4	40	8·6683862, 6	70	9·8132681, 7
11	4·9838044, 1	41	8·7301629, 1	71	9·8318354, 8
12	5·2446251, 8	42	8·7897858, 2	72	9·8493766, 1
13	5·4839379, 7	43	8·8473387, 1	73	9·8659038, 6
14	5·7048857, 1	44	8·9028993, 7	74	9·8814286, 8
15	5·9099598, 1	45	8·9565401, 9	75	9·8959617, 2
16	6·1011686, 8	46	9·0083286, 0	76	9·9095128, 5
17	6·2801549, 3	47	9·0583275, 3	77	9·9220912, 1
18	6·4482791, 8	48	9·1065957, 5	78	9·9337051, 9
19	6·6066810, 8	49	9·1531882, 2	79	9·9443624, 9
20	6·7563246, 8	50	9·1981563, 8	80	9·9540701, 3
21	6·8980327, 4	51	9·2415484, 5	81	9·9628344, 6
22	7·0325130, 2	52	9·2834096, 3	82	9·9706611, 8
23	7·1603787, 6	53	9·3237823, 3	83	9·9775553, 5
24	7·2821647, 3	54	9·3627064, 2	84	9·9835214, 2
25	7·3983401, 2	55	9·4002193, 3	85	9·9885632, 0
26	7·5093187, 4	56	9·4363562, 8	86	9·9926839, 4
27	7·6154674, 6	57	9·4711504, 0	87	9·9958862, 4
28	7·7171129, 8	58	9·5046328, 7	88	9·9981721, 4
29	7·8145474, 8	59	9·5368330, 4	89	9·9995431, 1
30	7·9080333, 0	60	9·5677785, 5	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^{8}_{8} = LOG ($\sin^8 \theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^{8}_{8}	θ	Log II'^{8}_{8}	θ	Log H'^{8}_{8}
1°	85·9581022, 9	31°	7·7117792, 6	61°	9·5399947, 8
2	88·3657916, 6	32	7·8103793, 4	62	9·5725808, 1
3	89·7736040, 3	33	7·9052027, 3	63	9·6038171, 2
4	0·7718290, 5	34	7·9964510, 7	64	9·6337285, 5
5	1·5454570, 2	35	8·0843082, 7	65	9·6623385, 1
6	2·1768874, 9	36	8·1689425, 7	66	9·6896690, 0
7	2·7100747, 9	37	8·2505082, 1	67	9·7157407, 4
8	3·1712556, 8	38	8·3291469, 4	68	9·7405731, 9
9	3·5773532, 9	39	8·4049892, 9	69	9·7641846, 3
10	3·9399225, 6	40	8·4781557, 3	70	9·7865922, 0
11	4·2672050, 4	41	8·5487576, 1	71	9·8078119, 7
12	4·5652859, 2	42	8·6168980, 8	72	9·8278589, 8
13	4·8387862, 5	43	8·6826728, 1	73	9·8467472, 7
14	5·0912979, 5	44	8·7461707, 1	74	9·8644899, 2
15	5·3256683, 6	45	8·8074745, 0	75	9·8810991, 1
16	5·5441927, 7	46	8·8666612, 6	76	9·8965861, 2
17	5·7487484, 9	47	8·9238028, 9	77	9·9109613, 8
18	5·9408904, 9	48	8·9789665, 7	78	9·9242345, 0
19	6·1219212, 4	49	9·0322151, 0	79	9·9364142, 7
20	6·2929425, 0	50	9·0836073, 0	80	9·9475087, 2
21	6·4548945, 6	51	9·1331982, 3	81	9·9575250, 9
22	6·6085863, 1	52	9·1810395, 7	82	9·9664699, 2
23	6·7547185, 8	53	9·2271798, 1	83	9·9743489, 7
24	6·8939025, 5	54	9·2716644, 8	84	9·9811673, 3
25	7·0266744, 2	55	9·3145363, 8	85	9·9869293, 8
26	7·1535071, 4	56	9·3558357, 5	86	9·9916387, 8
27	7·2748199, 6	57	9·3956004, 6	87	9·9952985, 6
28	7·3909862, 6	58	9·4338661, 4	88	9·9979110, 2
29	7·5023399, 7	59	9·4706663, 3	89	9·9994778, 4
30	7·6091809, 1	60	9·5060326, 2	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^{9}_{9} = LOG ($\sin^{9}\theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^{9}_{9}	θ	Log H'^{9}_{9}	θ	Log H'^{9}_{9}
1°	84·2028650, 7	31°	7·4257516, 7	61°	9·4824941, 2
2	86·9115156, 2	32	7·5366767, 5	62	9·5191534, 1
3	88·4953045, 3	33	7·6433530, 7	63	9·5542942, 6
4	89·6183076, 8	34	7·7460074, 5	64	9·5879446, 2
5	0·4886391, 5	35	7·8448468, 0	65	9·6201308, 2
6	1·1989984, 3	36	7·9400603, 9	66	9·6508776, 3
7	1·7988341, 4	37	8·0318217, 3	67	9·6802083, 3
8	2·3176626, 4	38	8·1202903, 1	68	9·7081448, 4
9	2·7745224, 5	39	8·2056129, 5	69	9·7347077, 1
10	3·1824128, 8	40	8·2879252, 0	70	9·7599162, 2
11	3·5506056, 8	41	8·3673523, 1	71	9·7837884, 7
12	3·8859466, 6	42	8·4440103, 4	72	9·8063413, 6
13	4·1936345, 3	43	8·5180069, 2	73	9·8275906, 8
14	4·4777101, 9	44	8·5894420, 5	74	9·8475511, 6
15	4·7413769, 0	45	8·6584088, 2	75	9·8662365, 0
16	4·9872168, 7	46	8·7249939, 1	76	9·8836593, 8
17	5·2173420, 5	47	8·7892782, 5	77	9·8998315, 5
18	5·4335018, 1	48	8·8513373, 9	78	9·9147638, 1
19	5·6371613, 9	49	8·9112419, 9	79	9·9284660, 5
20	5·8295603, 1	50	8·9690582, 1	80	9·9409473, 1
21	6·0117563, 8	51	9·0248480, 1	81	9·9522157, 3
22	6·1846596, 0	52	9·0786695, 2	82	9·9622786, 6
23	6·3490584, 0	53	9·1305772, 9	83	9·9711425, 9
24	6·5056403, 7	54	9·1806225, 4	84	9·9788132, 5
25	6·6550087, 2	55	9·2288534, 3	85	9·9852955, 5
26	6·7976955, 3	56	9·2753152, 2	86	9·9905936, 3
27	6·9341724, 5	57	9·3200505, 2	87	9·9947108, 8
28	7·0648595, 5	58	9·3630994, 0	88	9·9976499, 0
29	7·1901324, 7	59	9·4044996, 2	89	9·9994125, 7
30	7·3103285, 3	60	9·4442867, 0	90	0·0000000, 0

TABLE OF THE VALUES OF LOG H'^{10}_{10} = LOG ($\sin^{10}\theta'$), θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE.

θ	Log H'^{10}_{10}	θ	Log H'^{10}_{10}	θ	Log H'^{10}_{10}
1°	82·4476278, 6	31°	7·1397240, 8	61°	9·4249934, 7
2	85·4572395, 7	32	7·2629741, 7	62	9·4657260, 1
3	87·2170050, 4	33	7·3815034, 2	63	9·5047714, 0
4	88·4647863, 1	34	7·4955638, 4	64	9·5421606, 9
5	89·4318212, 8	35	7·6053853, 4	65	9·5779231, 3
6	0·2211093, 6	36	7·7111782, 1	66	9·6120862, 5
7	0·8875934, 9	37	7·8131352, 6	67	9·6446759, 3
8	1·4640696, 0	38	7·9114336, 7	68	9·6757164, 9
9	1·9716916, 1	39	8·0062366, 2	69	9·7052307, 8
10	2·4249032, 0	40	8·0976946, 6	70	9·7332402, 5
11	2·8340063, 1	41	8·1859470, 2	71	9·7597649, 7
12	3·2066074, 1	42	8·2711226, 0	72	9·7848237, 3
13	3·5484828, 1	43	8·3533410, 2	73	9·8084340, 9
14	3·8641224, 4	44	8·4327133, 9	74	9·8306124, 0
15	4·1570854, 5	45	8·5093431, 3	75	9·8513738, 9
16	4·4302409, 7	46	8·5833265, 7	76	9·8707326, 5
17	4·6859356, 1	47	8·6547536, 1	77	9·8887017, 3
18	4·9261131, 2	48	8·7237082, 1	78	9·9052931, 2
19	5·1524015, 5	49	8·7902688, 8	79	9·9205178, 4
20	5·3661781, 2	50	8·8545091, 2	80	9·9343859, 0
21	5·5686182, 0	51	8·9164977, 9	81	9·9469063, 7
22	5·7607328, 9	52	8·9762994, 7	82	9·9580874, 0
23	5·9433982, 2	53	9·0339747, 6	83	9·9679362, 1
24	6·1173781, 9	54	9·0895806, 0	84	9·9764591, 7
25	6·2833430, 2	55	9·1431704, 7	85	9·9836617, 2
26	6·4418839, 2	56	9·1947946, 9	86	9·9895484, 8
27	6·5935249, 5	57	9·2445005, 8	87	9·9941232, 0
28	6·7387328, 3	58	9·2923326, 7	88	9·9973887, 8
29	6·8779249, 7	59	9·3383329, 1	89	9·9993473, 0
30	7·0114761, 4	60	9·3825407, 8	90	0·0000000, 0

TABLE OF MULTIPLES OF LOG $\frac{a}{r}$ FOR EVERY DEGREE OF GEOGRAPHICAL COLATITUDE θ.

	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°
1	·0014541,8	·0014537	·0014524	·0014502	·0014470	·0014430,2	·0014381	·0014324	·0014257	·0014182
2	·0029083,6	·0029074	·0029048	·0029004	·0028940	·0028860,4	·0028762	·0028648	·0028514	·0028364
3	·0043625,4	·0043611	·0043572	·0043506	·0043410	·0043290,6	·0043143	·0042972	·0042771	·0042546
4	·0058167,2	·0058148	·0058096	·0058008	·0057880	·0057720,8	·0057524	·0057296	·0057028	·0056728
5	·0072709,0	·0072685	·0072620	·0072510	·0072350	·0072151,0	·0071905	·0071620	·0071285	·0070910
6	·0087250,8	·0087222	·0087144	·0087012	·0086820	·0086581,2	·0086286	·0085944	·0085542	·0085092
7	·0101792,6	·0101759	·0101668	·0101514	·0101290	·0101011,4	·0100667	·0100268	·0099799	·0099274
8	·0116334,4	·0116296	·0116192	·0116016	·0115760	·0115441,6	·0115048	·0114592	·0114056	·0113456
9	·0130876,2	·0130833	·0130716	·0130518	·0130230	·0129871,8	·0129429	·0128916	·0128313	·0127638
10	·0145418,0	·0145370	·0145240	·0145020	·0144700	·0144302,0	·0143810	·0143240	·0142570	·0141820
11	·0159959,8	·0159907	·0159764	·0159522	·0159170	·0158732,2	·0158191	·0157564	·0156827	·0156002
12	·0174501,6	·0174444	·0174288	·0174024	·0173640	·0173162,4	·0172572	·0171888	·0171084	·0170184

	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°
1	·0014099,0	·0014007	·0013907	·0013799	·0013683	·0013558,5	·0013427	·0013287	·0013141	·0012987
2	·0028198,0	·0028014	·0027814	·0027598	·0027366	·0027117,0	·0026854	·0026574	·0026282	·0025974
3	·0042297,0	·0042021	·0041721	·0041397	·0041049	·0040675,5	·0040281	·0039861	·0039423	·0038961
4	·0056396,0	·0056028	·0055628	·0055196	·0054732	·0054234,0	·0053708	·0053148	·0052564	·0051948
5	·0070495,0	·0070035	·0069535	·0068995	·0068415	·0067792,5	·0067135	·0066435	·0065705	·0064935
6	·0084594,0	·0084042	·0083442	·0082794	·0082098	·0081351,0	·0080562	·0079722	·0078846	·0077922
7	·0098693,0	·0098049	·0097349	·0096593	·0095781	·0094909,5	·0093989	·0093009	·0091987	·0090909
8	·0112792,0	·0112056	·0111256	·0110392	·0109464	·0108468,0	·0107416	·0106296	·0105128	·0103896
9	·0126891,0	·0126063	·0125163	·0124191	·0123147	·0122026,5	·0120843	·0119583	·0118269	·0116883
10	·0140990,0	·0140070	·0139070	·0137990	·0136830	·0135585,0	·0134270	·0132870	·0131410	·0129870
11	·0155089,0	·0154077	·0152977	·0151789	·0150513	·0149143,5	·0147697	·0146157	·0144551	·0142857
12	·0169188,0	·0168084	·0166884	·0165588	·0164196	·0162702,0	·0161124	·0159444	·0157692	·0155844

	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°
1	·0012825,6	·0012658	·0012483	·0012303	·0012116	·0011923,0	·0011725	·0011521	·0011312	·0011098
2	·0025651,2	·0025316	·0024966	·0024606	·0024232	·0023846,0	·0023450	·0023042	·0022624	·0022196
3	·0038476,8	·0037974	·0037449	·0036909	·0036348	·0035769,0	·0035175	·0034563	·0033936	·0033294
4	·0051302,4	·0050632	·0049932	·0049212	·0048464	·0047692,0	·0046900	·0046084	·0045248	·0044392
5	·0064128,0	·0063290	·0062415	·0061515	·0060580	·0059615,0	·0058625	·0057605	·0056560	·0055490
6	·0076953,6	·0075948	·0074898	·0073818	·0072696	·0071538,0	·0070350	·0069126	·0067872	·0066588
7	·0089779,2	·0088606	·0087381	·0086121	·0084812	·0083461,0	·0082075	·0080647	·0079184	·0077686
8	·0102604,8	·0101264	·0099864	·0098424	·0096928	·0095384,0	·0093800	·0092168	·0090496	·0088784
9	·0115430,4	·0113922	·0112347	·0110727	·0109044	·0107307,0	·0105525	·0103689	·0101808	·0099882
10	·0128256,0	·0126580	·0124830	·0123030	·0121160	·0119230,0	·0117250	·0115210	·0113120	·0110980
11	·0141081,6	·0139238	·0137313	·0135333	·0133276	·0131153,0	·0128975	·0126731	·0124432	·0122078
12	·0153907,2	·0151896	·0149796	·0147636	·0145392	·0143076,0	·0140700	·0138252	·0135744	·0133176

	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°
1	·0010878,9	·0010656	·0010429	·0010198	·0009963	·0009725,4	·0009485	·0009241	·0008995	·0008748
2	·0021757,8	·0021312	·0020858	·0020396	·0019926	·0019450,8	·0018970	·0018482	·0017990	·0017496
3	·0032636,7	·0031968	·0031287	·0030594	·0029889	·0029176,2	·0028455	·0027723	·0026985	·0026244
4	·0043515,6	·0042624	·0041716	·0040792	·0039852	·0038901,6	·0037940	·0036964	·0035980	·0034992
5	·0054394,5	·0053280	·0052145	·0050990	·0049815	·0048627,0	·0047425	·0046205	·0044975	·0043740
6	·0065273,4	·0063936	·0062574	·0061188	·0059778	·0058352,4	·0056910	·0055446	·0053970	·0052488
7	·0076152,3	·0074592	·0073003	·0071386	·0069741	·0068077,8	·0066395	·0064687	·0062965	·0061236
8	·0087031,2	·0085248	·0083432	·0081584	·0079704	·0077803,2	·0075880	·0073928	·0071960	·0069984
9	·0097910,1	·0095904	·0093861	·0091782	·0089667	·0087528,6	·0085365	·0083169	·0080955	·0078732
10	·0108789,0	·0106560	·0104290	·0101980	·0099630	·0097254,0	·0094850	·0092410	·0089950	·0087480
11	·0119667,9	·0117216	·0114719	·0112178	·0109593	·0106979,4	·0104335	·0101651	·0098945	·0096228
12	·0130546,8	·0127872	·0125148	·0122376	·0119556	·0116704,8	·0113820	·0110892	·0107940	·0104976

	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°
1	·0008498,0	·0008247	·0007995	·0007742	·0007488	·0007234,4	·0006981	·0006727	·0006475	·0006223
2	·0016996,0	·0016494	·0015990	·0015484	·0014976	·0014468,8	·0013962	·0013454	·0012950	·0012446
3	·0025494,0	·0024741	·0023985	·0023226	·0022464	·0021703,2	·0020943	·0020181	·0019425	·0018669
4	·0033992,0	·0032988	·0031980	·0030968	·0029952	·0028937,6	·0027924	·0026908	·0025900	·0024892
5	·0042490,0	·0041235	·0039975	·0038710	·0037440	·0036172,0	·0034905	·0033635	·0032375	·0031115
6	·0050988,0	·0049482	·0047970	·0046452	·0044928	·0043406,4	·0041886	·0040362	·0038850	·0037338
7	·0059486,0	·0057729	·0055965	·0054194	·0052416	·0050640,8	·0048867	·0047089	·0045325	·0043561
8	·0067984,0	·0065976	·0063960	·0061936	·0059904	·0057875,2	·0055848	·0053816	·0051800	·0049784
9	·0076482,0	·0074223	·0071955	·0069678	·0067392	·0065109,6	·0062829	·0060543	·0058275	·0056007
10	·0084980,0	·0082470	·0079950	·0077420	·0074880	·0072344,0	·0069810	·0067270	·0064750	·0062230
11	·0093478,0	·0090717	·0087945	·0085162	·0082368	·0079578,4	·0076791	·0073997	·0071225	·0068453
12	·0101976,0	·0098964	·0095940	·0092904	·0089856	·0086812,8	·0083772	·0080724	·0077700	·0074676

	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°
1	·0005972,9	·0005724	·0005478	·0005233	·0004991	·0004751,9	·0004516	·0004283	·0004054	·0003829
2	·0011945,8	·0011448	·0010956	·0010466	·0009982	·0009503,8	·0009032	·0008566	·0008108	·0007658
3	·0017918,7	·0017172	·0016434	·0015699	·0014973	·0014255,7	·0013548	·0012849	·0012162	·0011487
4	·0023891,6	·0022896	·0021912	·0020932	·0019964	·0019007,6	·0018064	·0017132	·0016216	·0015316
5	·0029864,5	·0028620	·0027390	·0026165	·0024955	·0023759,5	·0022580	·0021415	·0020270	·0019145
6	·0035837,4	·0034344	·0032868	·0031398	·0029946	·0028511,4	·0027096	·0025698	·0024324	·0022974
7	·0041810,3	·0040068	·0038346	·0036631	·0034937	·0033263,3	·0031612	·0029981	·0028378	·0026803
8	·0047783,2	·0045792	·0043824	·0041864	·0039928	·0038015,2	·0036128	·0034264	·0032432	·0030632
9	·0053756,1	·0051516	·0049302	·0047097	·0044919	·0042767,1	·0040644	·0038547	·0036486	·0034461
10	·0059729,0	·0057240	·0054780	·0052330	·0049910	·0047519,0	·0045160	·0042830	·0040540	·0038290
11	·0065701,9	·0062964	·0060258	·0057563	·0054901	·0052270,9	·0049676	·0047113	·0044594	·0042119
12	·0071674,8	·0068688	·0065736	·0062796	·0059892	·0057022,8	·0054192	·0051396	·0048648	·0045948

TABLE OF MULTIPLES OF $\text{LOG}\,\frac{a}{r}$ FOR EVERY DEGREE OF GEOGRAPHICAL COLATITUDE θ, *continued.*

	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°
1	·0003608,1	·0003392	·0003180	·0002973	·0002772	·0002575,9	·0002386	·0002201	·0002023	·0001851
2	·0007216,2	·0006784	·0006360	·0005946	·0005544	·0005151,8	·0004772	·0004402	·0004046	·0003702
3	·0010824,3	·0010176	·0009540	·0008919	·0008316	·0007727,7	·0007158	·0006603	·0006069	·0005553
4	·0014432,4	·0013568	·0012720	·0011892	·0011088	·0010303,6	·0009544	·0008804	·0008092	·0007404
5	·0018040,5	·0016960	·0015900	·0014865	·0013860	·0012879,5	·0011930	·0011005	·0010115	·0009255
6	·0021648,6	·0020352	·0019080	·0017838	·0016632	·0015455,4	·0014316	·0013206	·0012138	·0011106
7	·0025256,7	·0023744	·0022260	·0020811	·0019404	·0018031,3	·0016702	·0015407	·0014161	·0012957
8	·0028864,8	·0027136	·0025440	·0023784	·0022176	·0020607,2	·0019088	·0017608	·0016184	·0014808
9	·0032472,9	·0030528	·0028620	·0026757	·0024948	·0023183,1	·0021474	·0019809	·0018207	·0016659
10	·0036081,0	·0033920	·0031800	·0029730	·0027720	·0025759,0	·0023860	·0022010	·0020230	·0018510
11	·0039689,1	·0037312	·0034980	·0032703	·0030492	·0028334,9	·0026246	·0024211	·0022253	·0020361
12	·0043297,2	·0040704	·0038160	·0035676	·0033264	·0030910,8	·0028632	·0026412	·0024276	·0022212

	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°
1	·0001686,0	·0001528	·0001376	·0001232	·0001095	·0000965,0	·0000843	·0000729	·0000623	·0000524
2	·0003372,0	·0003056	·0002752	·0002464	·0002190	·0001930,0	·0001686	·0001458	·0001246	·0001048
3	·0005058,0	·0004584	·0004128	·0003696	·0003285	·0002895,0	·0002529	·0002187	·0001869	·0001572
4	·0006744,0	·0006112	·0005504	·0004928	·0004380	·0003860,0	·0003372	·0002916	·0002492	·0002096
5	·0008430,0	·0007640	·0006880	·0006160	·0005475	·0004825,0	·0004215	·0003645	·0003115	·0002620
6	·0010116,0	·0009168	·0008256	·0007392	·0006570	·0005790,0	·0005058	·0004374	·0003738	·0003144
7	·0011802,0	·0010696	·0009632	·0008624	·0007665	·0006755,0	·0005901	·0005103	·0004361	·0003668
8	·0013488,0	·0012224	·0011008	·0009856	·0008760	·0007720,0	·0006744	·0005832	·0004984	·0004192
9	·0015174,0	·0013752	·0012384	·0011088	·0009855	·0008685,0	·0007587	·0006561	·0005607	·0004716
10	·0016860,0	·0015280	·0013760	·0012320	·0010950	·0009650,0	·0008430	·0007290	·0006230	·0005240
11	·0018546,0	·0016808	·0015136	·0013552	·0012045	·0010615,0	·0009273	·0008019	·0006853	·0005764
12	·0020232,0	·0018336	·0016512	·0014784	·0013140	·0011580,0	·0010116	·0008748	·0007476	·0006288

	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	90°
1	·0000434,2	·0000352	·0000279	·0000214	·0000157	·0000109,4	·0000070	·0000039	·0000018	·0000004	·000000
2	·0000868,4	·0000704	·0000558	·0000428	·0000314	·0000218,8	·0000140	·0000078	·0000036	·0000008	·000000
3	·0001302,6	·0001056	·0000837	0000642	·0000471	·0000328,2	·0000210	·0000117	·0000054	·0000012	·000000
4	·0001736,8	·0001408	·0001116	·0000856	·0000628	·0000437,6	·0000280	·0000156	·0000072	·0000016	·000000
5	·0002171,0	·0001760	·0001395	·0001070	·0000785	·0000547,0	·0000350	·0000195	·0000090	·0000020	·000000
6	·0002605,2	·0002112	·0001674	·0001284	·0000942	·0000656,4	·0000420	·0000234	·0000108	·0000024	·000000
7	·0003039,4	·0002464	·0001953	·0001498	·0001099	·0000765,8	·0000490	·0000273	·0000126	·0000028	·000000
8	·0003473,6	·0002816	·0002232	·0001712	·0001256	·0000875,2	·0000560	·0000312	·0000144	·0000032	·000000
9	·0003907,8	·0003168	·0002511	·0001926	·0001413	·0000984,6	·0000630	·0000351	·0000162	·0000036	·000000
10	·0004342,0	·0003520	·0002790	·0002140	·0001570	·0001094,0	·0000700	·0000390	·0000180	·0000040	·000000
11	·0004776,2	·0003872	·0003069	·0002354	·0001727	·0001203,4	·0000770	·0000429	·0000198	·0000044	·000000
12	·0005210,4	·0004224	·0003348	·0002568	·0001884	·0001312,8	·0000840	·0000468	·0000216	·0000048	000000

TABLES OF THE VALUES OF LOG G'^{m}_{n} FOR THE EARTH'S SURFACE, REGARDED AS A SPHEROID, μ' BEING THE COSINE OF THE GEOCENTRIC COLATITUDE CORRESPONDING TO EACH DEGREE OF THE GEOGRAPHICAL COLATITUDE.

TABLE OF LOG $G'^0_0, \ldots G'^0_{10}$, FOR VALUES OF θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE, $\mu' = \cos\theta'$.

θ	G'^0_0 Log	G'^0_1 Log	Log G'^0_2	Log G'^0_3	Log G'^0_4	Log G'^0_5	Log G'^0_6	Log G'^0_7	Log G'^0_8	Log G'^0_9	Log G'^0_{10}
0°	0	$L\mu'$	9·8239087, 4	9·6020599, 9	9·3590219, 4	9·1037494, 4	8·8405080, 0	8·5716626, 9	8·2986614, 2	8·0224550, 1	7·7437014, 1
1			9·8237076, 0	9·6016576, 6	9·3583512, 9	9·1027432, 7	8·8390990, 8	8·5697836, 6	8·2962447, 4	8·0194332, 7	7·7400072, 1
2			9·8231038, 7	9·6004494, 9	9·3563361, 1	9·0997175, 8	8·8348580, 9	8·5641212, 6	8·2889531, 0	8·0103025, 1	7·7288251, 6
3			9·8220966, 5	9·5984320, 2	9·3529668, 6	9·0946508, 6	8·8277429, 8	8·5546003, 5	8·2766614, 4	7·9948659, 5	7·7098590, 9
4			9·8206843, 6	9·5955991, 9	9·3482269, 6	9·0875058, 9	8·8176801, 8	8·5410885, 1	8·2591464, 6	7·9727673, 3	7·6825629, 2
5			9·8188649, 4	9·5919428, 1	9·3420934, 2	9·0782299, 3	8·8045635, 6	8·5233908, 2	8·2360747, 6	7·9434628, 6	7·6460859, 8
6			9·8166354, 9	9·5874517, 0	9·3345350, 3	9·0667508, 9	8·7882463, 8	8·5012337, 7	8·2069661, 2	7·9061469, 5	7·5991207, 6
7			9·8139926, 1	9·5821122, 7	9·3255128, 5	9·0529768, 8	8·7685365, 8	8·4742470, 1	8·1711470, 1	7·8596443, 8	7·5396694, 0
8			9·8109321, 2	9·5759078, 3	9·3149785, 0	9·0367912, 0	8·7451832, 7	8·4419309, 8	8·1276696, 6	7·8022091, 1	7·4645581, 0
9			9·8074492, 2	9·5688186, 3	9·3028733, 5	9·0180483, 2	8·7178624, 9	8·4036101, 6	8·0751767, 5	7·7311510, 3	7·3684226, 4
10			9·8035382, 4	9·5608213, 9	9·2891267, 2	8·9965671, 3	8·6861531, 0	8·3583556, 0	8·0116583, 9	7·6420755, 6	7·2411952, 2
11			9·7991928, 5	9·5518891, 0	9·2736540, 3	8·9721220, 8	8·6495025, 1	8·3048593, 9	7·9339956, 9	7·5271270, 1	7·0604258, 6
12			9·7944058, 1	9·5419905, 3	9·2563543, 9	8·9444313, 9	8·6071743, 1	8·2412131, 6	7·8369877, 1	7·3700798, 7	6·7557376, 9
13			9·7891690, 3	9·5310896, 5	9·2371068, 1	8·9131388, 5	8·5581633, 2	8·1644905, 5	7·7110019, 8	7·1275971, 5	5·2455126, 7
14			9·7834734, 5	9·5191450, 5	9·2157662, 8	8·8777890, 8	8·5010541, 9	8·0698829, 7	7·5348963, 7	6·5749452, 2	6·6945086, 8 n
15			9·7773089, 2	9·5061091, 2	9·1921577, 8	8·8377902, 0	8·4337735, 1	7·9486545, 0	7·2445121, 5	6·7280289, 5 n	6·9806849, 7 n
16			9·7706643, 4	9·4919272, 6	9·1660687, 4	8·7923561, 8	8·3531120, 7	7·7822231, 5	6·1845778, 7	7·1377053, 3 n	7·1327718, 3 n
17			9·7635272, 8	9·4765365, 6	9·1372386, 0	8·7404149, 6	8·2537147, 0	7·5177446, 9	7·1330500, 6 n	7·3273855, 0 n	7·2279971, 9 n
18			9·7558840, 2	9·4598642, 7	9·1053436, 2	8·6804523, 7	8·1256094, 2	6·7994715, 1	7·4403816, 3 n	7·4437650, 7 n	7·2898653, 1 n
19			9·7477194, 9	9·4418264, 4	9·0699769, 9	8·6102353, 5	7·9466669, 1	7·2798357, 0 n	7·6044539, 4 n	7·5214283, 8 n	7·3284489, 3 n
20			9·7390168, 9	9·4223250, 3	9·0306174, 1	8·5262703, 8	7·6480417, 4	7·6316588, 4 n	7·7108025, 5 n	7·5739965, 2 n	7·3487506, 1 n
21			9·7297578, 4	9·4012455, 7	8·9865839, 2	8·4226343, 2	6·2996162, 4	7·8113061, 7 n	7·7846686, 4 n	7·6080883, 1 n	7·3533102, 2 n
22			9·7199218, 3	9·3784530, 8	8·9369634, 9	8·2880212, 9	7·5867369, 2 n	7·9275330, 6 n	7·8369074, 2 n	7·6272552, 2 n	7·3431977, 9 n
23			9·7094863, 4	9·3537875, 3	8·8804937, 0	8·0962347, 7	7·8865561, 4 n	8·0096051, 2 n	7·8730920, 6 n	7·6334240, 1 n	7·3183555, 0 n
24			9·6984263, 5	9·3270573, 0	8·8153588, 2	7·7572813, 6	8·0526381, 7 n	8·0696954, 7 n	7·8963691, 2 n	7·6275112, 8 n	7·2776183, 8 n
25			9·6867140, 8	9·2980304, 4	8·7388103, 5	6·9215148, 0 n	8·1642008, 2 n	8·1139911, 3 n	7·9085761, 6 n	7·6096701, 7 n	7·2183201, 5 n
26			9·6743187, 5	9·2664232, 8	8·6463956, 8	7·8536300, 9 n	8·2453831, 0 n	8·1460166, 5 n	7·9107513, 7 n	7·5793307, 8 n	7·1353076, 0 n
27			9·6612058, 7	9·2318832, 5	8·5301701, 0	8·1191321, 1 n	8·3067695, 4 n	8·1679107, 6 n	7·9033624, 3 n	7·5350623, 5 n	7·0182176, 8 n
28			9·6473371, 6	9·1939662, 3	8·3737029, 5	8·2753413, 0 n	8·3539256, 5 n	8·1810104, 9 n	7·8864140, 2 n	7·4741914, 9 n	6·8429273, 6 n
29			9·6326695, 0	9·1521004, 5	8·1328303, 8	8·3836577, 7 n	8·3901119, 9 n	8·1861327, 2 n	7·8594451, 4 n	7·3918539, 2 n	6·5293405, 1 n
30			9·6171545, 0	9·1055339, 2	7·5732998, 8	8·4645795, 6 n	8·4173746, 9 n	8·1837246, 9 n	7·8214440, 9 n	7·2786895, 9 n	5·2995073, 0
31			9·6007376, 1	9·0532497, 9	7·7648580, 6 n	8·5274856, 7 n	8·4370519, 7 n	8·1739350, 0 n	7·7706274, 4 n	7·1138097, 0 n	6·5644411, 7
32			9·5833568, 1	8·9938241, 0	8·1781714, 7 n	8·5774445, 8 n	8·4500357, 6 n	8·1566412, 1 n	7·7039745, 3 n	6·8334467, 4 n	6·8398361, 5
33			9·5649416, 5	8·9251770, 4	8·3798125, 9 n	8·6174932, 0 n	8·4569128, 1 n	8·1314293, 8 n	7·6161918, 4 n	5·7919712, 0 n	6·9931848, 4
34			9·5454113, 8	8·8440935, 9	8·5119090, 2 n	8·6495821, 2 n	8·4580457, 5 n	8·0975339, 1 n	7·4971425, 9 n	6·7346318, 4	7·0923240, 6
35			9·5246729, 7	8·7452100, 6	8·6086415, 5 n	8·6750255, 5 n	8·4536184, 2 n	8·0537056, 0 n	7·3238623, 6 n	7·0481542, 6	7·1589050, 2
36			9·5026186, 9	8·6185333, 0	8·6837114, 0 n	8·6947379, 2 n	8·4436579, 8 n	7·9979601, 2 n	7·0229024, 1 n	7·2167437, 0	7·2024501, 7
37			9·4791226, 4	8·4418583, 0	8·7439982, 7 n	8·7093658, 0 n	8·4280378, 7 n	7·9270745, 7 n	4·4099331, 2 n	7·3262948, 9	7·2277504, 9
38			9·4540366, 8	8·1459300, 7	8·7934440, 0 n	8·7193682, 5 n	8·4064652, 4 n	7·8354958, 9 n	7·0106687, 9	7·4020893, 5	7·2372596, 5
39			9·4271848, 8	6·7631283, 8	8·8345127, 7 n	8·7250645, 0 n	8·3784475, 1 n	7·7126160, 3 n	7·3025971, 4	7·4549545, 7	7·2320223, 0
40			9·3983561, 0	8·0986067, 0 n	8·8688425, 0 n	8·7266650, 4 n	8·3432303, 1 n	7·5341848, 1 n	7·4661169, 7	7·4903925, 6	7·2120197, 2

41	9·3672938,5	8·4041526,8n	8·8975721,7n	8·7242889,0n	8·2996874,7n	7·2203879,3n	7·5752028,7	7·5114167,5	7·1761346,9
42	9·3336826,6	8·5777609,1n	8·9215207,9n	8·7179747,3n	8·2461259,2n	5·9637878,3	7·6529199,9	7·5196665,8	7·1218191,0
43	9·2971284,7	8·6981772,2n	8·9412912,5n	8·7076831,6n	8·1799190,0n	7·2575873,9	7·7094344,9	7·5158844,3	7·0441319,4
44	9·2571297,5	8·7894949,0n	8·9573353,7n	8·6932945,5n	8·0967498,2n	7·5396606,2	7·7500507,2	7·5000937,0	6·9332391,9
45	9·2130349,6	8·8623293,9n	8·9699946,0n	8·6746001,2n	7·9888807,7n	7·7016828,0	7·7777935,4	7·4716085,2	6·7665095,1
46	9·1639762,8	8·9223039,9n	8·9795267,5n	8·6512840,2n	7·8403200,5n	7·8119615,5	7·7944524,9	7·4288715,0	6·4715557,3
47	9·1087610,6	8·9727582,2n	8·9861248,3n	8·6228949,9n	7·6084650,6n	7·8924402,3	7·8010461,1	7·3689968,0	4·5132176,0
48	9·0456867,4	9·0158359,6n	8·9899292,8n	8·5888007,1n	7·0757658,8n	7·9529575,1	7·7980516,1	7·2867527,9	6·4548448,6n
49	8·9722014,5	9·0529936,2n	8·9910360,1n	8·5481125,4n	7·2170178,2	7·9987317,5	7·7854942,6	7·1719311,2	6·7476525,8n
50	8·8842230,4	9·0852643,1n	8·9895021,4n	8·4995604,9n	7·6457287,2	8·0328149,1	7·7629537,6	7·0011537,0	6·9089887,2n
51	8·7746023,6	9·1134059,7n	8·9853492,7n	8·4412735,1n	7·8513427,5	8·0570968,1	7·7294831,0	6·6985615,8	7·0136965,3n
52	8·6289859,0	9·1379891,6n	8·9785640,9n	8·3703633,7n	7·9846114,6	8·0727662,6	7·6834064,0	5·0004340,8n	7·0848871,5n
53	8·4109419,8	9·1594527,7n	8·9690987,0n	8·2820642,2n	8·0808215,8	8·0805532,6	7·6218905,2	6·7095497,9n	7·1325702,5n
54	7·9607484,8	9·1781399,7n	8·9568673,0n	8·1676920,9n	8·1539690,5	8·0808453,8	7·5400064,5	6·9974606,5n	7·1617691,7n
55	7·8629228,8n	9·1943225,5n	8·9417424,3n	8·0087449,2n	8·2110222,4	8·0737564,7	7·4284603,9	7·1592512,8n	7·1750767,2n
56	8·3712132,8n	9·2082178,4n	8·9235475,0n	7·7526000,0n	8·2559357,5	8·0591338,4	7·2667325,2	7·2665115,7n	7·1736640,2n
57	8·5964702,0n	9·2200005,8n	8·9020460,5n	7·0467136,2n	8·2911443,2	8·0365426,8	6·9930215,8	7·3417687,7n	7·1576229,6n
58	8·7420639,2n	9·2298115,5n	8·8769264,0n	7·5299985,2	8·3182274,2	8·0051968,2	6·0355498,0	7·3949382,3n	7·1259884,6n
59	8·8492429,0n	9·2377638,7n	8·8477778,6n	7·8926794,0	8·3382394,9	7·9638237,0	6·8806821,4n	7·4311385,2n	7·0764656,7n
60	8·9336704,3n	9·2439477,6n	8·8140563,2n	8·0833420,6	8·3518882,6	7·9104056,0	7·2047926,2n	7·4531914,0n	7·0045792,9n
61	9·0029971,7n	9·2484338,5n	8·7750303,7n	8·2113232,2	8·3596332,7	7·8416622,4	7·3805711,9n	7·4626330,5n	6·9015400,5n
62	9·0615373,3n	9·2512759,4n	8·7296952,6n	8·3060491,3	8·3617455,9	7·7519323,6	7·4971702,4n	7·4601352,8n	6·7477690,4n
63	9·1119640,5n	9·2525126,7n	8·6766288,7n	8·3798102,4	8·3583361,9	7·6303424,4	7·5805440,9n	7·4456804,7n	6·4850396,5n
64	9·1560482,7n	9·2521688,6n	8·6137351,9n	8·4389130,5	8·3493696,3	7·4518386,1	7·6418021,9n	7·4185498,6n	5·6410773,1n
65	9·1950235,1n	9·2502563,1n	8·5377516,2n	8·4870154,3	8·3346595,9	7·1323365,5	7·6866586,1n	7·3771495,6n	6·3368398,3
66	9·2297830,2n	9·2467743,0n	8·4432008,5n	8·5264179,8	8·3138486,7	6·1092409,7n	7·7183521,9n	7·3185558,2n	6·6710339,2
67	9·2609940,9n	9·2417094,9n	8·3198020,4n	8·5586497,8	8·2863677,6	7·2040249,8n	7·7388035,4n	7·2374708,4n	6·8469923,9
68	9·2891678,7n	9·2350354,7n	8·1444667,9n	8·5847667,7	8·2513571,4	7·4818466,3n	7·7491230,8n	7·1235445,9n	6·9601614,8
69	9·3147043,9n	9·2267121,6n	7·8435541,8n	8·6055162,1	8·2075398,6	7·6439291,5n	7·7498645,8n	6·9529667,1n	7·0374504,1
70	9·3379221,4n	9·2166844,2n	4·9513375,2n	8·6214323,2	8·1529829,6	7·7553808,6n	7·7411303,2n	6·6477349,6n	7·0901169,1
71	9·3590784,9n	9·2048801,4n	7·8396477,1	8·6328946,0	8·0846406,2	7·8374792,2n	7·7225927,9n	5·2489536,2	7·1237536,7
72	9·3783841,1n	9·1912078,7n	8·1383295,5	8·6401647,9	7·9973959,9	7·8998336,3n	7·6934314,6n	6·6781084,7	7·1412459,1
73	9·3960133,0n	9·1755533,0n	8·3110599,2	8·6434089,5	7·8817257,9	7·9475670,3n	7·6521631,5n	6·9637359,0	7·1439729,4
74	9·4121114,9n	9·1577746,8n	8·4317589,5	8·6427114,7	7·7166918,2	7·9836849,1n	7·5962781,3n	7·1258129,6	7·1321988,3
75	9·4268011,7n	9·1376966,4n	8·5235702,4	8·6380811,3	7·4388649,8	8·0100539,0n	7·5214754,5n	7·2340082,9	7·1051932,9
76	9·4401858,6n	9·1151014,4n	8·5967925,4	8·6294518,1	6·4196749,4	8·0278525,3n	7·4198368,4n	7·3104723,9	7·0609734,3
77	9·4523537,9n	9·0897175,2n	8·6569111,1	8·6166775,4	7·3451325,0n	8·0378032,5n	7·2746796,1n	7·3649878,8	6·9956351,9
78	9·4633802,2n	9·0612026,8n	8·7071915,0	8·5995216,3	7·6676751,8n	8·0403000,5n	7·0404638,9n	7·4026207,1	6·9016272,1
79	9·4733296,2n	9·0291201,6n	8·7497316,0	8·5776350,5	7·8465362,8n	8·0354589,6n	6·4657545,2n	7·4261711,8	6·7627385,7
80	9·4822572,3n	8·9929032,4n	8·7859546,3	8·5505254,1	7·9683629,2n	8·0231376,0n	6·7131880,3	7·4371589,5	6·5346732,6
81	9·4902104,1n	8·9518011,5n	8·8168657,8	8·5175044,1	8·0587064,8n	8·0029216,5n	7·1178908,1	7·4362644,5	5·9795483,7
82	9·4972295,9n	8·9047920,2n	8·8431968,9	8·4776032,0	8·1286283,1n	7·9740532,1n	7·3173486,7	7·4234687,0	6·1879998,7n
83	9·5033492,1n	8·8504390,8n	8·8654925,3	8·4294289,4	8·1839064,4n	7·9353046,8n	7·4463118,8	7·3980737,5	6·5999267,2n
84	9·5085983,0n	8·7866335,4n	8·8841635,1	8·3709100,1	8·2279180,5n	7·8847342,0n	7·5378996,6	7·3585173,3	6·7981394,1n
85	9·5130010,8n	8·7101010,2n	8·8995221,9	8·2988026,6	8·2627839,2n	7·8192022,9n	7·6054149,5	7·3019345,6	6·9229122,1n
86	9·5165774,0n	8·6153520,4n	8·9118058,5	8·2076421,8	8·2898971,7n	7·7333224,8n	7·6554717,1	7·2231453,5	7·0079780,9n
87	9·5193430,7n	8·4920902,8n	8·9211919,8	8·0871511,6	8·3101929,0n	7·6168796,9n	7·6917558,5	7·1120785,7	7·0666687,5n
88	9·5213101,2n	8·3171947,2n	8·9278096,7	7·9142246,5	8·3242972,8n	7·4468151,1n	7·7164407,6	6·9457836,1	7·1054216,2n
89	9·5224870,0n	8·0168813,1n	8·9317465,2	7·6150900,7	8·3326107,3n	7·1493823,3n	7·7308032,6	6·6505891,9	7·1275842,0n
90	9·5228787,5n	$-\infty$	8·9330532,1	$-\infty$	8·3353580,2n	$-\infty$	7·7355194,9	$-\infty$	7·1348019,9n

TABLE OF LOG $G'^{1}_{3}, \ldots G'^{1}_{10}$, FOR VALUES OF θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE, $\mu' \doteq \cos\theta'$.

θ	Log G'^{1}_{3}	Log G'^{1}_{4}	Log G'^{1}_{5}	Log G'^{1}_{6}	Log G'^{1}_{7}	Log G'^{1}_{8}	Log G'^{1}_{9}	Log G'^{1}_{10}
0°	9·9030899, 9	9·7569619, 5	9·5808706, 9	9·3845760, 5	9·1737226, 8	8·9518739, 3	8·7214250, 1	8·4840641, 0
1	9·9029223, 8	9·7566602, 5	9·5804013, 4	9·3839054, 8	9·1728173, 5	8·9507002, 3	8·7199493, 0	8·4822527, 0
2	9·9024193, 7	9·7557545, 8	9·5789920, 7	9·3818914, 6	9·1700971, 5	8·9471721, 9	8·7155114, 8	8·4768028, 0
3	9·9015804, 3	9·7542435, 0	9·5766395, 4	9·3785273, 1	9·1655502, 2	8·9412700, 7	8·7080803, 8	8·4676672, 9
4	9·9004046, 8	9·7521243, 9	9·5733377, 9	9·3738013, 3	9·1591555, 0	8·9329587, 7	8·6976006, 9	8·4547628, 7
5	9·8988908, 9	9·7493937, 7	9·5690787, 9	9·3676973, 5	9·1508838, 2	8·9221892, 9	8·6839945, 2	8·4379711, 4
6	9·8970374, 4	9·7460469, 3	9·5638516, 9	9·3601937, 6	9·1406959, 0	8·9088950, 4	8·6671551, 9	8·4171277, 8
7	9·8948423, 3	9·7420781, 0	9·5576431, 5	9·3512636, 0	9·1285422, 0	8·8929912, 2	8·6469444, 4	8·3920170, 2
8	9·8923032, 5	9·7374804, 4	9·5504370, 4	9·3408738, 9	9·1143614, 4	8·8743712, 9	8·6231866, 2	8·3623593, 7
9	9·8894173, 9	9·7322457, 7	9·5422141, 0	9·3289849, 2	9·0980788, 5	8·8529034, 1	8·5956595, 4	8·3277932, 2
10	9·8861816, 1	9·7263646, 7	9·5329518, 4	9·3155496, 7	9·0796043, 0	8·8284250, 3	8·5640825, 1	8·2878489, 6
11	9·8825923, 3	9·7198264, 6	9·5226242, 5	9·3005127, 0	9·0588295, 0	8·8007361, 8	8·5281005, 4	8·2419109, 0
12	9·8786455, 2	9·7126188, 4	9·5112013, 0	9·2838090, 1	9·0356243, 9	8·7695902, 1	8·4872603, 0	8·1891589, 2
13	9·8743366, 6	9·7047280, 3	9·4986485, 3	9·2653624, 2	9·0098328, 1	8·7346806, 4	8·4409737, 5	8·1284710, 0
14	9·8696607, 6	9·6961384, 7	9·4849264, 4	9·2450836, 2	8·9812658, 5	8·6956222, 8	8·3884647, 9	8·0582635, 2
15	9·8646123, 7	9·6868328, 6	9·4699900, 7	9·2228678, 6	8·9496938, 5	8·6519243, 2	8·3286800, 4	7·9761972, 4
16	9·8591854, 0	9·6767917, 4	9·4537878, 6	9·1985917, 0	8·9148352, 5	8·6029512, 2	8·2601477, 4	7·8786142, 7
17	9·8533732, 4	9·6659934, 4	9·4362609, 1	9·1721089, 3	8·8763404, 9	8·5478603, 3	8·1807192, 6	7·7593006, 1
18	9·8471686, 4	9·6544137, 9	9·4173415, 7	9·1432452, 9	8·8337693, 5	8·4855019, 6	8·0870744, 1	7·6063168, 6
19	9·8405636, 9	9·6420258, 6	9·3969521, 9	9·1117914, 5	8·7865581, 5	8·4142521, 6	7·9736773, 0	7·3916107, 4
20	9·8335497, 5	9·6287995, 1	9·3750029, 5	9·0774931, 2	8·7339685, 1	8·3317054, 0	7·8301901, 2	7·0148480, 4
21	9·8261174, 2	9·6147011, 6	9·3513897, 0	9·0400382, 3	8·6750085, 7	8·2340751, 7	7·6334937, 0	6·3704575, 7 *n*
22	9·8182564, 8	9·5996931, 6	9·3259908, 9	8·9990377, 5	8·6082999, 5	8·1148650, 8	7·3101025, 2	7·1254617, 5 *n*
23	9·8099557, 9	9·5837333, 7	9·2986638, 0	8·9539993, 8	8·5318480, 3	7·9614277, 4	4·8419848, 0	7·3540892, 2 *n*
24	9·8012032, 0	9·5667742, 7	9·2692392, 0	8·9042862, 3	8·4426025, 1	7·7433184, 5	7·2534058, 6 *n*	7·4781392, 1 *n*
25	9·7919855, 2	9·5487624, 0	9·2375150, 9	8·8490557, 9	8·3355473, 6	7·3450814, 9	7·5245544, 9 *n*	7·5535553, 6 *n*
26	9·7822883, 7	9·5296371, 5	9·2032472, 4	8·7871580, 6	8·2015148, 4	6·9044504, 5 *n*	7·6674212, 0 *n*	7·5993284, 0 *n*
27	9·7720960, 4	9·5093296, 6	9·1661369, 9	8·7169659, 1	8·0207498, 3	7·5434782, 4 *n*	7·7559860, 8 *n*	7·6240024, 5 *n*
28	9·7613913, 9	9·4877612, 5	9·1258140, 7	8·6360655, 3	7·7354368, 1	7·7674364, 0 *n*	7·8132085, 8 *n*	7·6319204, 2 *n*
29	9·7501557, 2	9·4648416, 6	9·0818111, 4	8·5406448, 8	6·8889204, 8	7·8960619, 1 *n*	7·8491158, 6 *n*	7·6252970, 2 *n*
30	9·7383685, 1	9·4404665, 0	9·0335268, 9	8·4241420, 5	7·5417576, 5 *n*	7·9798437, 2 *n*	7·8688489, 0 *n*	7·6050603, 5 *n*
31	9·7260073, 0	9·4145145, 0	8·9801693, 7	8·2737177, 9	7·8641163, 6 *n*	8·0366524, 6 *n*	7·8752361, 1 *n*	7·5711251, 1 *n*
32	9·7130474, 0	9·3868434, 7	8·9206644, 5	8·0582380, 7	8·0297833, 9 *n*	8·0748201, 0 *n*	7·8698135, 2 *n*	7·5224011, 9 *n*
33	9·6994616, 1	9·3572853, 2	8·8535034, 0	7·6575109, 5	8·1355692, 8 *n*	8·0988246, 8 *n*	7·8532697, 9 *n*	7·4564616, 8 *n*
34	9·6852199, 2	9·3256395, 3	8·7764694, 5	7·2708487, 9 *n*	8·2087656, 1 *n*	8·1112564, 7 *n*	7·8256247, 2 *n*	7·3686921, 6 *n*
35	9·6702891, 3	9·2916639, 7	8·6861082, 7	7·8907506, 7 *n*	8·2609345, 8 *n*	8·1136472, 6 *n*	7·7862380, 5 *n*	7·2500782, 9 *n*
36	9·6546323, 9	9·2550627, 8	8·5765853, 9	8·1195544, 5 *n*	8·2980168, 5 *n*	8·1068556, 1 *n*	7·7336746, 5 *n*	7·0805326, 9 *n*
37	9·6382086, 5	9·2154690, 7	8·4368188, 4	8·2560501, 6 *n*	8·3234189, 8 *n*	8·0912486, 4 *n*	7·6653079, 5 *n*	6·7996782, 0 *n*
38	9·6209720, 5	9·1724198, 7	8·2413272, 8	8·3494042, 0 *n*	8·3392269, 8 *n*	8·0667801, 2 *n*	7·5764668, 9 *n*	5·8786366, 7 *n*
39	9·6028711, 2	9·1253195, 9	7·9040888, 0	8·4172504, 2 *n*	8·3467548, 0 *n*	8·0329824, 0 *n*	7·4582706, 2 *n*	6·6490816, 1
40	9·5838479, 7	9·0733835, 4	6·9406111, 0 *n*	8·4679247, 9 *n*	8·3468206, 7 *n*	7·9888887, 9 *n*	7·2912822, 5 *n*	6·9639247, 0

41	9·5638369, 4	9·0155473, 8	7·9646645, 0 n	8·5060078, 2 n	8·3398932, 7 n	7·9328287, 7 n	7·0183634, 2 n	7·1260761, 8
42	9·5427634, 1	8·9503162, 6	8·2304857, 1 n	8·5342595, 7 n	8·3261641, 6 n	7·8620198, 1 n	6·1955399, 0 n	7·2269605, 9
43	9·5205420, 2	8·8754941, 9	8·3838930, 6 n	8·5544483, 3 n	8·3055792, 0 n	7·7716734, 5 n	6·8347702, 9	7·2927919, 7
44	9·4970744, 7	8·7876610, 0	8·4887448, 3 n	8·5677470, 0 n	8·2778339, 9 n	7·6528485, 2 n	7·1644539, 9	7·3345944, 7
45	9·4722468, 1	8·6810470, 3	8·5660741, 1 n	8·5749444, 1 n	8·2423301, 9 n	7·4860874, 5 n	7·3340836, 4	7·3578319, 8
46	9·4459259, 7	8·5447148, 7	8·6254465, 3 n	8·5765643, 2 n	8·1980871, 9 n	7·2144199, 4 n	7·4415741, 7	7·3653625, 5
47	9·4179551, 9	8·3536206, 4	8·6720111, 8 n	8·5729322, 1 n	8·1435643, 6 n	6·3984608, 5 n	7·5143325, 4	7·3585116, 2
48	9·3881480, 3	8·0240643, 1	8·7088490, 4 n	8·5642153, 3 n	8·0763294, 5 n	7·0322157, 0	7·5638713, 0	7·3375391, 2
49	9·3562803, 1	6·9731555, 7 n	8·7379371, 5 n	8·5504410, 5 n	7·9923754, 3 n	7·3656639, 2	7·5959864, 6	7·3016939, 6
50	9·3220793, 1	8·0767813, 0 n	8·7606083, 7 n	8·5315003, 8 n	7·8845665, 5 n	7·5392568, 1	7·6138587, 3	7·2490148, 1
51	9·2852084, 5	8·3500996, 8 n	8·7777921, 6 n	8·5071388, 3 n	7·7384340, 9 n	7·6513846, 4	7·6192425, 9	7·1756277, 6
52	9·2452455, 7	8·5091742, 8 n	8·7901503, 6 n	8·4769294, 0 n	7·5169542, 7 n	7·7295996, 8	7·6129829, 5	7·0740773, 7
53	9·2016513, 2	8·6194528, 3 n	8·7981567, 0 n	8·4402223, 3 n	7·0551450, 6 n	7·7855241, 6	7·5952238, 8	6·9284112, 2
54	9·1537209, 0	8·7023253, 1 n	8·8021481, 7 n	8·3960604, 9 n	6·9788877, 8	7·8250825, 7	7·5654475, 6	6·6954116, 2
55	9·1005085, 3	8·7674922, 6 n	8·8023553, 7 n	8·3430288, 9 n	7·4694479, 2	7·8516221, 5	7·5223516, 3	6·1493115, 1
56	9·0407030, 0	8·8201774, 1 n	8·7989228, 8 n	8·2789814, 3 n	7·6833568, 8	7·8671255, 7	7·4635032, 9	6·3131076, 0 n
57	8·9724105, 8	8·8635121, 9 n	8·7919201, 6 n	8·2005076, 8 n	7·8167058, 4	7·8727522, 2	7·3845380, 0	6·7273134, 8 n
58	8·8927481, 7	8·8995177, 0 n	8·7813479, 5 n	8·1017745, 6 n	7·9099543, 0	7·8690996, 6	7·2772377, 8	6·9210358, 5 n
59	8·7970046, 7	8·9295695, 7 n	8·7671383, 4 n	7·9716079, 4 n	7·9784945, 9	7·8563185, 6	7·1239202, 1	7·0403570, 5 n
60	8·6766674, 6	8·9546418, 8 n	8·7491502, 5 n	7·7841199, 6 n	8·0298077, 2	7·8341478, 4	6·8748411, 7	7·1198000, 7 n
61	8·5137610, 6	8·9754452, 3 n	8·7271594, 9 n	7·4508508, 5 n	8·0680522, 1	7·8018583, 5	6·2217792, 6	7·1729122, 1 n
62	8·2579381, 1	8·9925094, 3 n	8·7008407, 4 n	6·5750838, 8	8·0957323, 5	7·7581136, 9	6·6064673, 6 n	7·2062239, 2 n
63	7·5937799, 1	9·0062352, 3 n	8·6697389, 7 n	7·5420407, 5	8·1144224, 8	7·7006596, 0	6·9770877, 0 n	7·2231297, 6 n
64	7·9956957, 3 n	9·0169283, 4 n	8·6332247, 8 n	7·8107982, 1	8·1251174, 1	7·6256766, 1	7·1625584, 4 n	7·2253144, 5 n
65	8·3682917, 2 n	9·0248220, 5 n	8·5904217, 1 n	7·9685305, 7	8·1284160, 7	7·5262785, 3	7·2807149, 2 n	7·2133211, 6 n
66	8·5611754, 7 n	9·0300926, 0 n	8·5400858, 5 n	8·0775361, 4	8·1246183, 3	7·3884990, 6	7·3621562, 6 n	7·1866936, 4 n
67	8·6906722, 1 n	9·0328700, 4 n	8·4803953, 4 n	8·1585048, 2	8·1137666, 0	7·1769300, 8	7·4193873, 3 n	7·1438607, 0 n
68	8·7873038, 1 n	9·0332451, 4 n	8·4085543, 8 n	8·2208744, 9	8·0956550, 2	6·7394694, 9	7·4586212, 4 n	7·0815993, 1 n
69	8·8637043, 3 n	9·0312747, 3 n	8·3199829, 3 n	8·2697258, 4	8·0697986, 6	6·5984841, 0 n	7·4832562, 8 n	6·9937844, 1 n
70	8·9263353, 0 n	9·0269842, 8 n	8·2064172, 8 n	8·3080856, 8	8·0353604, 9	7·1190182, 2 n	7·4951778, 7 n	6·8677855, 8 n
71	8·9789464, 1 n	9·0203694, 9 n	8·0505174, 1 n	8·3378803, 7	7·9910093, 1	7·3402675, 8 n	7·4953250, 6 n	6·6723195, 7 n
72	9·0239068, 3 n	9·0113961, 4 n	7·8044590, 7 n	8·3603851, 1	7·9346433, 6	7·4773658, 4 n	7·4839266, 1 n	6·2845435, 2 n
73	9·0628075, 7 n	8·9999984, 0 n	7·1886923, 0 n	8·3764607, 0	7·8628484, 5	7·5726311, 7 n	7·4605595, 9 n	5·9378186, 8
74	9·0967675, 4 n	8·9860761, 3 n	7·5082414, 5	8·3866840, 6	7·7697007, 1	7·6419151, 2 n	7·4240399, 9 n	6·5546950, 5
75	9·1266020, 3 n	8·9694893, 6 n	7·8971969, 8	8·3914235, 0	7·6437230, 4	7·6928284, 2 n	7·3721218, 2 n	6·7889457, 3
76	9·1529223, 2 n	8·9500509, 5 n	8·0944938, 4	8·3908814, 3	7·4579817, 6	7·7295316, 8 n	7·3007823, 8 n	6·9284726, 7
77	9·1761972, 8 n	8·9275157, 7 n	8·2254765, 1	8·3851166, 1	7·1179537, 0	7·7544645, 4 n	7·2025210, 1 n	7·0215448, 6
78	9·1967930, 2 n	8·9015642, 9 n	8·3219961, 0	8·3740505, 4	6·3962865, 5 n	7·7690766, 2 n	7·0615806, 4 n	7·0853940, 5
79	9·2149997, 9 n	8·8717795, 2 n	8·3970717, 7	8·3574587, 7	7·2525957, 7 n	7·7741745, 0 n	6·8357856, 6 n	7·1279563, 5
80	9·2310502, 4 n	8·8376117, 4 n	8·4573064, 9	8·3349472, 8	7·5164586, 8 n	7·7700865, 4 n	6·3110329, 7 n	7·1533330, 6
81	9·2451324, 3 n	8·7983246, 3 n	8·5065020, 5	8·3059067, 8	7·6733140, 2 n	7·7567250, 8 n	6·4421974, 6	7·1636234, 6
82	9·2573992, 2 n	8·7529091, 1 n	8·5470387, 3	8·2694319, 2	7·7820452, 9 n	7·7335703, 9 n	6·8733554, 8	7·1596814, 7
83	9·2679751, 6 n	8·6999391, 4 n	8·5804945, 7	8·2241809, 8	7·8625654, 6 n	7·6995697, 9 n	7·0770807, 1	7·1413494, 3
84	9·2769615, 4 n	8·6373147, 8 n	8·6079591, 6	8·1681234, 3	7·9240124, 2 n	7·6529122, 6 n	7·2060348, 0	7·1074338, 5
85	9·2844401, 6 n	8·5617696, 2 n	8·6302042, 6	8·0980491, 6	7·9713036, 9 n	7·5905554, 0 n	7·2956462, 6	7·0553133, 3
86	9·2904763, 8 n	8·4678202, 5 n	8·6477839, 0	8·0085188, 4	8·0073427, 3 n	7·5071877, 3 n	7·3598221, 1	6·9799942, 8
87	9·2951211, 2 n	8·3451754, 6 n	8·6610982, 1	7·8892770, 4	8·0339430, 0 n	7·3926477, 1 n	7·4053531, 9	6·8715204, 0
88	9·2984124, 6 n	8·1707180, 8 n	8·6704249, 1	7·7172318, 3	8·0522509, 6 n	7·2239162, 8 n	7·4359080, 4	6·7070248, 4
89	9·3003768, 9 n	7·8706660, 4 n	8·6759503, 8	7·4186194, 3	8·0629784, 9 n	6·9272728, 2 n	7·4535446, 1	6·4128803, 6
90	9·3010300, 0 n	− ∞	8·6777807, 1	− ∞	8·0665127, 1 n	− ∞	7·4593130, 8	− ∞

TABLE OF LOG G'^2_n FOR VALUES OF θ FROM 0° TO 90°.

$\mu' = \cos\theta'$.

θ	Log G'^2_4	Log G'^2_5	Log G'^2_6	Log G'^2_7	Log G'^2_8	Log G'^2_9	Log G'^2_{10}
0°	9·9330532, 1	9·8239087, 4	9·6856060, 4	9·5259052, 0	9·3498139, 4	9·1607577, 0	8·9611853, 5
1	9·9328967, 7	9·8236405, 5	9·6852037, 5	9·5253464, 6	9·3490763, 9	9·1598189, 4	8·9600229, 8
2	9·9324273, 3	9·8228356, 6	9·6839961, 8	9·5236688, 7	9·3468613, 7	9·1569989, 9	8·9565304, 6
3	9·9316444, 6	9·8214930, 4	9·6819812, 3	9·5208686, 8	9·3431626, 4	9·1522879, 8	8·9506929, 3
4	9·9305474, 5	9·8196109, 1	9·6791552, 9	9·5169392, 9	9·3379690, 9	9·1456682, 8	8·9424836, 3
5	9·9291353, 2	9·8171869, 1	9·6755134, 6	9·5118717, 4	9·3312655, 1	9·1371156, 8	8·9318658, 8
6	9·9274068, 0	9·8142178, 6	9·6710492, 2	9·5056539, 5	9·3230314, 7	9·1265975, 5	8·9187899, 0
7	9·9253603, 0	9·8106998, 2	9·6657544, 5	9·4982710, 6	9·3132416, 4	9·1140731, 2	8·9031929, 6
8	9·9229939, 8	9·8066282, 0	9·6596195, 0	9·4897050, 3	9·3018649, 2	9·0994919, 7	8·8849967, 8
9	9·9203056, 4	9·8019975, 1	9·6526330, 0	9·4799346, 0	9·2888645, 3	9·0827938, 4	8·8641069, 7
10	9·9172927, 8	9·7968014, 3	9·6447815, 8	9·4689346, 0	9·2741964, 3	9·0639057, 9	8·8404074, 2
11	9·9139525, 8	9·7910328, 5	9·6360501, 4	9·4566761, 9	9·2578095, 7	9·0427421, 1	8·8137598, 7
12	9·9102818, 8	9·7846836, 9	9·6264213, 4	9·4431259, 8	9·2396440, 3	9·0192006, 5	8·7839963, 6
13	9·9062771, 4	9·7777448, 5	9·6158755, 5	9·4282457, 2	9·2196303, 1	8·9931608, 8	8·7509145, 7
14	9·9019344, 7	9·7702062, 5	9·6043906, 7	9·4119917, 1	9·1976874, 2	8·9644792, 7	8·7142673, 8
15	9·8972495, 9	9·7620566, 5	9·5919418, 2	9·3943139, 3	9·1737212, 0	8·9329851, 8	8·6737525, 0
16	9·8922177, 9	9·7532836, 3	9·5785011, 0	9·3751554, 6	9·1476214, 3	8·8984732, 5	8·6289940, 6
17	9·8868339, 5	9·7438734, 4	9·5640371, 1	9·3544508, 6	9·1192594, 2	8·8606963, 8	8·5795205, 4
18	9·8810925, 1	9·7338109, 6	9·5485147, 9	9·3321254, 9	9·0884832, 2	8·8193517, 5	8·5247265, 6
19	9·8749873, 5	9·7230794, 4	9·5318946, 6	9·3080933, 8	9·0551128, 2	8·7740655, 6	8·4638217, 9
20	9·8685119, 0	9·7116604, 9	9·5141323, 9	9·2822552, 0	9·0189324, 3	8·7243677, 2	8·3957417, 9
21	9·8616590, 1	9·6995338, 7	9·4951780, 7	9·2544958, 8	8·9796817, 8	8·6696573, 9	8·3190075, 3
22	9·8544209, 5	9·6866772, 1	9·4749753, 1	9·2246809, 9	8·9370422, 6	8·6091480, 4	8·2314698, 4
23	9·8467893, 3	9·6730658, 8	9·4534602, 9	9·1926526, 3	8·8906192, 8	8·5417831, 1	8·1298284, 0
24	9·8387550, 9	9·6586726, 6	9·4305604, 9	9·1582237, 7	8·8399149, 5	8·4660908, 3	8·0085984, 6
25	9·8303084, 3	9·6434674, 9	9·4061930, 5	9·1211702, 5	8·7842882, 0	8·3799314, 0	7·8575911, 8
26	9·8214387, 3	9·6274170, 2	9·3802630, 1	9·0812213, 6	8·7228953, 2	8·2800148, 8	7·6541775, 0
27	9·8121344, 8	9·6104841, 3	9·3526604, 3	9·0380444, 3	8·6545892, 7	8·1608660, 2	7·3277516, 3
28	9·8023832, 6	9·5926277, 0	9·3232580, 2	8·9912262, 6	8·5777563, 1	8·0123060, 5	5·7254215, 5
29	9·7921715, 8	9·5738016, 7	9·2919065, 2	8·9402426, 2	8·4900131, 5	7·8116896, 0	7·2322615, 0 *n*
30	9·7814847, 9	9·5539543, 8	9·2584297, 1	8·8844152, 6	8·3876199, 1	7·4879016, 1	7·4998107, 6 *n*
31	9·7703070, 1	9·5330279, 0	9·2226175, 9	8·8228434, 0	8·2641829, 2	5·8627871, 0	7·6352414, 0 *n*
32	9·7586209, 4	9·5109566, 3	9·1842168, 3	8·7542925, 9	8·1073083, 5	7·4024350, 7 *n*	7·7151849, 3 *n*
33	9·7464077, 7	9·4876662, 6	9·1429181, 9	8·6770047, 3	7·8873995, 4	7·6745019, 9 *n*	7·7630669, 2 *n*
34	9·7336469, 6	9·4630719, 6	9·0983384, 0	8·5883469, 1	7·4944426, 4	7·8153359, 7 *n*	7·7890000, 9 *n*
35	9·7203160, 4	9·4370763, 8	9·0499938, 7	8·4841050, 1	6·9888442, 3 *n*	7·9014506, 9 *n*	7·7980806, 5 *n*
36	9·7063904, 3	9·4095670, 9	8·9972625, 9	8·3568807, 9	7·6602733, 7 *n*	7·9563298, 2 *n*	7·7930153, 7 *n*
37	9·6918431, 8	9·3804134, 3	8·9393240, 7	8·1917250, 9	7·8853318, 2 *n*	7·9902038, 4 *n*	7·7751384, 0 *n*
38	9·6766445, 4	9·3494618, 8	8·8750639, 2	7·9501457, 0	8·0128443, 9 *n*	8·0083335, 5 *n*	7·7448302, 4 *n*
39	9·6607617, 9	9·3165310, 0	8·8029127, 4	7·4551145, 0	8·0951062, 1 *n*	8·0136515, 0 *n*	7·7016344, 6 *n*
40	9·6441586, 4	9·2814036, 8	8·7205554, 5	7·4208383, 0 *n*	8·1504383, 5 *n*	8·0078104, 9 *n*	7·6441499, 5 *n*

41	9·6267948, 3	9·2438171, 0	8·6243627, 6	7·8775895, 9 n	8·1873545, 8 n	7·9916473, 1 n	7·5696243, 5 n
42	9·6086255, 4	9·2034491, 4	8·5081518, 2	8·0746449, 6 n	8·2104359, 8 n	7·9653926, 9 n	7·4729757, 3 n
43	9·5896005, 8	9·1598983, 2	8·3600120, 1	8·1941092, 8 n	8·2223545, 8 n	7·9287250, 4 n	7·3442763, 2 n
44	9·5696636, 3	9·1126555, 0	8·1518581, 2	8·2749264, 3 n	8·2247236, 2 n	7·8807101, 5 n	7·1608975, 2 n
45	9·5487511, 2	9·0610609, 1	7·7828315, 3	8·3320658, 2 n	8·2184942, 8 n	7·8195919, 6 n	6·8507503, 6 n
46	9·5267909, 8	9·0042377, 8	7·1593686, 5 n	8·3728420, 7 n	8·2041560, 9 n	7·7423189, 2 n	4·9995654, 9 n
47	9·5037010, 2	8·9409839, 0	7·9207456, 1 n	8·4013364, 3 n	8·1818276, 0 n	7·6434793, 3 n	6·8015340, 5
48	9·4793870, 0	8·8695867, 0	8·1642505, 2 n	8·4200188, 3 n	8·1512803, 4 n	7·5126043, 6 n	7·0817324, 6
49	9·4537400, 5	8·7874830, 8	8·3061246, 7 n	8·4304543, 3 n	8·1119077, 6 n	7·3256839, 3 n	7·2307349, 5
50	9·4266335, 3	8·6905790, 7	8·4022942, 3 n	8·4336496, 9 n	8·0626150, 7 n	7·0039299, 1 n	7·3237758, 2
51	9·3979189, 7	8·5717130, 2	8·4720078, 5 n	8·4302370, 0 n	8·0016039, 2 n	5·6584883, 8	7·3840432, 3
52	9·3674205, 1	8·4165173, 6	8·5241910, 7 n	8·4205732, 3 n	7·9259249, 4 n	7·0117607, 0	7·4216055, 8
53	9·3349279, 0	8·1885454, 6	8·5636955, 7 n	8·4047901, 2 n	7·8305200, 2 n	7·2832175, 3	7·4415349, 0
54	9·3001867, 1	7·7280590, 1	8·5934324, 0 n	8·3828146, 2 n	7·7058850, 8 n	7·4326486, 6	7·4465107, 6
55	9·2628849, 3	7·6195597, 4 n	8·6152648, 4 n	8·3543616, 4 n	7·5310150, 0 n	7·5291004, 0	7·4378218, 8
56	9·2226340, 6	8·1167720, 8 n	8·6304306, 3 n	8·3188985, 4 n	7·2420293, 2 n	7·5945825, 8	7·4157543, 8
57	9·1789416, 8	8·3305292, 3 n	8·6397970, 6 n	8·2755711, 7 n	6·1916745, 3 n	7·6388555, 0	7·3796807, 1
58	9·1311704, 4	8·4641779, 5 n	8·6439268, 8 n	8·2230648, 0 n	7·1305974, 8	7·6669801, 1	7·3278859, 0
59	9·0784746, 6	8·5589416, 2 n	8·6432140, 5 n	8·1593509, 0 n	7·4395490, 0	7·6817882, 9	7·2570110, 3
60	9·0196977, 3	8·6304555, 4 n	8·6378955, 2 n	8·0811830, 6 n	7·6050118, 3	7·6848543, 3	7·1607235, 7
61	8·9531974, 0	8·6863401, 2 n	8·6280844, 1 n	7·9830161, 3 n	7·7128617, 4	7·6769431, 4	7·0259977, 3
62	8·8765277, 4	8·7308759, 0 n	8·6137802, 2 n	7·8543169, 8 n	7·7884490, 9	7·6581812, 6	6·8202868, 6
63	8·7858088, 7	8·7666986, 8 n	8·5948707, 8 n	7·6711283, 9 n	7·8427047, 0	7·6280935, 1	6·4122084, 7
64	8·6743234, 9	8·7955377, 3 n	8·5711220, 8 n	7·3552522, 4 n	7·8812762, 1	7·5854968, 3	6·1301728, 9 n
65	8·5288216, 1	8·8185799, 8 n	8·5421559, 2 n	5·9576551, 7	7·9073882, 5	7·5282068, 6	6·7057012, 5 n
66	8·3167386, 0	8·8366664, 2 n	8·5074107, 0 n	7·3715573, 2	7·9229708, 2	7·4523600, 0	6·9295670, 7 n
67	7·9079007, 4	8·8504052, 6 n	8·4660742, 0 n	7·6526466, 0	7·9291761, 7	7·3508525, 1	7·0620891, 3 n
68	7·6164786, 6 n	8·8602407, 1 n	8·4169690, 2 n	7·8123604, 5	7·9266253, 7	7·2090026, 8	7·1494192, 8 n
69	8·2019782, 5 n	8·8664965, 8 n	8·3583537, 0 n	7·9203550, 3	7·9155296, 3	6·9887283, 6	7·2082773, 9 n
70	8·4354581, 4 n	8·8694037, 0 n	8·2875464, 5 n	7·9988624, 0	7·8957219, 1	6·5151715, 2	7·2464247, 1 n
71	8·5812098, 5 n	8·8691184, 4 n	8·2001599, 5 n	8·0578235, 1	7·8666364, 1	6·4878168, 7 n	7·2678965, 1 n
72	8·6858869, 6 n	8·8657341, 6 n	8·0883289, 2 n	8·1025150, 0	7·8272025, 0	6·9619571, 1 n	7·2747949, 6 n
73	8·7665403, 7 n	8·8592873, 6 n	7·9357606, 1 n	8·1360308, 6	7·7756177, 3	7·1717381, 5 n	7·2680043, 1 n
74	8·8313330, 7 n	8·8497608, 2 n	7·6987336, 8 n	8·1603028, 2	7·7088931, 8	7·3014792, 6 n	7·2474675, 3 n
75	8·8848072, 6 n	8·8370826, 4 n	7·1476917, 8 n	8·1765684, 3	7·6218583, 6	7·3903132, 4 n	7·2121503, 0 n
76	8·9297487, 6 n	8·8211218, 3 n	7·3262928, 4	8·1856155, 3	7·5047141, 1	7·4531898, 7 n	7·1597536, 2 n
77	8·9679865, 4 n	8·8016797, 1 n	7·7440284, 2	8·1879111, 3	7·3355361, 3	7·4973126, 0 n	7·0858932, 5 n
78	9·0007828, 9 n	8·7784757, 0 n	7·9467528, 2	8·1836703, 5	7·0460618, 3	7·5265783, 3 n	6·9820721, 4 n
79	9·0290428, 7 n	8·7511253, 0 n	8·0788394, 7	8·1728859, 5	5·7463227, 7	7·5431849, 1 n	6·8296124, 9 n
80	9·0534344, 9 n	8·7191062, 9 n	8·1747515, 7	8·1553266, 3	6·9929244, 8 n	7·5483120, 1 n	6·5742859, 8 n
81	9·0744621, 0 n	8·6817051, 4 n	8·2482560, 8	8·1305071, 3	7·2984884, 6 n	7·5424220, 5 n	6·8318697, 7 n
82	9·0925127, 4 n	8·6379325, 1 n	8·3062394, 7	8·0976182, 0	7·4683443, 8 n	7·5253649, 0 n	6·3753709, 4
83	9·1078870, 1 n	8·5863782, 1 n	8·3526245, 0	8·0553936, 3	7·5823145, 5 n	7·4963469, 8 n	6·7255359, 5
84	9·1208197, 7 n	8·5249560, 0 n	8·3898388, 1	8·0018613, 1	7·6646203, 5 n	7·4537402, 0 n	6·9063997, 0
85	9·1314945, 9 n	8·4504107, 9 n	8·4194698, 2	7·9338571, 5	7·7258267, 3 n	7·3946355, 1 n	7·0225026, 0
86	9·1400540, 3 n	8·3572676, 5 n	8·4425904, 4	7·8459769, 0	7·7714338, 8 n	7·3138208, 4 n	7·1023170, 6
87	9·1466068, 3 n	8·2352427, 9 n	8·4599369, 8	7·7279915, 8	7·8045884, 9 n	7·2011976, 9 n	7·1576078, 5
88	9·1512329, 9 n	8·0612243, 3 n	8·4720094, 7	7·5568317, 7	7·8271831, 2 n	7·0338056, 1 n	7·1941867, 6
89	9·1539873, 3 n	7·7614354, 1 n	8·4791315, 8	7·2588024, 5	7·8403407, 5 n	6·7380032, 1 n	7·2151271, 1
90	9·1549019, 6 n	∞	8·4814860, 6	∞	7·8446639, 6 n	∞	7·2219521, 7

TABLE OF LOG G'^3_n FOR VALUES OF θ FROM 0° TO 90°.

$\mu' = \cos\theta'$.

θ	Log G'^3_5	Log G'^3_6	Log G'^3_7	Log G'^3_8	Log G'^3_9	Log G'^3_{10}
0°	9·9488474, 8	9·8616973, 0	9·7477539, 5	9·6130553, 8	9·4617877, 0	9·2969774, 5
1	9·9486966, 3	9·8614458, 8	9·7473851, 8	9·6125524, 9	9·4611339, 4	9·2961560, 4
2	9·9482439, 7	9·8606913, 6	9·7462784, 0	9·6110430, 2	9·4591712, 8	9·2936896, 5
3	9·9474891, 2	9·8594329, 0	9·7444320, 2	9·6085242, 2	9·4558953, 6	9·2895716, 8
4	9·9464314, 5	9·8576690, 9	9·7418433, 5	9·6049914, 5	9·4512987, 2	9·2837907, 6
5	9·9450701, 2	9·8553980, 4	9·7385087, 4	9·6004384, 5	9·4453712, 6	9·2763314, 3
6	9·9434039, 8	9·8526171, 9	9·7344233, 3	9·5948567, 2	9·4380992, 6	9·2671726, 7
7	9·9414316, 8	9·8493234, 4	9·7295811, 8	9·5882359, 9	9·4294660, 4	9·2562887, 5
8	9·9391515, 5	9·8455130, 7	9·7239751, 1	9·5805637, 6	9·4194512, 4	9·2436480, 8
9	9·9365616, 7	9·8411817, 5	9·7175967, 8	9·5718253, 1	9·4080306, 9	9·2292130, 3
10	9·9336598, 8	9·8363245, 2	9·7104364, 6	9·5620034, 9	9·3951760, 8	9·2129390, 8
11	9·9304436, 7	9·8309356, 5	9·7024830, 3	9·5510786, 3	9·3808547, 6	9·1947744, 7
12	9·9269103, 0	9·8250088, 3	9·6937240, 6	9·5390283, 5	9·3650290, 6	9·1746588, 6
13	9·9230566, 6	9·8185369, 0	9·6841453, 5	9·5258271, 1	9·3476557, 4	9·1525221, 2
14	9·9188793, 7	9·8115119, 8	9·6737311, 4	9·5114461, 5	9·3286854, 8	9·1282833, 2
15	9·9143747, 0	9·8039253, 0	9·6624637, 7	9·4958529, 2	9·3080617, 5	9·1018481, 6
16	9·9095385, 4	9·7957672, 0	9·6503236, 3	9·4790108, 9	9·2857202, 2	9·0731075, 2
17	9·9043664, 7	9·7870271, 2	9·6372889, 4	9·4608788, 8	9·2615871, 0	9·0419338, 8
18	9·8988536, 2	9·7776933, 5	9·6233353, 7	9·4414103, 4	9·2355777, 4	9·0081775, 9
19	9·8929947, 7	9·7677532, 0	9·6084361, 5	9·4205530, 6	9·2075949, 7	8·9716626, 0
20	9·8867842, 2	9·7571926, 8	9·5925614, 0	9·3982478, 7	9·1775264, 0	8·9321793, 0
21	9·8802158, 3	9·7459965, 4	9·5756780, 4	9·3744279, 2	9·1452416, 7	8·8894766, 8
22	9·8732829, 5	9·7341480, 5	9·5577491, 4	9·3490168, 9	9·1105879, 6	8·8432496, 0
23	9·8659784, 3	9·7216289, 5	9·5387336, 7	9·3219281, 0	9·0733857, 7	8·7931233, 5
24	9·8582945, 5	9·7084192, 7	9·5185858, 6	9·2930620, 4	9·0334217, 8	8·7386302, 8
25	9·8502229, 5	9·6944970, 3	9·4972543, 0	9·2623038, 7	8·9904396, 0	8·6791748, 7
26	9·8417546, 9	9·6798382, 5	9·4746814, 5	9·2295205, 8	8·9441280, 8	8·6139824, 5
27	9·8328800, 4	9·6644164, 8	9·4508023, 2	9·1945565, 5	8·8941035, 2	8·5420168, 1
28	9·8235885, 8	9·6482027, 1	9·4255435, 1	9·1572286, 2	8·8398855, 6	8·4618441, 4
29	9·8138690, 1	9·6311648, 9	9·3988214, 6	9·1173187, 1	8·7808602, 3	8·3713935, 5
30	9·8037091, 6	9·6132676, 5	9·3705407, 1	9·0745643, 1	8·7162242, 0	8·2675030, 8
31	9·7930958, 7	9·5944718, 0	9·3405916, 3	9·0286460, 4	8·6448971, 7	8·1449666, 0
32	9·7820148, 8	9·5747337, 4	9·3088470, 1	8·9791681, 3	8·5653713, 7	7·9942326, 6
33	9·7704507, 9	9·5540050, 0	9·2751589, 2	8·9256333, 6	8·4754506, 5	7·7945312, 6
34	9·7583868, 6	9·5322312, 6	9·2393531, 5	8·8674024, 9	8·3717391, 2	7·4836571, 7
35	9·7458049, 2	9·5093514, 8	9·2012225, 7	8·8036338, 8	8·2485548, 8	6·4821157, 6
36	9·7326852, 2	9·4852968, 9	9·1605188, 9	8·7331866, 7	8·0952300, 4	7·3029496, 7 n
37	9·7190062, 1	9·4599894, 5	9·1169397, 5	8·6544547, 5	7·8876465, 6	7·5920445, 6 n
38	9·7047443, 5	9·4333401, 7	9·0701120, 8	8·5650694, 0	7·5472465, 9	7·7337740, 6 n
39	9·6898739, 5	9·4052471, 1	9·0195678, 9	8·4613163, 1	6·1501114, 3 n	7·8167601, 6 n
40	9·6743667, 4	9·3755924, 2	8·9647076, 5	8·3368526, 3	7·5176115, 2 n	7·8667143, 3 n

41	9·6581916, 7	9·3442390, 2	8·9047452, 4	8·1794047, 1	7·7742541, 2 n	7·8945431, 2 n
42	9·6413145, 1	9·3110261, 4	8·8386204, 3	7·9597623, 9	7·9096111, 1 n	7·9057322, 2 n
43	9·6236973, 6	9·2757633, 1	8·7648515, 7	7·5703128, 9	7·9930026, 0 n	7·9032511, 9 n
44	9·6052981, 6	9·2382224, 9	8·6812737, 2	7·0344279, 1 n	8·0464904, 8 n	7·8886820, 4 n
45	9·5860700, 8	9·1981273, 7	8·5845354, 3	7·7239635, 6 n	8·0798560, 5 n	7·8626917, 8 n
46	9·5659607, 2	9·1551384, 7	8·4690235, 1	7·9511528, 8 n	8·0981977, 9 n	7·8252196, 3 n
47	9·5449112, 2	9·1088317, 4	8·3241953, 8	8·0798669, 0 n	8·1044044, 8 n	7·7754728, 2 n
48	9·5228551, 7	9·0586678, 5	8·1262593, 7	8·1632794, 8 n	8·1001529, 9 n	7·7117363, 9 n
49	9·4997172, 2	9·0039450, 6	7·7981891, 8	8·2199382, 6 n	8·0863594, 8 n	7·6308843, 0 n
50	9·4754114, 2	8·9437261, 9	6·2455126, 7 n	8·2584591, 3 n	8·0633930, 4 n	7·5272289, 3 n
51	9·4498391, 4	8·8767178, 5	7·7786475, 1 n	8·2834900, 6 n	8·0311530, 7 n	7·3895399, 5 n
52	9·4228863, 0	8·8010618, 3	8·0497938, 6 n	8·2977655, 4 n	7·9890620, 7 n	7·1913671, 7 n
53	9·3944201, 3	8·7139454, 3	8·1988879, 2 n	8·3029654, 5 n	7·9359478, 9 n	6·8404824, 9 n
54	9·3642846, 3	8·6108044, 8	8·2966331, 7 n	8·3001233, 3 n	7·8697883, 1 n	6·0523476, 0
55	9·3322948, 8	8·4834667, 7	8·3654672, 0 n	8·2898334, 6 n	7·7871521, 8 n	6·9278834, 1
56	9·2982292, 1	8·3149110, 1	8·4153995, 9 n	8·2723552, 6 n	7·6819427, 8 n	7·1752972, 3
57	9·2618188, 1	8·0587038, 2	8·4517385, 6 n	8·2476588, 5 n	7·5421442, 1 n	7·3125539, 2
58	9·2227330, 1	7·4537188, 1	8·4776068, 2 n	8·2154220, 9 n	7·3389622, 1 n	7·3996045, 1
59	9·1805586, 6	7·7125217, 6 n	8·4949649, 4 n	8·1749864, 6 n	6·9666438, 2 n	7·4564480, 2
60	9·1347698, 9	8·0987046, 2 n	8·5050902, 6 n	8·1252469, 8 n	6·4201043, 4	7·4920811, 9
61	9·0846824, 7	8·2863405, 8 n	8·5088248, 0 n	8·0644411, 4 n	7·1353877, 5	7·5111384, 1
62	9·0293826, 1	8·4066619, 3 n	8·5067103, 6 n	7·9897300, 6 n	7·3739615, 1	7·5161187, 3
63	8·9676104, 5	8·4923408, 4 n	8·4990651, 5 n	7·8963156, 3 n	7·5117764, 4	7·5082535, 7
64	8·8975584, 2	8·5566201, 4 n	8·4860239, 1 n	7·7753169, 4 n	7·6026675, 8	7·4878747, 8
65	8·8164973, 5	8·6061783, 6 n	8·4675563, 1 n	7·6075558, 1 n	7·6653032, 6	7·4544988, 2
66	8·7200160, 1	8·6448413, 8 n	8·4434645, 2 n	7·3373853, 7 n	7·7082934, 0	7·4067053, 5
67	8·6002663, 5	8·6749907, 7 n	8·4133619, 8 n	6·5546102, 9 n	7·7362171, 7	7·3417292, 3
68	8·4410821, 4	8·6981944, 4 n	8·3766271, 4 n	7·1375693, 1	7·7516785, 6	7·2544489, 8
69	8·1993365, 8	8·7155240, 1 n	8·3323180, 5 n	7·4766852, 9	7·7561597, 2	7·1348366, 6
70	7·6522734, 3	8·7277292, 5 n	8·2790227, 8 n	7·6526862, 1	7·7504111, 9	6·9599472, 2
71	7·8047505, 6	8·7353366, 5 n	8·2145808, 0 n	7·7670095, 9	7·7346134, 3	6·6560310, 9
72	8·2245379, 8 n	8·7387116, 9 n	8·1355374, 9 n	7·8477416, 6	7·7084243, 0	3·5440680, 4 n
73	8·4260412, 5 n	8·7380950, 6 n	8·0359565, 8 n	7·9066916, 0	7·6709023, 6	6·6396159, 6 n
74	8·5571474, 6 n	8·7336258, 9 n	7·9043985, 0 n	7·9498918, 4	7·6202974, 6	6·9260593, 4 n
75	8·6527883, 4 n	8·7253533, 9 n	7·7141268, 2 n	7·9807798, 6	7·5535735, 7	7·0836674, 3 n
76	8·7268783, 3 n	8·7132424, 9 n	7·3718968, 5 n	8·0014329, 7	7·4653501, 3	7·1859810, 4 n
77	8·7863803, 3 n	8·6971707, 3 n	6·6184037, 6	8·0131259, 6	7·3452286, 2	7·2558897, 3 n
78	8·8352744, 7 n	8·6769187, 5 n	7·4918349, 6	8·0166051, 1	7·1693775, 5	7·3033818, 2 n
79	8·8760446, 1 n	8·6521521, 3 n	7·7545455, 6	8·0122283, 8	6·8599184, 9	7·3335379, 6 n
80	8·9103396, 7 n	8·6223887, 4 n	7·9099569, 8	8·0000225, 9	5·4418521, 8	7·3490811, 2 n
81	8·9393049, 5 n	8·5869491, 0 n	8·0176355, 0	7·9796903, 8	6·8825814, 5 n	7·3513826, 8 n
82	8·9637629, 0 n	8·5448708, 7 n	8·0976567, 1	7·9505605, 3	7·1663916, 3 n	7·3408544, 2 n
83	8·9843187, 6 n	8·4947662, 1 n	8·1592329, 2	7·9114718, 9	7·3272955, 7 n	7·3170725, 2 n
84	9·0014260, 9 n	8·4345681, 1 n	8·2073332, 5	7·8605314, 5	7·4351194, 0 n	7·2786667, 3 n
85	9·0154238, 5 n	8·3610346, 8 n	8·2449101, 6	7·7946337, 1	7·5120517, 1 n	7·2228958, 7 n
86	9·0265723, 6 n	8·2687045, 4 n	8·2738338, 6	7·7084234, 5	7·5679422, 4 n	7·1446707, 0 n
87	9·0350629, 4 n	8·1473027, 7 n	8·2953233, 8	7·5917044, 4	7·6079223, 1 n	7·0339903, 9 n
88	9·0410345, 2 n	7·9737248, 2 n	8·3101788, 6	7·4214328, 1	7·6348881, 9 n	6·8679445, 4 n
89	9·0445811, 8 n	7·6741977, 2 n	8·3189062, 5	7·1238745, 0	7·6504949, 9 n	6·5728948, 3 n
90	9·0457574, 9 n	∞	8·3217852, 2	∞	7·6556077, 3 n	∞

TABLE OF LOG G'^{4}_{n} FOR VALUES OF θ FROM 0° TO 90°.

$\mu' = \cos\theta'$.

θ	Log G'^{4}_{6}	Log G'^{4}_{7}	Log G'^{4}_{8}	Log G'^{4}_{9}	Log G'^{4}_{10}
0°	9·9586073, 1	9·8860566, 5	9·7891466, 3	9·6726410, 7	9·5400155, 0
1	9·9584598, 2	9·8858152, 9	9·7887980, 1	9·6721717, 7	9·5394121, 0
2	9·9580172, 2	9·8850909, 7	9·7877516, 5	9·6707630, 1	9·5376006, 0
3	9·9572791, 8	9·8838829, 8	9·7860063, 2	9·6684128, 4	9·5345779, 5
4	9·9562451, 3	9·8821901, 2	9·7835598, 0	9·6651175, 3	9·5303383, 7
5	9·9549142, 6	9·8800107, 2	9·7804091, 0	9·6608721, 4	9·5248741, 8
6	9·9532855, 3	9·8773425, 8	9·7765502, 2	9·6556700, 2	9·5181750, 3
7	9·9513576, 9	9·8741830, 4	9·7719783, 2	9·6495031, 5	9·5102283, 6
8	9·9491292, 1	9·8705288, 7	9·7666874, 9	9·6423616, 8	9·5010187, 8
9	9·9465983, 4	9·8663763, 8	9·7606709, 9	9·6342342, 7	9·4905283, 4
10	9·9437630, 5	9·8617212, 4	9·7539208, 8	9·6251076, 2	9·4787360, 4
11	9·9406210, 7	9·8565586, 2	9·7464282, 2	9·6149665, 5	9·4656177, 1
12	9·9371698, 6	9·8508830, 4	9·7381828, 4	9·6037938, 3	9·4511457, 1
13	9·9334065, 9	9·8446884, 4	9·7291734, 6	9·5915701, 1	9·4352887, 7
14	9·9293281, 4	9·8379680, 5	9·7193873, 6	9·5782734, 7	9·4180112, 8
15	9·9249310, 9	9·8307144, 3	9·7088104, 8	9·5638794, 9	9·3992730, 1
16	9·9202117, 1	9·8229194, 1	9·6974272, 9	9·5483609, 0	9·3790286, 8
17	9·9151659, 2	9·8145739, 9	9·6852205, 2	9·5316871, 7	9·3572270, 4
18	9·9097893, 1	9·8056681, 7	9·6721711, 6	9·5138242, 8	9·3338101, 5
19	9·9040771, 0	9·7961918, 5	9·6582583, 0	9·4947343, 4	9·3087126, 3
20	9·8980241, 2	9·7861327, 7	9·6434589, 0	9·4743750, 5	9·2818601, 9
21	9·8916247, 9	9·7754783, 8	9·6277474, 5	9·4526989, 5	9·2531682, 3
22	9·8848730, 8	9·7642148, 3	9·6110959, 6	9·4296531, 6	9·2225403, 0
23	9·8777625, 1	9·7523270, 1	9·5934734, 0	9·4051779, 6	9·1898654, 3
24	9·8702860, 8	9·7397985, 0	9·5748455, 8	9·3792063, 5	9·1550157, 9
25	9·8624363, 2	9·7266114, 2	9·5551746, 3	9·3516625, 0	9·1178428, 0
26	9·8542051, 2	9·7127462, 5	9·5344184, 9	9·3224603, 0	9·0781724, 8
27	9·8455837, 9	9·6981816, 8	9·5125304, 2	9·2915015, 6	9·0357994, 7
28	9·8365629, 7	9·6828944, 4	9·4894582, 9	9·2586736, 5	8·9904788, 4
29	9·8271326, 1	9·6668590, 7	9·4651436, 9	9·2238463, 8	8·9419149, 3
30	9·8172818, 5	9·6500476, 7	9·4395211, 1	9·1868685, 4	8·8897460, 3
31	9·8069990, 1	9·6324295, 7	9·4125165, 6	9·1475628, 0	8·8335229, 6
32	9·7962715, 0	9·6139710, 7	9·3840462, 8	9·1057193, 5	8·7726767, 7
33	9·7850857, 2	9·5946349, 3	9·3540147, 3	9·0610868, 3	8·7064703, 6
34	9·7734269, 5	9·5743799, 4	9·3223124, 1	9·0133612, 2	8·6339245, 3
35	9·7612793, 3	9·5531604, 3	9·2888130, 2	8·9621686, 7	8·5536926, 5
36	9·7486256, 2	9·5309254, 6	9·2533696, 1	8·9070423, 2	8·4638468, 7
37	9·7354471, 4	9·5076181, 6	9·2158100, 0	8·8473873, 7	8·3614753, 0
38	9·7217235, 7	9·4831745, 8	9·1759301, 7	8·7824281, 2	8·2418564, 0
39	9·7074328, 1	9·4575226, 4	9·1334859, 0	8·7111240, 5	8·0965193, 2
40	9·6925507, 0	9·4305807, 3	9·0881815, 7	8·6320315, 0	7·9077203, 1

41	9·6770508, 9	9·4022556, 7	9·0396532, 9	8·5430560, 8	7·6262869, 8
42	9·6609044, 3	9·3724406, 1	8·9874465, 1	8·4409853, 9	6·9553413, 3
43	9·6440795, 4	9·3410122, 1	8·9309819, 7	8·3204991, 8	7·2826153, 2 n
44	9·6265411, 8	9·3078267, 9	8·8695039, 5	8·1717615, 5	7·6286453, 4 n
45	9·6082505, 9	9·2727156, 1	8·8019990, 4	7·9731287, 7	7·7875554, 0 n
46	9·5891648, 2	9·2354787, 2	8·7270614, 8	7·6582451, 8	7·8798318, 9 n
47	9·5692359, 9	9·1958763, 1	8·6426553, 1	6·4684950, 2	7·9364328, 1 n
48	9·5484106, 2	9·1536172, 6	8·5456639, 3	7·5305288, 2 n	7·9699254, 0 n
49	9·5266286, 4	9·1083430, 1	8·4309458, 3	7·8147848, 1 n	7·9865325, 8 n
50	9·5038223, 1	9·0596046, 5	8·2890580, 2	7·9607380, 3 n	7·9896500, 1 n
51	9·4799148, 1	9·0068288, 2	8·0994672, 5	8·0508014, 7 n	7·9811768, 8 n
52	9·4548185, 0	8·9492663, 9	7·8008915, 9	8·1096399, 6 n	7·9620874, 5 n
53	9·4284328, 7	8·8859109, 3	6·8421222, 6	8·1479287, 9 n	7·9326852, 8 n
54	9·4006417, 3	8·8153637, 3	7·6352745, 9 n	8·1711821, 4 n	7·8926805, 1 n
55	9·3713097, 9	8·7355953, 1	7·9409789, 2 n	8·1825384, 8 n	7·8411353, 8 n
56	9·3402782, 2	8·6434945, 3	8·0987734, 8 n	8·1838620, 8 n	7·7762455, 9 n
57	9·3073586, 6	8·5339208, 2	8·1988912, 2 n	8·1762501, 9 n	7·6948525, 4 n
58	9·2723252, 8	8·3974180, 2	8·2674779, 7 n	8·1602713, 0 n	7·5912895, 0 n
59	9·2349042, 6	8·2133918, 3	8·3157363, 6 n	8·1360829, 2 n	7·4544850, 8 n
60	9·1947586, 2	7·9202266, 7	8·3494603, 4 n	8·1034647, 0 n	7·2585487, 0 n
61	9·1514670, 9	6·9555530, 6	8·3719901, 8 n	8·0617818, 5 n	6·9145969, 9 n
62	9·1044930, 2	7·7717175, 3 n	8·3853806, 7 n	8·0098743, 7 n	6·0410372, 1
63	9·0531375, 1	8·0835943, 5 n	8·3909349, 3 n	7·9458116, 5 n	6·9842482, 0
64	8·9964658, 3	8·2487702, 1 n	8·3894735, 3 n	7·8663913, 3 n	7·2376543, 2
65	8·9331867, 7	8·3569101, 9 n	8·3814788, 0 n	7·7660386, 3 n	7·3786558, 5
66	8·8614430, 0	8·4340939, 6 n	8·3671691, 1 n	7·6340367, 1 n	7·4691369, 1
67	8·7784201, 4	8·4915390, 2 n	8·3465311, 5 n	7·4457536, 2 n	7·5295573, 9
68	8·6795461, 7	8·5350830, 0 n	8·3193226, 3 n	7·1193251, 9 n	7·5691139, 6
69	8·5566252, 0	8·5681291, 8 n	8·2850399, 0 n	5·9800488, 5	7·5925442, 5
70	8·3925615, 5	8·5928127, 4 n	8·2428459, 3 n	7·1533239, 1	7·6025016, 6
71	8·1406038, 8	8·6105383, 7 n	8·1914348, 1 n	7·4230704, 8	7·6003618, 4
72	7·5355257, 7	8·6222545, 5 n	8·1287720, 1 n	7·5747429, 3	7·5866626, 8
73	7·8108438, 6 n	8·6286052, 1 n	8·0515873, 8 n	7·6750107, 7	7·5612373, 8
74	8·2005246, 5 n	8·6300166, 8 n	7·9542729, 1 n	7·7454214, 6	7·5231713, 7
75	8·3936462, 7 n	8·6267488, 1 n	7·8261337, 9 n	7·7956249, 3	7·4705677, 8
76	8·5200448, 2 n	8·6189234, 4 n	7·6427038, 6 n	7·8306906, 5	7·3999885, 2
77	8·6122084, 1 n	8·6065384, 0 n	7·3231282, 9 n	7·8535199, 6	7·3050933, 0
78	8·6833386, 9 n	8·5894670, 3 n	6·1310730, 5	7·8658238, 5	7·1730580, 1
79	8·7401098, 5 n	8·5674473, 9 n	7·3617165, 1	7·8685651, 0	6·9721611, 7
80	8·7863583, 2 n	8·5400552, 9 n	7·6399593, 8	7·8621707, 5	6·5771814, 8
81	8·8244841, 9 n	8·5066570, 8 n	7·7997828, 9	7·8466209, 8	6·2488556, 8 n
82	8·8560802, 8 n	8·4663270, 1 n	7·9086721, 0	7·8214375, 4	6·8538259, 3 n
83	8·8822492, 1 n	8·4177074, 7 n	7·9884024, 2	7·7856010, 0	7·0864950, 8 n
84	8·9037772, 0 n	8·3587544, 7 n	8·0487489, 7	7·7373198, 9	7·2264246, 1 n
85	8·9212351, 5 n	8·2862466, 8 n	8·0949066, 1	7·6735701, 2	7·3213083, 9 n
86	8·9350412, 3 n	8·1947365, 7 n	8·1299172, 9	7·5890499, 4	7·3883001, 8 n
87	8·9455000, 0 n	8·0739612, 7 n	8·1556648, 3	7·4736081, 2	7·4353951, 9 n
88	8·9528277, 8 n	7·9008246, 8 n	8·1733421, 8	7·3042254, 8	7·4668201, 9 n
89	8·9571691, 8 n	7·6015602, 7 n	8·1836823, 0	7·0071928, 2	7·4848932, 0 n
90	8·9586073, 1 n	∞	8·1870866, 4	∞	7·4907974, 8 n

TABLE OF LOG G'^{5}_{n} FOR VALUES OF θ FROM 0° TO 90°.

$$\mu' = \cos\theta'.$$

θ	Log G'^{5}_{7}	Log G'^{5}_{8}	Log G'^{5}_{9}	Log G'^{5}_{10}
0°	9·9652378, 9	9·9030899, 9	9·8187691, 0	9·7161067, 6
1	9·9650926, 3	9·9028553, 4	9·8184338, 9	9·7156598, 0
2	9·9646567, 5	9·9021511, 6	9·8174278, 3	9·7143182, 7
3	9·9639299, 1	9·9009768, 2	9·8157498, 3	9·7120804, 0
4	9·9629115, 9	9·8993312, 3	9·8133980, 0	9·7089431, 7
5	9·9616010, 2	9·8972128, 8	9·8103697, 3	9·7049024, 1
6	9·9599972, 4	9·8946197, 8	9·8066615, 3	9·6999525, 1
7	9·9580990, 2	9·8915495, 6	9·8022692, 2	9·6940867, 7
8	9·9559049, 5	9·8879992, 9	9·7971877, 3	9·6872970, 0
9	9·9534133, 7	9·8839656, 8	9·7914112, 2	9·6795737, 2
10	9·9506223, 7	9·8794448, 1	9·7849328, 1	9·6709058, 4
11	9·9475298, 1	9·8744323, 4	9·7777448, 1	9·6612808, 0
12	9·9441332, 7	9·8689233, 7	9·7698385, 4	9·6506843, 8
13	9·9404301, 2	9·8629124, 7	9·7612042, 8	9·6391005, 8
14	9·9364173, 9	9·8563935, 9	9·7518312, 2	9·6265115, 2
15	9·9320918, 9	9·8493601, 0	9·7417073, 4	9·6128971, 9
16	9·9274501, 0	9·8418046, 9	9·7308194, 7	9·5982355, 0
17	9·9224881, 9	9·8337196, 1	9·7191530, 2	9·5825017, 9
18	9·9172020, 3	9·8250955, 6	9·7066920, 1	9·5656687, 9
19	9·9115871, 2	9·8159236, 2	9·6934188, 3	9·5477062, 9
20	9·9056386, 2	9·8061933, 2	9·6793142, 5	9·5285807, 8
21	9·8993513, 1	9·7958934, 5	9·6643571, 4	9·5082551, 6
22	9·8927195, 8	9·7850118, 6	9·6485242, 9	9·4866880, 8
23	9·8857373, 4	9·7735353, 1	9·6317902, 6	9·4638338, 0
24	9·8783981, 2	9·7614495, 1	9·6141271, 6	9·4396411, 7
25	9·8706949, 1	9·7487389, 1	9·5955042, 9	9·4140531, 5
26	9·8626202, 0	9·7353866, 2	9·5758878, 6	9·3870057, 8
27	9·8541659, 4	9·7213742, 8	9·5552405, 2	9·3584271, 2
28	9·8453234, 5	9·7066819, 5	9·5335211, 4	9·3282360, 2
29	9·8360834, 2	9·6912878, 9	9·5106839, 9	9·2963403, 8
30	9·8264358, 6	9·6751683, 9	9·4866782, 9	9·2626350, 9
31	9·8163700, 0	9·6582881, 1	9·4614475, 2	9·2269998, 7
32	9·8058742, 6	9·6406472, 2	9·4349282, 1	9·1892953, 1
33	9·7949362, 0	9·6221862, 3	9·4070493, 7	9·1493597, 3
34	9·7835423, 7	9·6028805, 1	9·3777305, 4	9·1070029, 6
35	9·7716782, 7	9·5826925, 1	9·3468805, 6	9·0619995, 3
36	9·7593282, 5	9·5615808, 1	9·3143954, 8	9·0140792, 2
37	9·7464753, 5	9·5394993, 9	9·2801558, 5	8·9629136, 9
38	9·7331012, 3	9·5163972, 0	9·2440235, 0	8·9080980, 6
39	9·7191859, 9	9·4922172, 5	9·2058374, 5	8·8491242, 1
40	9·7047079, 4	9·4668955, 6	9·1654080, 7	8·7853408, 0

41	9·6896435, 0	9·4403602, 1	9·1225101, 5	8·7158923, 3
42	9·6739669, 2	9·4125298, 6	9·0768728, 6	8·6396218, 4
43	9·6576500, 4	9·3833119, 1	9·0281661, 7	8·5549026, 7
44	9·6406619, 2	9·3526002, 6	8·9759812, 9	8·4593395, 5
45	9·6229685, 6	9·3202727, 6	8·9198035, 4	8·3491795, 8
46	9·6045324, 5	9·2861874, 6	8·8589703, 1	8·2180168, 8
47	9·5853120, 4	9·2501780, 7	8·7926069, 6	8·0534487, 8
48	9·5652612, 0	9·2120479, 2	8·7195229, 5	7·8258848, 4
49	9·5443285, 0	9·1715618, 9	8·6380347, 6	7·4254510, 7
50	9·5224562, 9	9·1284353, 9	8·5456436, 6	6·9100691, 8 n
51	9·4995798, 0	9·0823190, 6	8·4383956, 6	7·5748065, 1 n
52	9·4756257, 3	9·0327769, 9	8·3094523, 7	7·7920041, 6 n
53	9·4505108, 6	8·9792548, 3	8·1453151, 5	7·9114261, 0 n
54	9·4241399, 1	8·9210316, 3	7·9126667, 6	7·9853779, 7 n
55	9·3964033, 0	8·8571440, 2	7·4762314, 9	8·0321378, 7 n
56	9·3671739, 1	8·7862615, 0	7·2023902, 2 n	8·0601540, 7 n
57	9·3363032, 0	8·7064696, 8	7·7482221, 5 n	8·0739231, 8 n
58	9·3036158, 7	8·6148623, 2	7·9564124, 1 n	8·0760047, 4 n
59	9·2689030, 6	8·5067033, 2	8·0773024, 8 n	8·0678661, 5 n
60	9·2319129, 9	8·3734525, 1	8·1563186, 2 n	8·0502625, 8 n
61	9·1923380, 9	8·1971040, 5	8·2101060, 1 n	8·0234129, 8 n
62	9·1497969, 9	7·9272764, 1	8·2465788, 3 n	7·9870510, 3 n
63	9·1038084, 4	7·2485151, 8	8·2700601, 4 n	7·9403790, 0 n
64	9·0537525, 9	7·6351850, 7 n	8·2831075, 7 n	7·8819019, 6 n
65	8·9988112, 7	7·9918482, 6 n	8·2872894, 5 n	7·8090638, 7 n
66	8·9378723, 6	8·1680588, 4 n	8·2835578, 8 n	7·7174541, 3 n
67	8·8693676, 8	8·2800832, 5 n	8·2724373, 7 n	7·5989042, 7 n
68	8·7909816, 6	8·3583765, 4 n	8·2541181, 6 n	7·4360703, 7 n
69	8·6990809, 1	8·4154965, 2 n	8·2284894, 1 n	7·1807937, 3 n
70	8·5874674, 9	8·4578169, 8 n	8·1951274, 7 n	6·5483157, 0 n
71	8·4441863, 1	8·4889864, 8 n	8·1532280, 9 n	6·8691671, 6
72	8·2409943, 0	8·5112570, 9 n	8·1014680, 0 n	7·2422133, 5
73	7·8760393, 5	8·5260816, 0 n	8·0377264, 2 n	7·4236324, 8
74	7·2746380, 9 n	8·5344169, 3 n	7·9585256, 6 n	7·5371753, 6
75	8·0321366, 6 n	8·5368835, 4 n	7·8577930, 9 n	7·6143381, 8
76	8·2817267, 8 n	8·5338582, 0 n	7·7236764, 5 n	7·6678963, 7
77	8·4307150, 6 n	8·5255231, 8 n	7·5280279, 3 n	7·7041917, 2
78	8·5346145, 4 n	8·5118884, 4 n	7·1691922, 7 n	7·7267068, 3
79	8·6125929, 4 n	8·4927954, 4 n	6·5698068, 2	7·7374145, 1
80	8·6735610, 9 n	8·4678992, 2 n	7·3315325, 7	7·7373604, 5
81	8·7223792, 7 n	8·4366272, 4 n	7·5824236, 1	7·7269202, 8
82	8·7619781, 0 n	8·3981020, 6 n	7·7310389, 1	7·7058867, 9
83	8·7942484, 6 n	8·3510040, 8 n	7·8329130, 0	7·6734255, 1
84	8·8204686, 6 n	8·2933192, 3 n	7·9071645, 8	7·6278787, 1
85	8·8415300, 3 n	8·2218496, 2 n	7·9626207, 4	7·5663149, 1
86	8·8580649, 9 n	8·1311668, 2 n	8·0040245, 7	7·4835059, 9
87	8·8705231, 2 n	8·0110207, 8 n	8·0341518, 0	7·3693569, 0
88	8·8792180, 0 n	7·8383269, 9 n	8·0546892, 6	7·2008614, 4
89	8·8843565, 4 n	7·5393258, 1 n	8·0666508, 7	6·9043584, 4
90	8·8860566, 5 n	∞	8·0705810, 7	∞

TABLE OF LOG G'^{6}_{n} FOR VALUES OF θ FROM 0° TO 90°.

$$\mu' = \cos \theta'.$$

θ	Log G'^{6}_{8}	Log G'^{6}_{9}	Log G'^{6}_{10}	θ	Log G'^{6}_{8}	Log G'^{6}_{9}	Log G'^{6}_{10}
0°	9·9700367, 8	9·9156791, 1	9·8410455, 0	46°	9·6154661, 9	9·3213258, 4	8·9424973, 1
1	9·9698931, 1	9·9154492, 6	9·8407198, 6	47	9·5967340, 9	9·2875561, 3	8·8848536, 2
2	9·9694620, 2	9·9147594, 7	9·8397425, 9	48	9·5772156, 0	9·2519630, 1	8·8225442, 6
3	9·9687431, 9	9·9136091, 6	9·8381126, 9	49	9·5568647, 5	9·2143708, 4	8·7547032, 7
4	9·9677361, 0	9·9119973, 0	9·8358284, 4	50	9·5356303, 5	9·1745717, 8	8·6801552, 3
5	9·9664400, 3	9·9099225, 4	9·8328875, 1				
6	9·9648540, 5	9·9073830, 2	9·8292868, 1	51	9·5134551, 5	9·1323186, 0	8·5972432, 2
7	9·9629770, 0	9·9043765, 3	9·8250225, 5	52	9·4902747, 7	9·0873027, 5	8·5035168, 2
8	9·9608074, 9	9·9009003, 7	9·8200901, 6	53	9·4660166, 1	9·0391577, 9	8·3951159, 3
9	9·9583439, 6	9·8969514, 9	9·8144844, 5	54	9·4405982, 3	8·9874151, 3	8·2654051, 5
10	9·9555845, 7	9·8925262, 9	9·8081992, 4	55	9·4139253, 8	8·9314783, 1	8·1014036, 9
				56	9·3858899, 5	8·8705674, 1	7·8715799, 4
11	9·9525272, 6	9·8876207, 4	9·8012276, 9	57	9·3563666, 7	8·8036320, 3	7·4532587, 0
12	9·9491697, 4	9·8822303, 5	9·7935620, 3	58	9·3252093, 3	8·7292082, 2	7·0752987, 2 n
13	9·9455094, 6	9·8763501, 1	9·7851936, 5	59	9·2922458, 6	8·6451630, 0	7·6690033, 7 n
14	9·9415436, 1	9·8699744, 4	9·7761128, 7	60	9·2572716, 1	8·5482057, 1	7·8816245, 6 n
15	9·9372691, 1	9·8630972, 4	9·7663090, 9				
16	9·9326826, 2	9·8557118, 5	9·7557706, 6	61	9·2200404, 5	8·4328492, 5	8·0024183, 5 n
17	9·9277804, 7	9·8478109, 2	9·7444846, 7	62	9·1802525, 1	8·2888517, 3	8·0797906, 4 n
18	9·9225587, 2	9·8393865, 2	9·7324370, 5	63	9·1375370, 0	8·0933383, 2	8·1311077, 3 n
19	9·9170130, 8	9·8304299, 7	9·7196124, 0	64	9·0914275, 9	7·7739224, 2	8·1645232, 7 n
20	9·9111389, 4	9·8209318, 9	9·7059938, 7	65	9·0413258, 4	5·8767949, 8	8·1844378, 8 n
				66	8·9864453, 1	7·7166909, 9 n	8·1934123, 6 n
21	9·9049313, 3	9·8108820, 9	9·6915630, 5	67	8·9257221, 6	7·9963658, 7 n	8·1929725, 7 n
22	9·8983848, 9	9·8002695, 1	9·6762997, 8	68	8·8576667, 7	8·1479723, 4 n	8·1839877, 6 n
23	9·8914938, 8	9·7890821, 9	9·6601820, 9	69	8·7800952, 9	8·2467019, 0 n	8·1668553, 2 n
24	9·8842521, 0	9·7773071, 9	9·6431859, 8	70	8·6896171, 1	8·3158790, 7 n	8·1415780, 1 n
25	9·8766529, 3	9·7649304, 8	9·6252850, 9				
26	9·8686892, 6	9·7519369, 1	9·6064506, 8	71	8·5805311, 9	8·3657918, 9 n	8·1077714, 7 n
27	9·8603534, 6	9·7383100, 2	9·5866511, 3	72	8·4420758, 6	8·4018501, 1 n	8·0645979, 8 n
28	9·8516373, 1	9·7240319, 6	9·5658517, 2	73	8·2497685, 6	8·4272063, 9 n	8·0106060, 6 n
29	9·8425320, 5	9·7090834, 3	9·5440143, 0	74	7·9229500, 8	8·4438139, 9 n	7·9433995, 4 n
30	9·8330282, 6	9·6934434, 0	9·5210966, 7	75	6·7089560, 0 n	8·4529181, 2 n	7·8589370, 3 n
				76	7·9481248, 1 n	8·4553061, 9 n	7·7499202, 0 n
31	9·8231157, 8	9·6770889, 8	9·4970521, 9	77	8·2228020, 2 n	8·4514413, 2 n	7·6013700, 0 n
32	9·8127837, 6	9·6599953, 2	9·4718292, 0	78	8·3800295, 3 n	8·4415317, 6 n	7·3743138, 3 n
33	9·8020205, 0	9·6421352, 3	9·4453700, 3	79	8·4874999, 5 n	8·4255632, 4 n	6·8865696, 8 n
34	9·7908134, 1	9·6234790, 0	9·4176103, 7	80	8·5670523, 7 n	8·4032973, 0 n	6·8890438, 5
35	9·7791489, 6	9·6039941, 0	9·3884779, 9				
36	9·7670125, 3	9·5836448, 1	9·3578916, 1	81	8·6285180, 8 n	8·3742419, 2 n	7·3509744, 7
37	9·7543883, 6	9·5623917, 3	9·3257589, 6	82	8·6771530, 8 n	8·3375812, 5 n	7·5583016, 8
38	9·7412594, 1	9·5401914, 6	9·2919750, 8	83	8·7160804, 2 n	8·2920428, 7 n	7·6878962, 2
39	9·7276072, 4	9·5169958, 0	9·2564194, 8	84	8·7472895, 8 n	8·2356493, 7 n	7·7780940, 6
40	9·7134118, 1	9·4927512, 0	9·2189531, 2	85	8·7721078, 1 n	8·1652315, 6 n	7·8436699, 4
				86	8·7914459, 3 n	8·0753823, 4 n	7·8917994, 0
41	9·6986514, 2	9·4673979, 2	9·1794140, 7	87	8·8059353, 4 n	7·9558690, 3 n	7·9264324, 8
42	9·6833024, 0	9·4408689, 2	9·1376120, 8	88	8·8160083, 8 n	7·7836192, 0 n	7·9498704, 0
43	9·6673389, 8	9·4130888, 3	9·0933216, 5	89	8·8219464, 9 n	7·4848804, 1 n	7·9634624, 5
44	9·6507329, 1	9·3839721, 1	9·0462716, 0	90	8·8239087, 4 n	∞	7·9679187, 3
45	9·6334533, 2	9·3534216, 3	8·9961325. 5				

Table of Log G'^{7}_{n} and Log G'^{8}_{n} for Values of θ from 0° to 90°.

$\mu' = \cos\theta'$.

θ	Log G'^{7}_{9}	Log G'^{7}_{10}	Log G'^{8}_{10}	θ	Log G'^{7}_{9}	Log G'^{7}_{10}	Log G'^{8}_{10}
0°	9·9736710, 6	9·9253663, 8	9·9765189, 0	46°	9·6236454, 2	9·3471885, 0	9·6299956, 1
1	9·9735286, 0	9·9251401, 2	9·9763773, 7	47	9·6052703, 0	9·3149486, 2	9·6118928, 7
2	9·9731011, 0	9·9244611, 1	9·9759526, 6	48	9·5861402, 1	9·2810724, 5	9·5930586, 4
3	9·9723882, 7	9·9233288, 2	9·9752445, 1	49	9·5662129, 4	9·2454184, 4	9·5734535, 1
4	9·9713896, 1	9·9217422, 7	9·9742524, 0	50	9·5454417, 0	9·2078213, 8	9·5530339, 2
5	9·9701044, 2	9·9197001, 9	9·9729756, 7	51	9·5237744, 2	9·1680866, 0	9·5317514, 7
6	9·9685317, 8	9·9172008, 3	9·9714134, 1	52	9·5011527, 8	9·1259820, 6	9·5095522, 5
7	9·9666705, 8	9·9142420, 8	9·9695645, 5	53	9·4775113, 1	9·0812279, 2	9·4863759, 4
8	9·9645195, 0	9·9108214, 3	9·9674277, 9	54	9·4527761, 2	9·0334823, 1	9·4621547, 2
9	9·9620769, 9	9·9069360, 2	9·9650016, 3	55	9·4268632, 9	8·9823201, 8	9·4368119, 5
10	9·9593412, 9	9·9025824, 3	9·9622843, 5	56	9·3996770, 4	8·9272041, 6	9·4102606, 1
11	9·9563104, 1	9·8977569, 1	9·9592737, 7	57	9·3711073, 1	8·8674386, 7	9·3324012, 5
12	9·9529821, 3	9·8924552, 3	9·9559684, 4	58	9·3410266, 3	8·8021006, 9	9·3531193, 8
13	9·9493539, 7	9·8866726, 6	9·9523652, 5	59	9·3092862, 7	8·7299244, 7	9·3222823, 2
14	9·9454232, 3	9·8804040, 1	9·9484617, 9	60	9·2757111, 6	8·6491047, 0	9·2897351, 0
15	9·9411869, 4	9·8736435, 3	9·9442551, 9	61	9·2400929, 7	8·5569314, 1	9·2552948, 1
16	9·9366418, 5	9·8663850, 1	9·9397422, 5	62	9·2021811, 2	8·4490511, 4	9·2187435, 3
17	9·9317844, 4	9·8586215, 7	9·9349195, 9	63	9·1616701, 9	8·3177730, 1	9·1798184, 9
18	9·9266108, 9	9·8503458, 0	9·9297834, 8	64	9·1181823, 3	8·1473977, 3	9·1381985, 8
19	9·9211170, 8	9·8415496, 4	9·9243299, 1	65	9·0712419, 2	7·8966739, 0	9·0934852, 9
20	9·9152985, 6	9·8322253, 5	9·9185545, 6	66	9·0202380, 2	7·3626860, 0	9·0451751, 0
21	9·9091505, 5	9·8223596, 8	9·9124527, 9	67	8·9643667, 5	7·4313010, 3 n	8·9926183, 3
22	9·9026678, 7	9·8119474, 6	9·9060195, 9	68	8·9025390, 6	7·8537547, 7 n	8·9349553, 8
23	9·8958450, 1	9·8009746, 0	9·8992495, 9	69	8·8332256, 7	8·0425774, 8 n	8·8710137, 7
24	9·8886760, 3	9·7894298, 1	9·8921370, 5	70	8·7541790, 2	8·1577533, 5 n	8·7991323, 9
25	9·8811545, 5	9·7773001, 2	9·8846757, 9	71	8·6618941, 4	8·2356458, 4 n	8·7168423, 9
26	9·8732737, 5	9·7645715, 2	9·8768592, 0	72	8·5504410, 5	8·2904918, 3 n	8·6202333, 2
27	9·8650263, 1	9·7512289, 3	9·8686802, 2	73	8·4085191, 1	8·3292758, 1 n	8·5025419, 6
28	9·8564043, 8	9·7372559, 8	9·8601312, 4	74	8·2099831, 1	8·3559493, 2 n	8·3504306, 2
29	9·8473995, 6	9·7226349, 7	9·8512041, 5	75	7·8651958, 6	8·3728383, 9 n	8·1310878, 2
30	9·8380028, 3	9·7073467, 3	9·8418902, 7	76	7·0132460, 3 n	8·3813795, 5 n	7·7127321, 7
31	9·8282045, 3	9·6913704, 8	9·8321802, 8	77	7·9474410, 7 n	8·3824734, 4 n	7·4262183, 8 n
32	9·8179943, 0	9·6746836, 5	9·8220641, 7	78	8·2080870, 1 n	8·3766160, 0 n	7·9980355, 9 n
33	9·8073609, 8	9·6572616, 8	9·8115312, 5	79	8·3595020, 4 n	8·3639904, 6 n	8·2224688, 3 n
34	9·7962926, 3	9·6390778, 3	9·8005700, 1	80	8·4632817, 2 n	8·3445007, 4 n	8·3592114, 7 n
35	9·7847763, 7	9·6201028, 9	9·7891681, 2	81	8·5399475, 8 n	8·3177583, 9 n	8·4545026, 3 n
36	9·7727983, 6	9·6003050, 0	9·7773122, 9	82	8·5988686, 1 n	8·2830247, 4 n	8·5252416, 8 n
37	9·7603436, 8	9·5796491, 3	9·7649882, 4	83	8·6450889, 6 n	8·2390851, 6 n	8·5794879, 5 n
38	9·7473962, 2	9·5580968, 2	9·7521805, 5	84	8·6816132, 2 n	8·1840068, 1 n	8·6216869, 5 n
39	9·7339386, 0	9·5356057, 5	9·7388726, 6	85	8·7103518, 9 n	8·1146543, 5 n	8·6562492, 7 n
40	9·7199519, 7	9·5121290, 0	9·7250465, 7	86	8·7325705, 8 n	8·0256457, 9 n	8·6797007, 4 n
41	9·7054159, 2	9·4876146, 3	9·7106828, 8	87	8·7491240, 1 n	7·9067691, 2 n	8·6983517, 4 n
42	9·6903083, 0	9·4620047, 7	9·6957605, 5	88	8·7605870, 5 n	7·7349645, 6 n	8·7112147, 5 n
43	9·6746050, 1	9·4352347, 9	9·6802566, 9	89	8·7673265, 1 n	7·4364870, 6 n	8·7187593, 6 n
44	9·6582797, 1	9·4072318, 9	9·6641464, 2	90	8·7695510, 8 n	∞	∞
45	9·6413036, 9	9·3779141, 5	9·6474026, 3				

VALUES OF G'^0_0, G'^0_1, G'^0_2, G'^0_3 ... G'^0_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^0_0	G'^0_1	G'^0_2	G'^0_3	G'^0_4	G'^0_5	G'^0_6	G'^0_7	G'^0_8	G'^0_9	G'^0_{10}
0°	1	μ'	+0·66666667	+0·40000000	+0·22857143	+0·12698413	+0·06926407	+0·03729604	+0·01989122	+0·01053065	+0·00554245
1			·66635798	·39962961	·22821873	·12669027	·06903973	·03713502	·01978084	·01045763	·00549550
2			·66543228	·39851942	·22716222	·12581070	·06836882	·03665399	·01945150	·01024006	·00535581
3			·66389080	·39667243	·22540672	·12435145	·06725785	·03585918	·01890869	·00988248	·00512695
4			·66173540	·39409343	·22296000	·12232237	·06571737	·03476070	·01816128	·00939220	·00481463
5			·65896893	·39078943	·21983327	·11973743	·06376224	·03337266	·01722165	·00877936	·00442676
6			·65559478	·38676904	·21604043	·11661405	·06141103	·03171274	·01610520	·00805651	·00397302
7			·65161731	·38204302	·21159863	·11297358	·05868628	·02980211	·01483020	·00723843	·00346473
8			·64704148	·37662386	·20652779	·10884067	·05561389	·02766502	·01341744	·00634175	·00291446
9			·64187316	·37052595	·20085070	·10424334	·05222308	·02532854	·01188986	·00538457	·00233573
10			+0·63611881	+0·36376540	+0·19459278	+0·09921267	+0·04854596	+0·02282210	+0·01027208	+0·00438607	+0·00174259
11			+0·62978578	+0·35636012	+0·18778203	+0·09378256	+0·04461722	+0·02017713	+0·00859005	+0·00336610	+0·00114928
12			·62288204	·34832972	·18044896	·08798961	·04047383	·01742662	·00687049	·00234466	·00056982
13			·61541635	·33969539	·17262624	·08187265	·03615458	·01460463	·00514046	·00134152	+0·00001760
14			·60739812	·33047990	·16434870	·07547256	·03169963	·01174581	·00342686	+0·00037579	−0·00049489
15			·59883740	·32070750	·15565310	·06883197	·02715023	·00888494	·00175595	−0·00053460	·00095650
16			·58974509	·31040397	·14657798	·06199493	·02254821	·00605652	+0·00015296	·00137311	·00135760
17			·58013262	·29959637	·13716351	·05500662	·01793555	·00329416	−0·00135847	·00212513	·00169043
18			·57001204	·28831303	·12745111	·04791289	·01335394	+0·00063019	·00275665	·00277821	·00194924
19			·55939617	·27658361	·11748353	·04076011	·00884437	−0·00190474	·00402211	·00332222	·00213034
20			+0·54829829	+0·26443870	+0·10730437	+0·03359467	+0·00444674	−0·00428212	−0·00513810	−0·00374970	−0·00223229
21			+0·53673244	+0·25191009	+0·09695806	+0·02646271	+0·00019935	−0·00647599	−0·00609072	−0·00405591	−0·00225585
22			·52471301	·23903037	·08648952	·01940981	−0·00386133	·00846317	·00686922	·00423892	·00220393
23			·51225516	·22583307	·07594404	·01248058	·00770116	·01022363	·00746607	·00429956	·00208140
24			·49937449	·21235246	·06536704	+0·00571849	·01128855	·01174074	·00787714	·00424142	·00189504
25			·48608709	·19862341	·05480376	−0·00083467	·01459489	·01300143	·00810170	·00407071	·00165319
26			·47240964	·18468145	·04429918	·00713888	·01759475	·01399641	·00814238	·00379603	·00136555
27			·45835911	·17056238	·03389769	·01315625	·02026607	·01472010	·00800502	·00342817	·00104284
28			·44395317	·15630261	·02364302	·01885130	·02259049	·01517087	·00769864	·00297983	·00069651
29			·42920967	·14193858	·01357783	·02419122	·02455342	·01535086	·00723511	·00246521	−0·00033833
30			+0·41414698	+0·12750697	+0·00374369	−0·02914604	−0·02614416	−0·01526598	−0·00662894	−0·00189972	+0·00001993
31			+0·39878389	+0·11304459	−0·00581913	−0·03368881	−0·02735596	−0·01492571	−0·00589695	−0·00129960	+0·00036681
32			·38313939	·09858801	·01507202	·03779589	·02818615	·01434304	·00505795	·00068147	·00069157
33			·36723296	·08417382	·02397798	·04144701	·02863603	·01353410	·00413230	−0·00006194	·00098443
34			·35108428	·06983829	·03250192	·04462540	·02871083	·01251797	·00314154	+0·00054279	·00123687
35			·33471330	·05561732	·04061080	·04731791	·02841963	·01131633	·00210796	·00111726	·00144180
36			·31814030	·04154639	·04827379	·04951513	·02777525	·00995314	·00105415	·00164719	·00159386
37			·30138570	·02766039	·05546235	·05121130	·02679402	·00845424	−0·00000257	·00211980	·00168947
38			·28447014	·01399362	·06215041	·05240446	·02549560	·00684693	+0·00102487	·00252400	·00172687
39			·26741446	+0·00057960	·06831448	·05309633	·02390273	·00515960	·00200723	·00285072	·00170617
40			+0·25023963	−0·01254893	−0·07393371	−0·05329237	−0·02204095	−0·00342125	+0·00292494	+0·00309309	+0·00162937

41	+0·23296670	−0·02536020	−0·07899001	−0·05300159	−0·01993827	−0·00166107	+0·00376013	+0·00324651	+0·00150015
42	·21561683	·03782343	·08346815	·05223658	·01762487	+0·00009200	·00449697	·00330877	·00132379
43	·19821133	·04990881	·08735570	·05101327	·01513279	·00180962	·00512194	·00328008	·00110696
44	·18077141	·06158783	·09064323	·04935084	·01249539	·00346466	·00562407	·00316296	·00085751
45	·16331834	·07283320	·09332427	·04727158	·00974722	·00503133	·00599506	·00296216	·00058413
46	·14587346	·08361881	·09539525	·04480062	·00692341	·00648577	·00622949	·00268455	·00029618
47	·12845797	·09392003	·09685562	·04196575	·00405943	·00780621	·00632479	·00233882	+0·00000326
48	·11109301	·10371366	·09770781	·03879723	−0·00119060	·00897341	·00628133	·00193532	−0·00028500
49	·09379970	·11297793	·09795712	·03532747	+0·00164823	·00997084	·00610231	·00148570	·00055931
50	+0·07659899	−0·12169264	−0·09761176	−0·03159079	+0·00442312	+0·01078487	+0·00579367	+0·00100266	−0·00081094
51	+0·05951170	−0·12983924	−0·09668281	−0·02762317	+0·00710138	+0·01140504	+0·00536393	+0·00049953	−0·00103204
52	·04255846	·13740077	·09518403	·02346191	·00965187	·01182405	·00482399	−0·00001001	·00121587
53	·02575977	·14436196	·09313195	·01914539	·01204541	·01203796	·00418688	·00051233	·00135697
54	+0·00913584	·15070927	·09054559	·01471269	·01425506	·01204607	·00346742	·00099417	·00145134
55	−0·00729328	·15643090	·08744650	·01020340	·01625632	·01185104	·00268201	·00144295	·00149650
56	·02350787	·16151685	·08385858	·00565718	·01802751	·01145866	·00184813	·00184719	·00149164
57	·03948846	·16595891	·07980793	−0·00111356	·01954989	·01087784	·00098406	·00219669	·00143755
58	·05521587	·16975069	·07532279	+0·00338843	·02080786	·01012038	+0·00010853	·00248278	·00133656
59	·07067127	·17288761	·07043327	·00781051	·02178911	·00920076	−0·00075977	·00269860	·00119252
60	−0·08583619	−0·17536695	−0·06517129	+0·01211552	+0·02248476	+0·00813590	−0·00160248	−0·00283917	−0·00101060
61	−0·10069251	−0·17718781	−0·05957038	+0·01626759	+0·02288934	+0·00694484	−0·00240199	−0·00290157	−0·00079715
62	·11522251	·17835116	·05366551	·02023248	·02300094	·00564849	·00314174	·00288493	·00055946
63	·12940887	·17885977	·04749292	·02397785	·02282108	·00426916	·00380666	·00279049	·00030552
64	·14323471	·17871823	·04108991	·02747344	·02235474	·00283034	·00438331	·00262150	−0·00004376
65	·15668359	·17793292	·03449464	·03069131	·02161024	+0·00135624	·00486025	·00238314	+0·00021719
66	·16973954	·17651203	·02774603	·03360609	·02059912	−0·00012860	·00522820	·00208236	·00046885
67	·18238709	·17446547	·02088344	·03619510	·01933605	·00159965	·00548029	·00172771	·00070306
68	·19461122	·17180487	·01394655	·03843353	·01783845	·00303282	·00561207	·00132906	·00091235
69	·20639748	·16854356	·00697516	·04031960	·01612649	·00440483	·00562166	·00089736	·00109006
70	−0·21773194	−0·16469652	−0·00000894	+0·04182465	+0·01422273	−0·00569352	−0·00550973	−0·00044436	+0·00123060
71	−0·22860119	−0·16028030	+0·00691270	+0·04294322	+0·01215180	−0·00687827	−0·00527950	+0·00001774	+0·00132970
72	·23899241	·15531302	·01375085	·04366815	·00994022	·00794024	·00493664	·00047655	·00138435
73	·24889335	·14981431	·02046727	·04399557	·00761598	·00886272	·00448914	·00091989	·00139307
74	·25829232	·14380523	·02702458	·04392497	·00520825	·00963130	·00394710	·00133602	·00135581
75	·26717829	·13730825	·03338645	·04345914	·00274704	·01023420	·00332258	·00171399	·00127407
76	·27554077	·13034712	·03951778	·04260414	+0·00026283	·01066234	·00262928	·00204396	·00115073
77	·28336995	·12294688	·04538487	·04136924	−0·00221377	·01090946	·00188226	·00231733	·00099000
78	·29065662	·11513376	·05095555	·03976689	·00465238	·01097236	·00109765	·00252709	·00079731
79	·29739223	·10693507	·05619939	·03781247	·00702322	·01085072	−0·00029225	·00266791	·00057908
80	−0·30356887	−0·09837919	+0·06108782	+0·03552429	−0·00929743	−0·01054721	+0·00051664	+0·00273627	+0·00034251
81	−0·30917930	−0·08949549	+0·06559425	+0·03292338	−0·01144739	−0·01006750	+0·00131187	+0·00273064	+0·00009540
82	·31421694	·08031414	·06969424	·03003331	·01344709	·00942005	·00207658	·00265136	−0·00015417
83	·31867589	·07086619	·07336561	·02687998	·01527237	·00861598	·00279455	·00250077	·00039804
84	·32255093	·06118339	·07658849	·02349146	·01690122	·00766892	·00345064	·00228306	·00062826
85	·32583751	·05129807	·07934548	·01989769	·01831403	·00659481	·00403102	·00200417	·00083736
86	·32853179	·04124317	·08162174	·01613029	·01949383	·00541156	·00452347	·00167165	·00101854
87	·33063062	·03105205	·08340498	·01222225	·02042645	·00413885	·00491763	·00129443	·00116592
88	·33213154	·02075844	·08468562	·00820776	·02110072	·00279779	·00520524	·00088264	·00127474
89	·33303279	·01039636	·08545678	·00412183	·02150853	·00141053	·00538026	·00044729	·00134148
90	−0·33333333	−0·00000000	+0·08571429	+0·00000000	−0·02164502	−0·00000000	+0·00543901	+0·00000000	−0·00136396

VALUES OF G'^1_1, G'^1_2, G'^1_3, G'^1_4 ... G'^1_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^1_1	G'^1_2	G'^1_3	G'^1_4	G'^1_5	G'^1_6	G'^1_7	G'^1_8	G'^1_9	G'^1_{10}
0°	1	μ'	+0·80000000	+0·57142857	+0·38095238	+0·24242424	+0·14918415	+0·08951049	+0·05265323	+0·03048345
1			·79969131	·57103173	·38054090	·24205022	·14887348	·08926891	·05247462	·03035657
2			·79876562	·56984217	·37930806	·24093032	·14794393	·08854666	·05194114	·02997801
3			·79722412	·56786290	·37725893	·23907123	·14640308	·08735144	·05105995	·02935400
4			·79506875	·56509881	·37440168	·23648376	·14426318	·08569565	·04984260	·02849462
5			·79230224	·56155690	·37074797	·23318325	·14154151	·08359673	·04830527	·02741392
6			·78892811	·55724597	·36631246	·22918899	·13825979	·08107651	·04646813	·02612930
7			·78495062	·55217672	·36111302	·22452443	·13444424	·07816120	·04435519	·02466136
8			·78037481	·54636194	·35517062	·21921683	·13012521	·07488094	·04199394	·02303347
9			·77520647	·53981602	·34850908	·21329709	·12533687	·07126945	·03941482	·02127126
10			+0·76945215	+0·53255525	+0·34115507	+0·20679959	+0·12011695	+0·06736356	+0·03665072	+0·01940211
11			+0·76311912	+0·52459779	+0·33313806	+0·19976192	+0·11450633	+0·06320278	+0·03373654	+0·01745464
12			·75621539	·51596334	·32448999	·19222462	·10854864	·05882883	·03070862	·01545820
13			·74874968	·50667331	·31524523	·18423088	·10228991	·05428510	·02760411	·01344222
14			·74073142	·49675068	·30544037	·17582621	·09577802	·04961606	·02446047	·01143572
15			·73217074	·48622005	·29511417	·16705822	·08906229	·04486672	·02131474	·00946667
16			·72307842	·47510734	·28430720	·15797621	·08219308	·04008217	·01820320	·00756161
17			·71346593	·46343993	·27306177	·14863084	·07522124	·03530696	·01516070	·00574514
18			·70334538	·45124644	·26142166	·13907379	·06819764	·03058454	·01222009	·00403940
19			·69272950	·43855681	·24943201	·12935745	·06117277	·02595686	·00941190	·00246383
20			+0·68163164	+0·42540198	+0·23713898	+0·11953446	+0·05419616	+0·02146374	+0·00676379	+0·00103478
21			+0·67006575	+0·41181405	+0·22458963	+0·10965747	+0·04731606	+0·01714254	+0·00430025	−0·00023467
22			·65804635	·39782600	·21183167	·09977868	·04057887	·01302762	·00204222	·00133494
23			·64558850	·38347175	·19891329	·08994963	·03402891	·00915014	+0·00000695	·00225990
24			·63270782	·36878587	·18588280	·08022066	·02770783	·00553756	−0·00179228	·00300704
25			·61942043	·35380372	·17278860	·07064083	·02165446	+0·00221351	·00334622	·00357730
26			·60574295	·33856117	·15967879	·06125733	·01590431	−0·00080251	·00464966	·00397492
27			·59169246	·32309457	·14660102	·05211538	·01048938	·00349525	·00570146	·00420729
28			·57728649	·30744062	·13360234	·04325791	·00543797	·00585378	·00650442	·00428470
29			·56254299	·29163635	·12072887	·03472521	+0·00077432	·00787158	·00706506	·00421985
30			+0·54748032	+0·27571888	+0·10802565	+0·02655474	−0·00348143	−0·00954649	−0·00739348	−0·00402773
31			+0·53211721	+0·25972545	+0·09553651	+0·01878096	−0·00731335	−0·01088059	−0·00750302	−0·00372499
32			·51647273	·24369323	·08330373	·01143505	·01070985	·01188010	·00740992	·00332967
33			·50056630	·22765926	·07136798	+0·00454476	·01366373	·01255523	·00713296	·00286063
34			·48441761	·21166036	·05976810	−0·00186573	·01617207	·01291982	·00669306	·00233718
35			·46804664	·19573296	·04854095	·00777590	·01823621	·01299114	·00611277	·00177860
36			·45147363	·17991310	·03772119	·01316905	·01986172	·01278956	·00541595	·00120374
37			·43471903	·16423627	·02734128	·01803226	·02105809	·01233811	·00462709	·00063049
38			·41780347	·14873729	·01743120	·02235652	·02183871	·01166219	·00377109	−0·00007562
39			·40074780	·13345031	+0·00801842	·02613668	·02222055	·01078903	·00287257	+0·00044574
40			+0·38357295	+0·11840868	−0·00087219	−0·02937141	−0·02222392	−0·00974740	−0·00195561	+0·00092029

41			+0·36630002	+0·10364477	−0·00321859	−0·03206327	−0·02187224	−0·00856700	−0·00104319	+0·00133683
42			·34895017	·08919002	·01700144	·03421839	·02119162	·00727813	−0·00015687	·00168640
43			·33154465	·07507480	·02420433	·03584663	·02021060	·00591117	+0·00068355	·00196242
44			·31410472	·06132831	·03081377	·03696128	·01895981	·00449623	·00146034	·00216070
45			·29665168	·04797854	·03681918	·03757893	·01747150	·00306258	·00215816	·00227946
46			·27920679	·03505221	·04221303	·03771936	·01577928	·00163840	·00276423	·00231933
47			·26179129	·02257463	·04699062	·03740522	·01391760	−0·00025030	·00326838	·00228303
48			·24442635	+0·01056974	·05115040	·03666193	·01192146	+0·00107700	·00366329	·00217540
49			·22713304	−0·00094006	·05469368	·03551739	·00982597	·00232094	·00394445	·00200306
50			+0·20993232	−0·01193387	−0·05762466	−0·03400168	−0·00766596	+0·00346144	+0·00411016	+0·00177425
51			+0·19284503	−0·02239236	−0·05995041	−0·03214688	−0·00547563	+0·00448110	+0·00416143	+0·00149840
52			·17589179	·03229790	·06168085	·02998675	·00328817	·00536537	·00410188	·00118598
53			·15909309	·04163445	·06282850	·02755639	−0·00113539	·00610273	·00393753	·00084803
54			·14246917	·05038779	·06340860	·02489204	+0·00095255	·00668471	·00367661	·00049592
55			·12604004	·05854533	·06343886	·02203073	·00294746	·00710595	·00332929	+0·00014103
56			·10982545	·06609634	·06293944	·01900997	·00482344	·00736420	·00290739	−0·00020564
57			·09384488	·07303183	·06193272	·01586747	·00655701	·00746023	·00242403	·00053372
58			·07811747	·07934466	·06044327	·01264080	·00812745	·00739775	·00189338	·00083375
59			·06266206	·08502949	·05849764	·00936716	·00951688	·00718321	·00133021	·00109738
60			+0·04749714	−0·09008280	−0·05612421	−0·00608303	+0·01071045	+0·00682571	+0·00074962	−0·00131765
61			+0·03264082	−0·09450292	−0·05335308	−0·00282391	+0·01169640	+0·00633663	+0·00016664	−0·00148906
62			·01811082	·09829002	·05021584	+0·00037591	·01246615	·00572946	−0·00040408	·00160777
63			+0·00392446	·10144607	·04674541	·00348370	·01301435	·00501949	·00094861	·00167159
64			−0·00990138	·10397486	·04297588	·00646842	·01333882	·00422354	·00145398	·00168002
65			·02335026	·10588198	·03894231	·00930102	·01344052	·00335953	·00190860	·00163426
66			·03640621	·10717478	·03468054	·01195463	·01332350	·00244624	·00230227	·00153707
67			·04905375	·10786239	·03022702	·01440472	·01299471	·00150290	·00262656	·00139271
68			·06127789	·10795559	·02561854	·01662932	·01246393	+0·00054887	·00287489	·00120670
69			·07306415	·10746690	·02089214	·01860912	·01174353	−0·00039672	·00304268	·00098579
70			−0·08439861	−0·10641045	−0·01608486	+0·02032758	+0·01084827	−0·00131528	−0·00312736	−0·00073754
71			−0·09526786	−0·10480198	−0·01123356	+0·02177110	+0·00979511	−0·00218911	−0·00312842	−0·00047024
72			·10565908	·10265879	·00637469	·02292900	·00860287	·00300169	·00304738	−0·00019255
73			·11556001	·09999963	−0·00154416	·02379363	·00729203	·00373793	·00288775	+0·00008666
74			·12495900	·09684476	+0·00322286	·02436038	·00588438	·00438445	·00265485	·00035867
75			·13384496	·09321576	·00789218	·02462768	·00440274	·00492979	·00235571	·00061510
76			·14220744	·08913555	·01243065	·02459696	·00287066	·00536453	·00199886	·00084815
77			·15003662	·08462833	·01680647	·02427262	+0·00131206	·00568152	·00159412	·00105086
78			·15732329	·07971945	·02098921	·02366195	−0·00024905	·00587593	·00115234	·00121729
79			·16405890	·07443540	·02495007	·02277502	·00178894	·00594531	·00068515	·00134263
80			−0·17023554	−0·06880369	+0·02866200	+0·02162456	−0·00328442	−0·00588961	−0·00020466	+0·00142342
81			−0·17584597	−0·06285280	+0·03209978	+0·02022585	−0·00471318	−0·00571117	+0·00027682	+0·00145755
82			·18088361	·05661208	·03524023	·01859653	·00605404	·00541465	·00074706	·00144438
83			·18534256	·05011170	·03806226	·01675641	·00728728	·00500691	·00119421	·00138468
84			·18921760	·04338252	·04054704	·01472731	·00839484	·00449689	·00160707	·00128066
85			·19250418	·03645605	·04267802	·01253283	·00936060	·00389543	·00197536	·00113583
86			·19519846	·02936434	·04444107	·01019809	·01017051	·00321505	·00228993	·00095498
87			·19729729	·02213989	·04582455	·00774956	·01081292	·00246972	·00254304	·00074391
88			·19879821	·01481556	·04681930	·00521473	·01127849	·00167462	·00272840	·00050936
89			·19969946	·00742448	·04741878	·00262192	·01156055	·00084581	·00284148	·00025875
90			−0·20000000	−0·00000000	+0·04761905	+0·00000000	−0·01165501	−0·00000000	+0·00287947	+0·00000000

VALUES OF G'^2_2, G'^2_3, G'^2_4, G'^2_5 ... G'^2_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^2_2	G'^2_3	G'^2_4	G'^2_5	G'^2_6	G'^2_7	G'^2_8	G'^2_9	G'^2_{10}
0°	1	μ'	+0·85714286	+0·66666667	+0·48484849	+0·33566434	+0·22377622	+0·14479638	+0·09145035
1			·85683417	·66625510	·48439957	·33523276	·22339651	·14448373	·09120591
2			·85590848	·66502146	·48305455	·33394033	·22226003	·14354861	·09047539
3			·85436698	·66296871	·48081857	·33179412	·22037516	·14199988	·08926741
4			·85221161	·66010180	·47770005	·32880566	·21775548	·13985187	·08759587
5			·84944510	·65642773	·47371099	·32499130	·21442011	·13712470	·08548027
6			·84607097	·65195536	·46886652	·32037156	·21039309	·13384358	·08294494
7			·84209348	·64669547	·46318496	·31497136	·20570348	·13003885	·08001897
8			·83751767	·64066087	·45668790	·30881972	·20038487	·12574536	·07673558
9			·83234933	·63386607	·44939993	·30194970	·19447534	·12100236	·07313192
10			+0·82659501	+0·62632743	+0·44134842	+0·29439783	+0·18801670	+0·11585260	+0·06924803
11			+0·82026198	+0·61806314	+0·43256376	+0·28620432	+0·18105460	+0·11034232	+0·06512682
12			·81335825	·60909311	·42307887	·27741247	·17363770	·10452030	·06081299
13			·80589254	·59943880	·41292916	·26806846	·16581748	·09843757	·05635268
14			·79787428	·58912337	·40215241	·25822109	·15764762	·09214659	·05179256
15			·78931360	·57817146	·39078854	·24792135	·14918364	·08570086	·04717941
16			·78022128	·56660921	·37887949	·23722227	·14048224	·07915407	·04255926
17			·77060879	·55446411	·36646888	·22617826	·13160107	·07255985	·03797699
18			·76048824	·54176502	·35360206	·21484512	·12259795	·06597080	·03347546
19			·74987236	·52854192	·34032563	·20327940	·11353057	·05943819	·02909523
20			+0·73877450	+0·51482603	+0·32668740	+0·19153811	+0·10445577	+0·05301121	+0·02487378
21			+0·72720861	+0·50064959	+0·31273614	+0·17967840	+0·09542931	+0·04673663	+0·02084527
22			·71518921	·48604582	·29852129	·16775713	·08650521	·04065819	·01704001
23			·70273136	·47104877	·28409284	·15583056	·07773548	·03481634	·01348430
24			·68985068	·45569332	·26950107	·14395401	·06916955	·02924764	·01019996
25			·67656329	·44001500	·25479626	·13218137	·06085387	·02398454	·00720429
26			·66288581	·42404994	·24002861	·12056503	·05283179	·01905526	·00451001
27			·64883532	·40783466	·22524773	·10915520	·04514288	·01448325	·00212516
28			·63442935	·39140620	·21050287	·09800004	·03782303	·01028741	+0·00005314
29			·61968585	·37480180	·19584231	·08714503	·03090389	·00648171	−0·00170711
30			+0·60462318	+0·35805882	+0·18131332	+0·07663290	+0·02441293	+0·00307540	−0·00316090
31			+0·58926007	+0·34121483	+0·16696198	+0·06650333	+0·01837312	+0·0007291	−0·00431759
32			·57361559	·32430723	·15283289	·05679271	·01280290	−0·00252601	·00519021
33			·55770916	·30737338	·13896908	·04753404	·00771613	·00472609	·00579518
34			·54156047	·29045039	·12541180	·03875671	+0·00312207	·00653636	·00615177
35			·52518950	·27357498	·11220026	·03048632	−0·00097464	·00796986	·00628175
36			·50861649	·25678349	·09937167	·02274473	·00457376	·00904336	·00620891
37			·49186189	·24011176	·08696091	·01554981	·00767948	·00977696	·00595852
38			·47494633	·22359489	·07500046	·00891550	·01030017	·01019374	·00555687
39			·45789066	·20726740	·06352033	+0·00285177	·01244819	·01031933	·00503077
40			+0·44071581	+0·19116293	+0·05254791	−0·00263535	−0·01413964	−0·01018147	−0·00440707

41	+0·42344288	+0·17531420	+0·04210782	−0·00754379	−0·01539411	−0·00980951	−0·00371214
42	·40609303	·15975304	·03222195	·01187531	·01623439	·00923406	·00297150
43	·38868751	·14451014	·02290931	·01563541	·01668609	·00848643	·00220941
44	·37124758	·12961507	·01418594	·01883330	·01677736	·00759819	·00144843
45	·35379454	·11509618	+0·00606501	·02148156	·01653843	·00660073	·00070917
46	·33634965	·10098056	−0·00144334	·02359620	·01600133	·00552483	−0·00000999
47	·31893415	·08729390	·00833193	·02519628	·01519944	·00440027	+0·00063319
48	·30156921	·07406051	·01459656	·02630382	·01416708	·00325540	·00120707
49	·28427590	·06130319	·02023600	·02694352	·01293921	·00211682	·00170112
50	+0·26707518	+0·04904323	−0·02525191	−0·02714249	−0·01155088	−0·00100909	+0·00210754
51	+0·24998789	+0·03730036	−0·02964885	−0·02693004	−0·01003700	+0·00004555	+0·00242127
52	·23303465	·02609260	·03343421	·02633742	·00843189	·00102745	·00264001
53	·21623595	·01543638	·03661808	·02539745	·00676893	·00191963	·00276398
54	·19961203	+0·00534637	·03921321	·02414430	·00508025	·00270800	·00279583
55	·18318290	−0·00416447	·04123489	·02261318	·00339637	·00338143	·00274045
56	·16696831	·01308495	·04270086	·02084004	·00174594	·00393172	·00260468
57	·15098774	·02140569	·04363119	·01886128	−0·00015548	·00435367	·00239707
58	·13526033	·02911910	·04404807	·01671340	+0·00135082	·00464494	·00212758
59	·11980492	·03621943	·04397583	·01443281	·00275137	·00480605	·00180722
60	+0·10464000	−0·04270272	−0·04344057	−0·01205544	+0·00402728	+0·00484010	+0·00144785
61	+0·08978368	−0·04856687	−0·04247021	−0·00961648	+0·00516252	+0·00475273	+0·00106169
62	·07525368	·05381160	·04109417	·00715018	·00614397	·00455178	·00066113
63	·06106732	·05843845	·03934330	·00468952	·00696153	·00424711	+0·00025835
64	·04724148	·06245076	·03724964	−0·00226596	·00760810	·00385032	−0·00013495
65	·03379260	·06585367	·03484624	+0·00009071	·00807957	·00337448	·00050781
66	·02073665	·06865409	·03216701	·00235265	·00837473	·00283374	·00085029
67	+0·00808911	·07086067	·02924652	·00449414	·00849525	·00224312	·00115369
68	−0·00413503	·07248376	·02611975	·00649173	·00844550	·00161809	·00141065
69	·01592129	·07353542	·02282200	·00832444	·00823246	·00097438	·00161539
70	−0·02725575	−0·07402931	−0·01938860	+0·00997384	+0·00786542	+0·00032747	−0·00176370
71	−0·03812500	−0·07398070	−0·01585477	+0·01142414	+0·00735591	−0·00030748	−0·00185309
72	·04851622	·07340644	·01225544	·01266237	·00671742	·00091613	·00188276
73	·05841715	·07232482	·00862503	·01367826	·00596510	·00148504	·00185355
74	·06781614	·07075560	·00499728	·01446448	·00511556	·00200207	·00176794
75	·07670210	·06871992	−0·00140505	·01501649	·00418657	·00245648	·00162986
76	·08506458	·06624023	+0·00211979	·01533259	·00319679	·00283916	·00144462
77	·09289376	·06334024	·00554662	·01541385	·00216539	·00314277	·00121869
78	·10018043	·06004484	·00884612	·01526407	·00111189	·00336185	·00095956
79	·10691604	·05638003	·01199056	·01488970	+0·00005576	·00349289	·00067548
80	−0·11309268	−0·05237286	+0·01495380	+0·01429969	−0·00098384	−0·00353437	−0·00037522
81	−0·11870311	−0·04805130	+0·01771153	+0·01350539	−0·00198833	−0·00348676	−0·00006790
82	·12374075	·04344427	·02024135	·01252040	·00293998	·00335247	+0·00023734
83	·12819970	·03858142	·02252291	·01136040	·00382221	·00313579	·00053154
84	·13207474	·03349315	·02453798	·01004295	·00461977	·00284276	·00080612
85	·13536132	·02821050	·02627059	·00858731	·00531896	·00248105	·00105318
86	·13805560	·02276500	·02770706	·00701418	·00590791	·00205978	·00126566
87	·14015443	·01718869	·02883613	·00534554	·00637659	·00158927	·00143750
88	·14165535	·01151395	·02964896	·00360439	·00671712	·00108095	·00156382
89	·14255660	·00577345	·03013919	·00181446	·00692374	·00054702	·00164107
90	−0·14285714	−0·00000000	+0·03030303	+0·00000000	−0·00699301	−0·00000000	+0·00166706

VALUES OF G'^3_3, G'^3_4, G'^3_5, G'^3_6 … G'^3_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^3_3	G'^3_4	G'^3_5	G'^3_6	G'^3_7	G'^3_8	G'^3_9	G'^3_{10}
0°	1	μ'	+0·88888889	+0·72727273	+0·55944056	+0·41025641	+0·28959276	+0·19814241
1			·88858020	·72685183	·55896573	·40978163	·28915715	·19776801
2			·88765451	·72559012	·55754304	·40835983	·28785335	·19664805
3			·88611301	·72349061	·55517771	·40599830	·28569021	·19479225
4			·88395764	·72055824	·55187833	·40270911	·28268237	·19221654
5			·88119113	·71680007	·54765712	·39850928	·27885039	·18894327
6			·87781700	·71222495	·54252947	·39342026	·27422008	·18500040
7			·87383951	·70684378	·53651415	·38746814	·26882276	·18042169
8			·86926370	·70066927	·52963309	·38068324	·26269466	·17524599
9			·86409536	·69371606	·52191139	·37310005	·25587667	·16951691
10			+0·85834104	+0·68600063	+0·51337705	+0·36475688	+0·24841401	+0·16328229
11			+0·85200801	+0·67754110	+0·50406092	+0·35569571	+0·24035588	+0·15659377
12			·84510428	·66835751	·49399671	·34596196	·23175497	·14950608
13			·83763857	·65847138	·48322050	·33560399	·22266694	·14207646
14			·82962031	·64790597	·47177089	·32467298	·21315007	·13436412
15			·82105963	·63668599	·45968864	·31322247	·20326460	·12642942
16			·81196731	·62483767	·44701658	·30130816	·19307241	·11833345
17			·80235482	·61238863	·43379939	·28898738	·18263630	·11013716
18			·79223427	·59936772	·42008325	·27631874	·17201952	·10190080
19			·78161839	·58580516	·40591598	·26336197	·16128537	·09368339
20			+0·77052053	+0·57173224	+0·39134645	+0·25017728	+0·15049650	+0·08554198
21			+0·75895464	+0·55718131	+0·37642464	+0·23682520	+0·13971456	+0·07753123
22			·74693524	·54218569	·36120116	·22336591	·12899948	·06970270
23			·73447739	·52677961	·34572730	·20985924	·11840929	·06210454
24			·72159671	·51099808	·33005465	·19636408	·10799951	·05478104
25			·70830932	·49487672	·31423481	·18293798	·09782269	·04777216
26			·69463184	·47845186	·29831937	·16963700	·08792818	·04111331
27			·68058135	·46176018	·28235945	·15651521	·07836164	·03483508
28			·66617538	·44483885	·26640570	·14362453	·06916487	·02896304
29			·65143188	·42772525	·25050792	·13101430	·06037543	·02351763
30			+0·63636921	+0·41045698	+0·23471493	+0·11873105	+0·05202645	+0·01851412
31			+0·62100610	+0·39307172	+0·21907440	+0·10681839	+0·04414659	+0·01396261
32			·60536162	·37560706	·20363246	·09531651	·03675965	·00986808
33			·58945519	·35810056	·18843385	·08426231	·02988482	·00623062
34			·57330650	·34058950	·17352144	·07368897	·02353635	·00304549
35			·55693553	·32311080	·15893611	·06362589	·01772372	+0·00030347
36			·54036252	·30570102	·14471678	·05409868	·01245174	−0·00200886
37			·52360792	·28839614	·13090003	·04512890	·00772052	·00390881
38			·50669236	·27123153	·11752008	·03673410	+0·00352571	·00541719
39			·48963669	·25424189	·10460872	·02892786	−0·00014129	·00655783
40			+0·47246184	+0·23746107	+0·09219506	+0·02171964	−0·00329315	−0·00735723

41	+0·45518891	+0·22092203	+0·08030549	+0·01511488	−0·00594640	−0·00784410
42	·43783906	·20465678	·06896368	·00911512	·00812103	·00804882
43	·42043354	·18869627	·05819043	+0·00371803	·00984017	·00800297
44	·40299361	·17307028	·04800359	−0·00108250	·01112988	·00773895
45	·38554057	·15780740	·03841806	·00529619	·01201866	·00728940
46	·36809568	·14293496	·02944581	·00893620	·01253712	·00668682
47	·35068018	·12847888	·02109577	·01201896	·01271758	·00596311
48	·33331524	·11446372	·01337394	·01456396	·01259369	·00514916
49	·31602193	·10091252	+0·00628332	·01659351	·01219999	·00427449
50	+0·29882121	+0·08784685	−0·00017600	−0·01813256	−0·01157159	−0·00336689
51	+0·28173392	+0·07528663	−0·00600686	−0·01920835	−0·01074368	−0·00245211
52	·26478068	·06325019	·01121486	·01985023	·00975129	·00155370
53	·24798198	·05175418	·01580840	·02008933	·00862875	−0·00069260
54	·23135806	·04081356	·01979854	·01995829	·00740949	+0·00011281
55	·21492893	·03044155	·02319889	·01949097	·00612565	·00084700
56	·19871434	·02064957	·02602553	·01872213	·00480776	·00149726
57	·18273377	·01144732	·02829688	·01768719	·00348453	·00205378
58	·16700636	+0·00284262	·03003356	·01642185	·00218254	·00250960
59	·15155095	−0·00515848	·03125827	·01496189	−0·00092607	·00286054
60	+0·13638603	−0·01255176	−0·03199560	−0·01334280	+0·00026309	+0·00310514
61	+0·12152971	−0·01933484	−0·03227192	−0·01159955	+0·00136582	+0·00324443
62	·10699971	·02550715	·03211518	·00976630	·00236571	·00328185
63	·09281335	·03106997	·03155478	·00787618	·00324920	·00322295
64	·07898751	·03602634	·03062132	·00596097	·00400560	·00307521
65	·06553863	·04038112	·02934650	·00405094	·00462704	·00284773
66	·05248268	·04414092	·02776288	·00217463	·00510850	·00255097
67	·03983514	·04731412	·02590371	−0·00035860	·00544775	·00219649
68	·02761100	·04991079	·02380275	+0·00137268	·00564519	·00179659
69	·01582474	·05194265	·02149404	·00299699	·00570374	·00136407
70	+0·00449028	−0·05342312	−0·01901178	+0·00449455	+0·00562874	+0·00091190
71	−0·00637897	−0·05436716	−0·01639007	+0·00584803	+0·00542767	+0·00045293
72	·01677019	·05479131	·01366273	·00704274	·00511004	−0·00000035
73	·02667112	·05471357	·01086317	·00806662	·00468708	·00043613
74	·03607011	·05415342	·00802414	·00891029	·00417155	·00084345
75	·04495607	·05313166	·00517758	·00956709	·00357745	·00121246
76	·05331855	·05167048	−0·00235449	·01003305	·00291978	·00153455
77	·06114773	·04979328	+0·00041534	·01030685	·00221426	·00180256
78	·06843440	·04752463	·00310338	·01038975	·00147699	·00201086
79	·07517001	·04489026	·00568258	·01028557	+0·00072430	·00215545
80	−0·08134665	−0·04191686	+0·00812750	+0·01000052	−0·00002766	−0·00223399
81	−0·08695708	−0·03863217	+0·01041443	+0·00954312	−0·00076310	−0·00224586
82	·09199472	·03506476	·01252151	·00892402	·00146687	·00219207
83	·09645367	·03124397	·01442889	·00815590	·00212469	·00207526
84	·10032871	·02719995	·01611882	·00725323	·00272345	·00189962
85	·10361529	·02296332	·01757560	·00623209	·00325126	·00167069
86	·10630957	·01856541	·01878598	·00511003	·00369779	·00139531
87	·10840840	·01403792	·01973892	·00390575	·00405436	·00108141
88	·10990932	·00941293	·02042579	·00263896	·00431408	·00073781
89	·11081057	·00472278	·02084041	·00133007	·00447193	·00037402
90	−0·11111111	−0·00000000	+0·02097902	+0·00000000	−0·00452489	−0·00000000

VALUES OF G'^4_4, G'^4_5, G'^4_6, G'^4_7 ... G'^4_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^4_4	G'^4_5	G'^4_6	G'^4_7	G'^4_8	G'^4_9	G'^4_{10}
0°	1	μ'	+0·90909091	+0·76923077	+0·61538462	+0·47058824	+0·34674923
1			·90878222	·76880340	·61489082	·47007999	·34626780
2			·90785653	·76752225	·61341113	·46855763	·34482647
3			·90631503	·76539036	·61095091	·46602888	·34243484
4			·90415966	·76241271	·60751891	·46250616	·33910826
5			·90139315	·75859631	·60312746	·45800702	·33486841
6			·89801902	·75395005	·59779216	·45255360	·32974258
7			·89404153	·74848489	·59153210	·44617286	·32376385
8			·88946572	·74221353	·58436943	·43889605	·31697045
9			·88429738	·73515071	·57632969	·43075891	·30940572
10			+0·87854306	+0·72731282	+0·56744121	+0·42180101	+0·30111753
11			+0·87221003	+0·71871816	+0·55773541	+0·41206578	+0·29215795
12			·86530630	·70938669	·54724631	·40160012	·28258279
13			·85784059	·69934011	·53601070	·39045421	·27245123
14			·84982233	·68860164	·52406766	·37868096	·26182510
15			·84126165	·67719606	·51145860	·36633590	·25076852
16			·83216933	·66514970	·49822703	·35347679	·23934738
17			·82255684	·65249020	·48441827	·34016308	·22762871
18			·81243629	·63924652	·47007933	·32645572	·21568014
19			·80182041	·62544893	·45525875	·31241677	·20356946
20			+0·79072255	+0·61112882	+0·44000630	+0·29810897	+0·19136398
21			+0·77915666	+0·59631864	+0·42437272	+0·28359525	+0·17912996
22			·76713726	·58105177	·40840962	·26893861	·16693227
23			·75467941	·56536251	·39216913	·25420141	·15483368
24			·74179873	·54928596	·37570379	·23944532	·14289459
25			·72851134	·53285792	·35906628	·22473075	·13117250
26			·71483386	·51611473	·34230913	·21011657	·11972159
27			·70078337	·49909323	·32548458	·19565978	·10859241
28			·68637740	·48183067	·30864432	·18141519	·09783153
29			·67163390	·46436457	·29183924	·16743505	·08748124
30			+0·65657123	+0·44673262	+0·27511933	+0·15376891	+0·07757933
31			+0·64120812	+0·42897262	+0·25853334	+0·14046328	+0·06815896
32			·62556364	·41112234	·24212870	·12756142	·05924841
33			·60965721	·39321939	·22595124	·11510305	·05087101
34			·59350852	·37530119	·21004503	·10312435	·04304518
35			·57713755	·35740484	·19445227	·09165764	·03578431
36			·56056454	·33956699	·17921304	·08073137	·02909691
37			·54380994	·32182380	·16436525	·07036997	·02298663
38			·52689438	·30421076	·14994437	·06059379	·01745245
39			·50983871	·28676269	·13598340	·05141905	·01248876
40			+0·49266386	+0·26951363	+0·12251283	+0·04285796	+0·00808575

41	+0·47539093	+0·25249668	+0·10956032	+0·03491854	+0·00422948
42	·45804108	·23574398	·09715083	·02760485	+0·00090228
43	·44063556	·21928666	·08530647	·02091699	−0·00191697
44	·42319563	·20315466	·07404640	·01485120	·00425251
45	·40574259	·18737671	·06338683	·00940002	·00613134
46	·38829770	·17198031	·05334104	·00455245	·00758284
47	·37088220	·15699156	·04391929	+0·00029410	·00863839
48	·35351726	·14243518	·03512885	−0·00339257	·00933094
49	·33622395	·12833438	·02697403	·00652807	·00969466
50	+0·31902323	+0·11471089	+0·01945620	−0·00913562	−0·00976450
51	+0·30193594	+0·10158482	+0·01257382	−0·01124091	−0·00957584
52	·28498270	·08897467	·00632254	·01287182	·00916405
53	·26818400	·07689727	+0·00069522	·01405817	·00856417
54	·25156008	·06536778	−0·00431792	·01483140	·00781053
55	·23513095	·05439955	·00872929	·01522434	·00693642
56	·21891636	·04400424	·01255375	·01527081	·00597373
57	·20293579	·03419171	·01580852	·01500549	·00495282
58	·18720838	·02496997	·01851305	·01446343	·00390202
59	·17175297	·01634526	·02068885	·01367990	·00284764
60	+0·15658805	+0·00832198	−0·02235941	−0·01269009	−0·00181363
61	+0·14173173	+0·00090272	−0·02354996	−0·01152874	−0·00082148
62	·12720173	−0·00591177	·02428738	·01022997	+0·00010991
63	·11301537	·01212256	·02459999	·00882697	·00096438
64	·09918953	·01773251	·02451735	·00735176	·00172844
65	·08574065	·02274627	·02407015	·00583497	·00239142
66	·07268470	·02717027	·02328998	·00430563	·00294535
67	·06003716	·03101266	·02220911	·00279096	·00338499
68	·04781302	·03428333	·02086040	−0·00131621	·00370768
69	·03602676	·03699382	·01927702	+0·00009551	·00391331
70	+0·02469230	−0·03915730	−0·01749226	+0·00142339	+0·00400407
71	+0·01382305	−0·04078856	−0·01553942	+0·00264893	+0·00398439
72	+0·00343183	·04190391	·01345154	·00375615	·00386067
73	−0·00646910	·04252117	·01126127	·00473163	·00364114
74	·01586809	·04265959	·00900063	·00556444	·00333558
75	·02475405	·04233980	·00670091	·00624633	·00295507
76	·03311653	·04158373	·00439242	·00677159	·00251182
77	·04094571	·04041461	−0·00210440	·00713707	·00201880
78	·04823238	·03885680	+0·00013523	·00734216	·00148956
79	·05496799	·03693579	·00229994	·00738865	·00093791
80	−0·06114463	−0·03467810	+0·00436475	+0·00728066	+0·00037773
81	−0·06675506	−0·03211124	+0·00630642	+0·00702459	−0·00017736
82	·07179270	·02926355	·00810349	·00662884	·00071421
83	·07625165	·02616420	·00973649	·00610381	·00122038
84	·08012669	·02284307	·01118791	·00546160	·00168432
85	·08341327	·01933066	·01244247	·00471596	·00209560
86	·08610755	·01565801	·01348706	·00388195	·00244512
87	·08820638	·01185663	·01431083	·00297583	·00272518
88	·08970730	·00795838	·01490535	·00201477	·00292968
89	·09060855	·00399540	·01526449	·00101670	·00305417
90	−0·09090909	−0·00000000	+0·01538462	+0·00000000	−0·00309598

VALUES OF G'^5_5, G'^5_6, G'^5_7 ... G'^5_{10}, AND G'^6_6, G'^6_7, G'^6_8 ... G'^6_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^5_5	G'^5_6	G'^5_7	G'^5_8	G'^5_9	G'^5_{10}
0°	1	μ'	+0·92307692	+0·80000000	+0·65882352	+0·52012383
1			·92276823	·79956787	·65831521	·51958882
2			·92184254	·79827248	·65679197	·51798629
3			·92030104	·79611686	·65425921	·51532403
4			·91814567	·79310599	·65072576	·51161489
5			·91537916	·78924688	·64620414	·50687679
6			·91200503	·78454848	·64071004	·50113244
7			·90802754	·77902171	·63426278	·49440946
8			·90345173	·77267933	·62688479	·48673995
9			·89828339	·76553611	·61860185	·47816053
10			+0·89252907	+0·75760844	+0·60944259	+0·46871175
11			+0·88619604	+0·74891467	+0·59943875	+0·45843820
12			·87929231	·73947478	·58862478	·44738804
13			·87182660	·72931051	·57703783	·43561275
14			·86380834	·71844511	·56471746	·42316673
15			·85524766	·70690345	·55170554	·41010701
16			·84615534	·69471183	·53804608	·39649298
17			·83654285	·68189799	·52378496	·38238583
18			·82642230	·66849099	·50896980	·36784833
19			·81580642	·65452105	·49364965	·35294440
20			+0·80470856	+0·64001967	+0·47787493	+0·33773866
21			+0·79314267	+0·62501933	+0·46169709	+0·32229618
22			·78112327	·60955354	·44516836	·30668186
23			·76866542	·59365662	·42834161	·29096034
24			·75578474	·57736375	·41127012	·27519540
25			·74249735	·56071079	·39400732	·25944968
26			·72881987	·54373416	·37660654	·24378433
27			·71476938	·52647079	·35912077	·22825858
28			·70036341	·50895800	·34160258	·21292959
29			·68561991	·49123340	·32410370	·19785197
30			+0·67055724	+0·47333475	+0·30667494	+0·18307755
31			+0·65519413	+0·45529997	+0·28936601	+0·16865525
32			·63954965	·43716684	·27222513	·15463055
33			·62364322	·41897318	·25529915	·14104566
34			·60749453	·40075644	·23863302	·12793900
35			·59112356	·38255379	·22226985	·11534520
36			·57455055	·36440205	·20625072	·10329498
37			·55779595	·34633740	·19061446	·09181501
38			·54088039	·32839550	·17539754	·08092786
39			·52382472	·31061130	·16063399	·07065196
40			+0·50664987	+0·29301885	+0·14635517	+0·06100154

θ	G'^6_6	G'^6_7	G'^6_8	G'^6_9	G'^6_{10}
0°	1	μ'	+0·93333333	+0·82352941	+0·69349845
1			·93302464	·82309365	·69297866
2			·93209895	·82178737	·69142103
3			·93055745	·81961360	·68883101
4			·92840208	·81657730	·68521749
5			·92563557	·81268555	·68059306
6			·92226144	·80794727	·67497363
7			·91828395	·80237342	·66837862
8			·91370814	·79597673	·66083062
9			·90853980	·78877201	·65235567
10			+0·90278548	+0·78077570	+0·64298263
11			+0·89645245	+0·77200611	+0·63274350
12			·88954872	·76248331	·62167303
13			·88208301	·75222905	·60980875
14			·87406475	·74126661	·59719047
15			·86550407	·72962085	·58386050
16			·85641175	·71731819	·56986327
17			·84679926	·70438633	·55524501
18			·83667871	·69085439	·54005383
19			·82606283	·67675266	·52433928
20			+0·81496497	+0·66211265	+0·50815227
21			+0·80339908	+0·64696694	+0·49154473
22			·79137968	·63134901	·47456945
23			·77892183	·61529330	·45727988
24			·76604115	·59883502	·43972988
25			·75275376	·58201004	·42197341
26			·73907628	·56485492	·40406448
27			·72502579	·54740659	·38605673
28			·71061982	·52970243	·36800330
29			·69587632	·51178014	·34995669
30			+0·68081365	+0·49367757	+0·33196834
31			+0·66545054	+0·47543263	+0·31408862
32			·64980606	·45708326	·29636656
33			·63389963	·43866727	·27884961
34			·61775094	·42022221	·26158351
35			·60137997	·40178536	·24461213
36			·58480696	·38339356	·22797730
37			·56805236	·36508310	·21171857
38			·55113680	·34688974	·19587323
39			·53408113	·32884845	·18047601
40			+0·51690628	+0·31099342	+0·16555912

41	+0·48937694	+0·27565140	+0·13258981	+0·05198671
42	·47202709	·25854126	·11936386	·04361359
43	·45462157	·24171962	·10670043	·03588415
44	·43718164	·22521653	·09461964	·02879649
45	·41972860	·20906087	·08313876	·02234496
46	·40228371	·19328024	·07227204	·01652026
47	·38486821	·17790087	·06203074	·01130964
48	·36750327	·16294758	·05242313	·00669707
49	·35020996	·14844374	·04345450	+0·00266349
50	+0·33300924	+0·13441118	+0·03512721	−0·00081296
51	+0·31592195	+0·12087015	+0·02744073	−0·00375670
52	·29896871	·10783928	·02039165	·00619447
53	·28217001	·09533554	·01397382	·00815504
54	·26554609	·08337419	·00817837	·00966892
55	·24911696	·07196876	+0·00299386	·01076807
56	·23290237	·06113100	−0·00159364	·01148561
57	·21692180	·05087093	·00560044	·01185559
58	·20119439	·04119669	·00904508	·01191255
59	·18573898	·03211466	·01194820	·01169139
60	+0·17057406	+0·02362939	−0·01433239	−0·01122697
61	+0·15571774	+0·01574360	−0·01622206	−0·01055390
62	·14118774	·00845817	·01764326	·00970624
63	·12700138	+0·00177221	·01862345	·00871724
64	·11317554	−0·00431703	·01919144	·00761907
65	·09972666	·00981405	·01937713	·00644264
66	·08667071	·01472512	·01921135	·00521740
67	·07402317	·01905826	·01872567	·00397104
68	·06179903	·02282320	·01795222	·00272942
69	·05001277	·02603134	·01692347	·00151633
70	+0·03867831	−0·02869571	−0·01567211	−0·00035344
71	+0·02780906	−0·03083092	−0·01423076	+0·00073989
72	·01741784	·03245316	·01263188	·00174668
73	+0·00751691	·03358007	·01090753	·00265236
74	−0·00188208	·03423079	·00908920	·00344489
75	·01076804	·03442576	·00720764	·00411470
76	·01913052	·03418678	·00529269	·00465475
77	·02695970	·03353692	·00337309	·00506048
78	·03424637	·03250038	−0·00147636	·00532975
79	·04098198	·03110251	+0·00037137	·00546279
80	−0·04715862	−0·02936968	+0·00214552	+0·00546211
81	−0·05276905	−0·02732922	+0·00382317	+0·00533237
82	·05780669	·02500933	·00538318	·00508027
83	·06226564	·02243903	·00680633	·00471439
84	·06614068	·01964804	·00807541	·00424501
85	·06942726	·01666670	·00917531	·00368396
86	·07212154	·01352592	·01009310	·00304443
87	·07422037	·01025701	·01081812	·00234076
88	·07572129	·00689171	·01134199	·00158804
89	·07662254	·00346199	·01165872	·00080234
90	−0·07692308	−0·00000000	+0·01176471	+0·00000000

41	+0·49963335	+0·29335799	+0·15115206
42	·48228350	·27597448	·13728152
43	·46487798	·25887424	·12397144
44	·44743805	·24208736	·11124272
45	·42998501	·22564288	·09911344
46	·41254012	·20956842	·08759861
47	·39512462	·19389032	·07671029
48	·37775968	·17863354	·06645754
49	·36046637	·16382148	·05684644
50	+0·34326565	+0·14947611	+0·04788012
51	+0·32617836	+0·13561777	+0·03955881
52	·30922512	·12226517	·03187989
53	·29242642	·10943539	·02483796
54	·27580250	·09714381	·01842490
55	·25937337	·08540402	·01263001
56	·24315878	·07422794	·00744012
57	·22717821	·06362562	+0·00283961
58	·21145080	·05360536	−0·00118932
59	·19599539	·04417362	·00466663
60	+0·18083047	+0·03533505	−0·00761424
61	+0·16597415	+0·02709251	−0·01005584
62	·15144415	·01944696	·01201685
63	·13725779	·01239762	·01352408
64	·12343195	·00594186	·01460573
65	·10998307	+0·00007530	·01529107
66	·09692712	−0·00520824	·01561034
67	·08427958	·00991667	·01559454
68	·07205544	·01405958	·01527523
69	·06026918	·01764826	·01468437
70	+0·04893472	−0·02069565	−0·01385409
71	+0·03806547	−0·02321624	−0·01281656
72	·02767425	·02522610	·01160374
73	·01777332	·02674277	·01024722
74	+0·00837433	·02778523	·00877808
75	−0·00051163	·02837384	·00722665
76	·00887411	·02853029	·00562238
77	·01670329	·02827752	·00399365
78	·02398996	·02763960	·00236763
79	·03072557	·02664178	−0·00077014
80	−0·03690221	−0·02531030	+0·00077454
81	−0·04251264	−0·02367238	+0·00224375
82	·04755028	·02175611	·00361661
83	·05200923	·01959038	·00487412
84	·05588427	·01720479	·00599921
85	·05917085	·01462957	·00697702
86	·06186513	·01189549	·00779470
87	·06396396	·00903377	·00844175
88	·06546488	·00607602	·00890985
89	·06636613	·00305408	·00919311
90	−0·06666667	−0·00000000	+0·00928793

VALUES OF G'^{7}_{7}, G'^{7}_{8} … G'^{7}_{10}, AND G'^{8}_{8} … G'^{8}_{10}, WITH θ FROM 0° TO 90°.

θ = GEOGRAPHICAL COLATITUDE, θ' = GEOCENTRIC COLATITUDE. $\mu' = \cos\theta'$.

θ	G'^{7}_{7}	G'^{7}_{8}	G'^{7}_{9}	G'^{7}_{10}	θ	G'^{7}_{7}	G'^{7}_{8}	G'^{7}_{9}	G'^{7}_{10}
0°	1	μ'	+0·94117647	+0·84210526	46°	1	μ'	+0·42038326	+0·22242751
1			·94086778	·84166664	47			·40296776	·20651358
2			·93994209	·84035176	48			·38560282	·19101719
3			·93840059	·83816364	49			·36830951	·17596182
4			·93624522	·83510728	50			+0·35110879	+0·16136947
5			·93347871	·83118977					
6			·93010458	·82642001	51			+0·33402150	+0·14726061
7			·92612709	·82080894	52			·31706826	·13365403
8			·92155128	·81436938	53			·30026956	·12056685
9			·91638294	·80711612	54			·28364564	·10801456
10			+0·91062862	+0·79906559	55			·26721651	·09601082
					56			·25100192	·08456763
11			+0·90429559	+0·79023618	57			·23502135	·07369511
12			·89739186	·78064796	58			·21929394	·06340167
13			·88992615	·77032264	59			·20383853	·05369384
14			·88190789	·75928358	60			+0·18867361	+0·04457637
15			·87334721	·74755565					
16			·86425489	·73516531	61			+0·17381729	+0·03605217
17			·85464240	·72214029	62			·15928729	·02812232
18			·84452185	·70850970	63			·14510093	·02078610
19			·83390597	·69430394	64			·13127509	·01404099
20			+0·82280811	+0·67955452	65			·11782621	·00788268
					66			·10477026	+0·00230508
21			+0·81124222	+0·66429402	67			·09212272	−0·00269961
22			·79922282	·64855597	68			·07989858	·00714093
23			·78676497	·63237487	69			·06811232	·01103005
24			·77388429	·61578600	70			+0·05677786	−0·01437982
25			·76059690	·59882526					
26			·74691942	·58152919	71			+0·04590861	−0·01720465
27			·73286893	·56393484	72			·03551739	·01952054
28			·71846296	·54607963	73			·02561646	·02134490
29			·70371946	·52800128	74			·01621747	·02269663
30			+0·68865679	+0·50973767	75			+0·00733151	·02359601
					76			−0·00103097	·02406465
31			+0·67329368	+0·49132683	77			·00886015	·02412534
32			·65764920	·47280673	78			·01614682	·02380214
33			·64174277	·45421522	79			·02288243	·02312014
34			·62559408	·43558993	80			−0·02905907	−0·02210552
35			·60922311	·41696816					
36			·59265010	·39838685	81			−0·03466950	−0·02078540
37			·57589550	·37988236	82			·03970714	·01918778
38			·55897994	·36149044	83			·04416609	·01734144
39			·54192427	·34324621	84			·04804113	·01527590
40			+0·52474942	+0·32518387	85			·05132771	·01302130
					86			·05402199	·01060830
41			+0·50747649	+0·30733684	87			·05612082	·00806806
42			·49012664	·28973754	88			·05762174	·00543206
43			·47272112	·27241736	89			·05852299	·00273204
44			·45528119	·25540647	90			−0·05882353	−0·00000000
45			+0·43782815	+0·23873394					

θ	G'^{8}_{8}	G'^{8}_{9}	G'^{8}_{10}	θ	G'^{8}_{8}	G'^{8}_{9}	G'^{8}_{10}
0°	1	μ'	+0·94736842	46°	1	μ'	+0·42657521
1			·94705973	47			·40915971
2			·94613404	48			·39179477
3			·94459254	49			·37450146
4			·94243717	50			+0·35730074
5			·93967066				
6			·93629653	51			+0·34021345
7			·93231904	52			·32326021
8			·92774323	53			·30646151
9			·92257489	54			·28983759
10			+0·91682057	55			·27340846
				56			·25719387
11			+0·91048754	57			·24121330
12			·90358381	58			·22548589
13			·89611810	59			·21003048
14			·88809984	60			+0·19486556
15			·87953916				
16			·87044684	61			+0·18000924
17			·86083435	62			·16547924
18			·85071380	63			·15129288
19			·84009792	64			·13746704
20			+0·82900006	65			·12401816
				66			·11096221
21			+0·81743417	67			·09831467
22			·80541477	68			·08609053
23			·79295692	69			·07430427
24			·78007624	70			+0·06296981
25			·76678885				
26			·75311137	71			+0·05210056
27			·73906088	72			·04170934
28			·72465491	73			·03180841
29			·70991141	74			·02240942
30			+0·69484874	75			·01352346
				76			+0·00516098
31			+0·67948563	77			−0·00266820
32			·66384115	78			·00995487
33			·64793472	79			·01669048
34			·63178603	80			−0·02286712
35			·61541506				
36			·59884205	81			−0·02847755
37			·58208745	82			·03351519
38			·56517189	83			·03797414
39			·54811622	84			·04184918
40			+0·53094137	85			·04513576
				86			·04783004
41			+0·51366844	87			·04992887
42			·49631859	88			·05142979
43			·47891307	89			·05233104
44			·46147314	90			−0·05263158
45			+0·44402010				

SECTION III.

ON THE DEFINITE INTEGRAL OF THE PRODUCT OF TWO LEGENDRE'S COEFFICIENTS.

1. Let $V = \frac{1}{\rho} = (1 - 2\mu r + r^2)^{-\frac{1}{2}}$.

Then $$\frac{1}{\rho} = 1 + P_1 r + P_2 r^2 + \&c. + P_n r^n + \&c.,$$

and $$VP_m = \frac{P_m}{\rho} = P_m + P_1 P_m r + P_2 P_m r^2 + \&c. + P_n P_m r^n + \&c.$$

Integrating from $\mu = 0$ to $\mu = 1$, we get

$$\int_0^1 VP_m d\mu = \int_0^1 \frac{P_m}{\rho} d\mu = \int_0^1 P_m d\mu + r\int_0^1 P_1 P_m d\mu + \ldots + r^n \int_0^1 P_n P_m d\mu + \&c.$$

Hence $\int_0^1 P_n P_m d\mu$ is the coefficient of r^n in the expansion of $\int_0^1 \frac{P_m}{\rho} d\mu$ or $\int_{x_0}^1 P_m dx$, where $\mu = x + \frac{r}{2}(1 - x^2)$ and $\frac{dx}{d\mu} = \frac{1}{(1 - 2\mu r + r^2)^{\frac{1}{2}}} = \frac{1}{\rho}$; (see p. 245)

x_0 is the value of x when $\mu = 0$;

$\int_0^1 P_m P_n d\mu$ is the coefficient of $r^m r_1^n$ in the expansion of

$$\int_0^1 \frac{d\mu}{(1 - 2r\mu + r^2)^{\frac{1}{2}} (1 - 2r_1\mu + r_1^2)^{\frac{1}{2}}},$$

in powers of r and r_1.

2. Now if P_m^x be what P_m becomes when x is substituted for μ, we get

$$P_m = P_m^x + \frac{r}{2}(1 - x^2)\frac{dP_m^x}{dx} + \left(\frac{r}{2}\right)^2 \frac{1}{1 \cdot 2}(1 - x^2)^2 \frac{d^2 P_m^x}{dx^2} + \ldots$$
$$+ \left(\frac{r}{2}\right)^n \frac{1}{n!}(1 - x^2)^n \frac{d^n P_m^x}{dx^n} + \&c.$$

Integrating with respect to x, we get

$$\int P_m dx = \int P_m^x dx + \frac{r}{2}\int (1-x^2)\frac{dP_m^x}{dx}dx + \ldots + \left(\frac{r}{2}\right)^n \frac{1}{n!}\int (1-x^2)^n \frac{d^n P_m^x}{dx^n}dx + \&c.;$$

but $$\int (1-x^2)^n \frac{d^n P_m^x}{dx^n}dx = -\frac{(1-x^2)^{n+1}}{(m-n)(m+n+1)}\frac{d^{n+1}P_m^x}{dx^{n+1}}, \text{ (see p. 249)}$$

hence $$\int P_m dx = -\frac{(1-x^2)}{m(m+1)}\frac{dP_m^x}{dx} - \frac{r}{2}\frac{(1-x^2)^2}{2(m-1)(m+2)}\frac{d^2P_m^x}{dx^2} - \ldots$$

$$- \left(\frac{r}{2}\right)^n \frac{1}{n!}\frac{(1-x^2)^{n+1}}{(m-n)(m+n+1)}\frac{d^{n+1}P_m^x}{dx^{n+1}} - \&c.$$

When $\mu = 1$, $x = 1$ and all the terms vanish; also when $\mu = -1$, $x = -1$ and all the terms vanish.

3. By means of equation (4) of Section I., viz.,

$$P_{n+1} = (2n+1)\int P_n d\mu + P_{n-1},$$

we may at once prove the theorem

$$\int_0^1 P_{2n}P_{2n-1}d\mu = \int_0^1 P_{2n}P_{2n+1}d\mu,$$

or $$\int_0^1 P_n \,.\, P_{n-1}d\mu = \int_0^1 P_n \,.\, P_{n+1}d\mu \text{ when } n \text{ is even.}$$

Write S_n for $\int P_n d\mu$, which vanishes when $\mu = 1$.

Then $$P_{n+1} = (2n+1)S_n + P_{n-1}.$$

Multiply by $P_n = \frac{dS_n}{d\mu}$ and integrate;

$$\therefore \int P_n \,.\, P_{n+1}d\mu = (2n+1)\int S_n \,.\, \frac{dS_n}{d\mu}d\mu + \int P_n \,.\, P_{n-1}d\mu$$

$$= \frac{2n+1}{2} \,.\, S_n^2 + \int P_n \,.\, P_{n-1}d\mu;$$

$$\therefore \int_0^1 P_n \,.\, P_{n+1}d\mu = -\frac{2n+1}{2}(S_n^2)_{\mu=0} + \int_0^1 P_n \,.\, P_{n-1}d\mu.$$

If n be even, S_n will involve only odd powers of μ and will consequently vanish when $\mu = 0$.

Therefore in this case

$$\int_0^1 P_n P_{n+1} d\mu = \int_0^1 P_n P_{n-1} d\mu.$$

But if n be odd, since

$$S_n = \frac{1}{2^n \cdot n\,!} \frac{d^{n-1}}{d\mu^{n-1}} (\mu^2 - 1)^n,$$

and the coefficient of μ^{n-1} in $(\mu^2-1)^n$ is

$$\frac{n\,!\,(-1)^{\frac{n+1}{2}}}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!},$$

the term independent of μ in S_n is

$$\frac{1}{2^n} \frac{(n-1)\,!\,(-1)^{\frac{n+1}{2}}}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!};$$

or
$$(S_n)_{\mu=0} = \frac{(-1)^{\frac{n+1}{2}}\, 1 \cdot 2 \cdot 3 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots (n-1)\, 2 \cdot 4 \cdot 6 \ldots (n+1)}$$

$$= \frac{(-1)^{\frac{n+1}{2}}\, 1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n+1)}, \text{ when } n \text{ is odd.}$$

Hence we see that, when n is odd

$$\int_0^1 P_n P_{n+1} d\mu = -\frac{2n+1}{2} \left(\frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n+1)}\right)^2 + \int_0^1 P_n P_{n-1} d\mu \;\ldots\ldots\; (1),$$

and when n is even

$$\int_0^1 P_n P_{n+1} d\mu = \int_0^1 P_n P_{n-1} d\mu^*.$$

4. Now take the general case, and suppose $m > n$,

$$\int_0^1 P_m P_n d\mu = \frac{1}{2^{m+n} \cdot m\,!\,n\,!} \int_0^1 \frac{d^m}{d\mu^m} (\mu^2-1)^m \cdot \frac{d^n}{d\mu^n} (\mu^2-1)^n d\mu.$$

* This result was given by Lord Rayleigh in *Phil. Trans.* Vol. CLX. p. 579, Part II. (1870).

Integrating repeatedly by parts we have

$$\int \frac{d^m}{d\mu^m}(\mu^2-1)^m \frac{d^n}{d\mu^n}(\mu^2-1)^n d\mu$$

$$= \frac{d^{m-1}}{d\mu^{m-1}}(\mu^2-1)^m \frac{d^n}{d\mu^n}(\mu^2-1)^n - \frac{d^{m-2}}{d\mu^{m-2}}(\mu^2-1)^m \frac{d^{n+1}}{d\mu^{n+1}}(\mu^2-1)^n$$

$$+ \frac{d^{m-3}}{d\mu^{m-3}}(\mu^2-1)^m \frac{d^{n+2}}{d\mu^{n+2}}(\mu^2-1)^n - \&c.$$

$$+(-r)^r \frac{d^{m-r-1}}{d\mu^{m-r-1}}(\mu^2-1)^n \frac{d^{n+r}}{d\mu^{n+r}}(\mu^2-1)^n + \&c.$$

$$+(-1)^n \frac{d^{m-n-1}}{d\mu^{m-n-1}}(\mu^2-1)^m \frac{d^{2n}}{d\mu^{2n}}(\mu^2-1)^n,$$

since the following terms vanish.

Now the last term is $(-1)^n \frac{d^{m-n-1}}{d\mu^{m-n-1}}(\mu^2-1)^m (2n)!$.

We may observe that all the quantities like $\frac{d^{m-r-1}}{d\mu^{m-r-1}}(\mu^2-1)^m$ vanish when $\mu=1$; we have therefore only to find the values of the terms on the right-hand side of the equation when $\mu=0$, and this with the sign changed will be the value of

$$\int_0^1 \frac{d^m}{d\mu^m}(\mu^2-1)^m \frac{d^n}{d\mu^n}(\mu^2-1)^n d\mu.$$

5. If both of the quantities m and n be even or both be odd, we have

$$\int_0^1 P_m P_n d\mu = \frac{1}{2}\int_{-1}^1 P_m P_n d\mu = 0, \text{ unless } m=n.$$

In the case when m and n are both even or both odd the integrated terms being of the form

$$(-1)^r \frac{d^{m+r}}{d\mu^{m+r}}(\mu^2-1)^m \frac{d^{n-r-1}}{d\mu^{n-r-1}}(\mu^2-1)^n$$

will consist of two factors, one consisting entirely of even and the other entirely of odd powers of μ, and therefore the product will consist entirely of odd powers of μ and will vanish when $\mu=0$.

Also the last factor vanishes when $\mu=1$.

If $m=n$, the last term becomes

$$\frac{n}{(-1)^n}(2n)!\int(\mu^2-1)^n d\mu.$$

By integration by parts

$$\int(\mu^2-1)^n\,d\mu=\mu(\mu^2-1)^n-2n\int(\mu^2-1)^n d\mu-2n\int(\mu^2-1)^{n-1}d\mu\,;$$

$$(2n+1)\int(\mu^2-1)^n d\mu=\mu(\mu^2-1)^n-2n\int(\mu^2-1)^{n-1}d\mu\,;$$

$$\int(\mu^2-1)^n d\mu=\frac{1}{2n+1}\mu(\mu^2-1)^n-\frac{2n}{2n+1}\int(\mu^2-1)^{n-1}d\mu.$$

Continuing this process we find

$$\int(\mu^2-1)^n d\mu=\frac{1}{2n+1}\mu(\mu^2-1)^n-\frac{2n}{(2n+1)(2n-1)}\mu(\mu^2-1)^{n-1}$$

$$+\frac{2n(2n-2)}{(2n+1)(2n-1)(2n-3)}\mu(\mu^2-1)^{n-2}\ \text{\&c.}+(-1)^n\frac{2n(2n-2)\dots 4\,.\,2}{(2n+1)(2n-1)\dots 5\,.\,3}\mu.$$

Hence between the limits 0 and 1

$$\int_0^1(\mu^2-1)^n d\mu=(-1)^n\frac{2n(2n-2)\dots 4\,.\,2}{(2n+1)(2n-1)\dots 5\,.\,3}.$$

Hence $$\int_0^1(P_n)^2 d\mu=\frac{(-1)^n(2n)!(-1)^n}{2^{2n}\{n!\}^2}\frac{\{2^n n!\}^2}{(2n+1)!}=\frac{1}{2n+1}\ \dots\dots\dots\dots (2).$$

6. We will now consider the case when one of the quantities m, n is even and the other odd.

First then let m—the greater of the two quantities m and n—be even and n odd, and let $m=2p$ and $n=2q-1$, then q may be equal to p, but cannot be greater than it.

The general term on the left-hand side of the equation is now

$$(-1)^r\frac{d^{2p-r-1}}{d\mu^{2p-r-1}}(\mu^2-1)^{2p}\frac{d^{2q+r-1}}{d\mu^{2q+r-1}}(\mu^2-1)^{2q-1}.$$

In order that this may not vanish when $\mu=0$, since all the powers of μ contained in $(\mu^2-1)^{2p}$ or $(\mu^2-1)^{2q-1}$ are even, we must have r odd and therefore $r+1$ and $r-1$ even, or $2p-r-1$ and $2q+r-1$ even.

$$\therefore\ (-1)^r=-1 \text{ in this case.}$$

The coefficient of μ^{2p-r-1} in $(\mu^2-1)^{2p}$ is

$$\frac{(2p)!}{\left(p-\frac{r+1}{2}\right)!\left(p+\frac{r+1}{2}\right)!}(-1)^{p+\frac{r+1}{2}};$$

and therefore the absolute term in $\frac{d^{2p-r-1}}{d\mu^{2p-r-1}}(\mu^2-1)^{2p}$ is

$$\frac{(2p)!\,(2p-r-1)!}{\left(p-\frac{r+1}{2}\right)!\left(p+\frac{r+1}{2}\right)!}(-1)^{p+\frac{r+1}{2}}.$$

Again the coefficient of μ^{2q+r-1} in $(\mu^2-1)^{2q-1}$ is

$$\frac{(2q-1)!}{\left(q+\frac{r-1}{2}\right)!\left(q-\frac{r+1}{2}\right)!}(-1)^{q-\frac{r+1}{2}};$$

and therefore the absolute term in $\frac{d^{2q+r-1}}{d\mu^{2q+r-1}}(\mu^2-1)^{2q-1}$ is

$$\frac{(2q-1)!\,(2q+r-1)!}{\left(q+\frac{r-1}{2}\right)!\left(q-\frac{r+1}{2}\right)!}(-1)^{q-\frac{r+1}{2}};$$

therefore the value of the above general term when $\mu=0$ is

$$(-1)^{p+q+r}\frac{(2p)!\ (2p-r-1)!\ (2q-1)!\ (2q+r-1)!}{\left(p-\frac{r+1}{2}\right)!\left(p+\frac{r+1}{2}\right)!\left(q-\frac{r+1}{2}\right)!\left(q+\frac{r-1}{2}\right)!},$$

i.e. $$(-1)^{\frac{m+n-1}{2}}\frac{m!\ n!\ (m-r-1)!\ (n+r)!}{\left(\frac{m-r-1}{2}\right)!\left(\frac{m+r+1}{2}\right)!\left(\frac{n-r}{2}\right)!\left(\frac{n+r}{2}\right)!}.$$

Dividing by $2^{m+n}m!\,n!$ and changing the sign, we have

$$\int_0^1 P_mP_n d\mu=\frac{(-1)^{\frac{m+n+1}{2}}}{2^{m+n}}\ \Sigma\ \frac{(m-r-1)!\ (n+r)!}{\left(\frac{m-r-1}{2}\right)!\left(\frac{m+r+1}{2}\right)!\left(\frac{n-r}{2}\right)!\left(\frac{n+r}{2}\right)!}\ldots(3),$$

where r is to be taken equal to the odd numbers from 1 to n.

This general term may be put under the form

$$(-1)^{\frac{m+n+1}{2}}\frac{1.3.5\ldots(m-r-2)\,1.3.5\ldots(n+r-1)}{2.4.6\ldots(m+r+1)\,2.4.6\ldots(n-r)}.$$

7. Now let $m=2p+1$ and $n=2q$, then q may be equal to but not greater than p.

Hence by reasoning similar to that in the last case it may be as readily proved that the value of the above general term when $\mu=0$ is

$$(-1)^{p+q+1}\frac{(2p+1)!\,(2p-r)!\,(2q)!\,(2q+r)!}{\left(p-\frac{r}{2}\right)!\left(p+\frac{r}{2}+1\right)!\left(q+\frac{r}{2}\right)!\left(q-\frac{r}{2}\right)!},$$

r being even; we may put this under the form

$$(-1)^{\frac{m+n+1}{2}}\frac{m!\,n!\,(m-r-1)!\,(n+r)!}{\left(\frac{m-r-1}{2}\right)!\left(\frac{m+r+1}{2}\right)!\left(\frac{n+r}{2}\right)!\left(\frac{n-r}{2}\right)!}.$$

Hence

$$\int_0^1 P_m P_n d\mu=\frac{(-1)^{\frac{m+n-1}{2}}}{2^{m+n}}\,\Sigma\,\frac{(m-r-1)!\,(n+r)!}{\left(\frac{m-r-1}{2}\right)!\left(\frac{m+r+1}{2}\right)!\left(\frac{n+r}{2}\right)!\left(\frac{n-r}{2}\right)!}\quad\ldots(4),$$

where m is odd and greater than n, and n is even, and r is equal to the even numbers in succession up to n, bearing in mind that $(0)!=1$.

The above is of exactly the same form as in the former case, except that the sign is changed, and the general term may be put under the form

$$(-1)^{\frac{m+n-1}{2}}\frac{1.3.5\ldots(m-r-2)\,1.3.5\ldots(n+r-1)}{2.4.6\ldots(m+r+1)\,2.4.6\ldots(n-r)}.$$

8. If m be large compared with n and even, the greatest term will be that for which $r=1$, which

$$=(-1)^{\frac{m+n+1}{2}}\frac{1.3.5\ldots(m-3)\,1.3.5\ldots n}{2.4.6\ldots(m+2)\,2.4.6\ldots(n-1)},$$

and all the other terms will be small compared with this term.

If m be large compared with n and odd, the greatest term will be that for which $r=0$, which

$$=(-1)^{\frac{m+n-1}{2}}\frac{1.3.5\ldots(m-2)\,1.3.5\ldots(n-1)}{2.4.6\ldots(m+1)\,2.4.6\ldots n},$$

and all the other terms will be small compared with this term.

Now we have approximately when m is very large and even,

$$\frac{1\,.\,3\,.\,5\,\ldots\,(m-3)}{2\,.\,4\,.\,6\,\ldots\,(m+2)}=\frac{1}{(m-1)\,(m+2)}\,\frac{1\,.\,3\,.\,5\ldots(m-1)}{2\,.\,4\,.\,6\ldots m}=\frac{1}{m^2}\,\frac{1}{\left(\frac{\pi m}{2}\right)^{\frac{1}{2}}},$$

and when m is very large and odd,

$$\frac{1\,.\,3\,.\,5\ldots(m-2)}{2\,.\,4\,.\,6\ldots(m+1)}=\frac{1}{m+1}\,\frac{1}{\left\{\frac{\pi\,(m-1)}{2}\right\}^{\frac{1}{2}}}=\frac{1}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}m^{\frac{3}{2}}}.$$

If n be also large though very small compared with m, we have

$$\int_0^1 P_m P_n d\mu=\frac{1}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}m^{\frac{5}{2}}}\cdot\frac{n}{n^{\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}}=\frac{2n^{\frac{1}{2}}}{\pi m^{\frac{5}{2}}}$$

approximately when m is even, and

$$\int_0^1 P_m P_n d\mu=\frac{1}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}m^{\frac{3}{2}}}\cdot\frac{1}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}n^{\frac{1}{2}}}=\frac{2}{\pi m^{\frac{3}{2}}n^{\frac{1}{2}}}$$

when m is odd.

9. The equation $P_{n+1}=(2n+1)\int P_n d\mu+P_{n-1}$ enables us to express $\int^m P_n d\mu^m$ by means of Legendre's Coefficients.

For
$$\int P_n d\mu=\frac{1}{2n+1}P_{n+1}-\frac{1}{2n+1}P_{n-1}.$$

Similarly
$$\iint P_n d\mu d\mu=\frac{1}{2n+1}\int P_{n+1}d\mu-\frac{1}{2n+1}\int P_{n-1}d\mu$$

$$=\frac{1}{(2n+1)\,(2n+3)}P_{n+2}-\frac{2}{(2n-1)\,(2n+3)}P_n+\frac{1}{(2n-1)\,(2n+1)}P_{n-2}.$$

Integrate again and by the same relation we get

$$\iiint P_n d\mu d\mu d\mu=\frac{1}{(2n+1)\,(2n+3)\,(2n+5)}P_{n+3}-\frac{3}{(2n-1)\,(2n+1)\,(2n+3)}P_{n+1}$$

$$+\frac{3}{(2n+1)\,(2n-3)\,(2n+3)}P_{n-1}-\frac{1}{(2n-3)\,(2n-1)\,(2n+1)}P_{n-3}.$$

Following out the law of formation of the terms, we see that the terms are alternately positive and negative, the numerical coefficients are those of a binomial raised to the power indicated by the number of integrations, the denominators are products of factors

$$(2n+1)(2n+3)(2n+5)\ldots\ldots,$$

and of those factors all diminished by 2, 4, 6 &c. with the omission in the case of any term involving P_{n+r} of the factor $2n+2r+1$.

Thus if we have m integrations the factors for the first denominator are

$$(2n+1)(2n+3)\ldots(2n+2m-1),$$

and the factors for the $(r+1)$th denominator would be

$$(2n-2r+1)(2n-2r+3)\ldots(2n+2m-2r+1),$$

of which the factor $(2n+2m-4r+1)$ is omitted.

Hence taking r from 0 to m, the general term of the expression of $\int^m P_n d\mu^m$ is

$$(-1)^r \frac{m!}{r!\,(m-r)!} \frac{2n+2m-4r+1}{(2n-2r+1)(2n-2r+3)\ldots(2n+2m-2r+1)} P_{n+m-2r} \ldots (5).$$

Hence we can find

$$\int_{-1}^{1} \{P_p \textstyle\int^m P_n d\mu^m\} d\mu,$$

for if $p=n+m-2r$, where r is not greater than m or less than 0, and if S_n^m be written for shortness instead of $\int^m P_n d\mu^m$, we shall have

$$\int_{-1}^{1} P_p S_n^m d\mu = \frac{2}{2p+1}(-1)^r \frac{m!}{r!\,(m-r)!} \frac{2n+2m-4r+1}{(2n-2r+1)(2n-2r+3)\ldots(2n+2m-2r+1)}$$

$$= 2(-1)^r \frac{m!}{r!\,(m-r)!} \frac{1}{(2n-2r+1)(2n-2r+3)\ldots(2n+2m-2r+1)} \ldots (6).$$

Hence we see that $n+m-p$ must be an even number not greater than $2m$ in order that

$$\int_{-1}^{1} P_p S_n^m d\mu \text{ may not vanish.}$$

10. We will now return to the consideration of the value of $\int_0^1 S_m P_n d\mu$, where $S_m = \int P_m d\mu$, the integral vanishing when $\mu = 1$.

First, consider the case when $m = n$,

$$\therefore \int S_n P_n d\mu = \int S_n \frac{dS_n}{d\mu} d\mu = \frac{1}{2}(S_n)^2,$$

$$\therefore \int_0^1 S_n P_n d\mu = -\frac{1}{2}(S_n)^2_{\mu=0};$$

therefore as before, if n be even there will be no term in S_n independent of μ and therefore $(S_n)_{\mu=0}$ vanishes;

$$\therefore \int_0^1 S_n P_n d\mu = 0,$$

but if n be odd, the constant term in S_n will be, as already shewn,

$$(-1)^{\frac{n+1}{2}} \frac{1 \,.\, 3 \,.\, 5 \ldots (n-2)}{2 \,.\, 4 \,.\, 6 \ldots (n+1)};$$

$$\therefore \int_0^1 S_n P_n d\mu = (-1)^n \cdot \frac{1}{2} \cdot \left(\frac{1 \,.\, 3 \,.\, 5 \ldots (n-2)}{2 \,.\, 4 \,.\, 6 \ldots (n+1)}\right)^2 = -\frac{1}{2}\left(\frac{1 \,.\, 3 \,.\, 5 \ldots (n-2)}{2 \,.\, 4 \,.\, 6 \ldots (n+1)}\right)^2.$$

Next, suppose $m = n + 1$ and consider

$$\int_0^1 S_m P_n d\mu = \frac{1}{2m+1}\int_0^1 P_{m+1} P_n d\mu - \frac{1}{2m+1}\int_0^1 P_{m-1} P_n d\mu$$

$$= \frac{1}{2n+3}\int_0^1 P_{n+2} P_n d\mu - \frac{1}{2n+3}\int_0^1 (P_n)^2 d\mu$$

$$= -\frac{1}{(2n+1)(2n+3)}, \text{ since } \int_0^1 P_{n+2} P_n d\mu = 0, \text{ as before proved.}$$

Similarly

$$\int_0^1 S_n P_m d\mu = \frac{1}{2n+1}\int_0^1 P_{n+1} P_m d\mu - \frac{1}{2n+1}\int_0^1 P_{n-1} P_m d\mu$$

$$= \frac{1}{2n+1}\int_0^1 (P_{n+1})^2 d\mu - \frac{1}{2n+1}\int_0^1 P_{n-1} P_{n+1} d\mu$$

$$= \frac{1}{(2n+1)(2n+3)}, \text{ since } \int_0^1 P_{n-1} P_{n+1} d\mu = 0.$$

This should evidently be the case since

$$\int (S_m P_n + S_n P_m)\, d\mu = S_m S_n;$$

and since $m = n + 1$, one of the quantities m and n will be even and the corresponding quantity S_m or S_n will be of odd dimensions in μ and will therefore vanish when $\mu = 0$, and both S_m and S_n vanish when $\mu = 1$; therefore $S_m S_n$ taken between the limits vanishes,

or $\displaystyle\int_0^1 (S_m P_n + S_n P_m)\, d\mu = 0$, or $\displaystyle\int_0^1 S_m P_n\, d\mu = -\int_0^1 S_n P_m\, d\mu$, as above found.

11. Now take the more general case in which $m > n + 1$. We have

$$\int_0^1 S_m P_n\, d\mu = \frac{1}{2m+1} \int_0^1 P_{m+1} P_n\, d\mu - \frac{1}{2m+1} \int_0^1 P_{m-1} P_n\, d\mu;$$

here $m - 1$ and n are not the same quantities, therefore by what we have proved, both these definite integrals vanish unless one of the quantities $m + 1$ and n is odd and the other even, i.e. unless m and n are both even or both odd.

First suppose m and n to be both even and $m > (n + 1)$. Then by what has been before proved we have

$$\int_0^1 P_{m+1} P_n\, d\mu = \frac{(-1)^{\frac{m+n}{2}}}{2^{m+n+1}} \Sigma \frac{(m-2r)!\ (n+2r)!}{\left(\frac{m}{2}-r\right)!\ \left(\frac{m}{2}+r+1\right)!\ \left(\frac{n}{2}-r\right)!\ \left(\frac{n}{2}+r\right)!}$$

for all values of r from 0 to $\frac{n}{2}$ and $0! = 1$.

Similarly

$$\int_0^1 P_{m-1} P_n\, d\mu = \frac{(-1)^{\frac{m+n}{2}-1}}{2^{m+n-1}} \Sigma \frac{(m-2r-2)!\ (n+2r)!}{\left(\frac{m}{2}-r-1\right)!\ \left(\frac{m}{2}+r\right)!\ \left(\frac{n}{2}-r\right)!\ \left(\frac{n}{2}+r\right)!}.$$

Now generally $$\frac{1}{4} \frac{(m-2r)(m-2r-1)}{\left(\frac{m}{2}-r\right)\left(\frac{m}{2}+r+1\right)} + 1 = \frac{1}{2} \frac{(2m+1)}{\frac{m}{2}+r+1}.$$

Hence

$$\int_0^1 S_m P_n\, d\mu = \frac{(-1)^{\frac{m+n}{2}}}{2^{m+n}} \Sigma \frac{(m-2r-2)!\ (n+2r)!}{\left(\frac{m}{2}-r-1\right)!\ \left(\frac{m}{2}+r+1\right)!\ \left(\frac{n}{2}-r\right)!\ \left(\frac{n}{2}+r\right)!} \quad \ldots(7).$$

Also
$$\int_0^1 S_n P_m d\mu + \int_0^1 S_m P_n d\mu = \Big[S_m S_n \Big]_0^1;$$
and S_m, S_n, being of odd dimensions in μ, will vanish when $\mu = 0$, and they also vanish when $\mu = 1$;
$$\therefore \int_0^1 S_n P_m d\mu = -\int_0^1 S_m P_n d\mu.$$

Next suppose m and n to be both odd, m being $> n+1$. By what has been before proved, since $m+1$ and $m-1$ are even, we may prove that
$$\int_0^1 S_m P_n d\mu = \frac{(-1)^{\frac{m+n}{2}+1}}{2^{m+n}} \Sigma \frac{(m-2r-1)!\,(n+2r-1)!}{\left(\frac{m-2r-1}{2}\right)!\left(\frac{m+2r+1}{2}\right)!\left(\frac{n-2r+1}{2}\right)!\left(\frac{n+2r-1}{2}\right)!} \quad \text{.........(8)},$$
for all values of r from 1 to $\frac{n+1}{2}$.

Also
$$\int_0^1 S_n P_m d\mu + \int_0^1 S_m P_n d\mu = \Big[S_m S_n \Big]_0^1 = \frac{(-1)^{\frac{m+n}{2}}}{2^{m+n}} \frac{(m-1)!\,(n-1)!}{\left(\frac{m-1}{2}\right)!\left(\frac{m+1}{2}\right)!\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}$$

Hence $\int_0^1 S_n P_m d\mu$ is found from $\int_0^1 S_m P_n d\mu$ by changing the sign and adding another term at the beginning, i.e. taking all values of r from 0 to $\frac{n+1}{2}$.

12. Having expressed the values of
$$\int_0^1 P_m P_n d\mu \text{ and } \int_0^1 S_m P_n d\mu$$
in series, we will now determine their values in the form of a single term.

The theory of these operations may perhaps be presented in a still more simple form.

First, suppose m to be even and n odd, and let m be the greater.

Then if $\quad f(r) = \frac{1.3.5\ldots(m-r-2)}{2.4.6\ldots(m+r+1)} \frac{1.3.5\ldots(n+r-1)}{2.4.6\ldots(n-r)}$,

we have $\quad \int_0^1 P_m P_n d\mu = (-1)^{\frac{m+n+1}{2}} \{f(1) + f(3) + f(5) + \text{\&c.} + f(n)\}.$

It is to be observed that the operation denoted by f, as above defined, has no meaning unless the subject of the operation is an *odd* integer.

Also let $$\phi(r)=\frac{1.3.5\ldots(m-r-1)}{2.4.6\ldots(m+r+2)}\,\frac{1.3.5\ldots(n+r)}{2.4.6\ldots(n-r+1)},$$

so that the operation or sign of functionality ϕ has no meaning unless the subject (r) of the operation be an *even* integer.

We have evidently

$$\phi(r-1)=\frac{1.3.5\ldots(m-r)}{2.4.6\ldots(m+r+1)}\,\frac{1.3.5\ldots(n+r-1)}{2.4.6\ldots(n-r+2)},$$

and $$\phi(r+1)=\frac{1.3.5\ldots(m-r-2)}{2.4.6\ldots(m+r+3)}\,\frac{1.3.5\ldots(n+r+1)}{2.4.6\ldots(n-r)},$$

where $r-1$, $r+1$ must be even and therefore r odd.

We may observe that

$$\phi(r-1)=f(r)\left\{\frac{m-r}{n-r+2}\right\},$$

and $$\phi(r+1)=f(r)\left\{\frac{n+r+1}{m+r+3}\right\}.$$

Assume $$f(r)=\lambda\phi(r-1)-\mu\phi(r+1),$$

and therefore $$f(r)=f(r)\left\{\lambda\frac{m-r}{n-r+2}-\mu\frac{n+r+1}{m+r+3}\right\},$$

or $$1=\lambda\frac{m-r}{n-r+2}-\mu\frac{n+r+1}{m+r+3},$$

and determine λ and μ by the condition that μ is the same function of $r+1$ that λ is of $r-1$.

This will evidently be the case if

$$\lambda=(n-r+2)(m+r+1)c \text{ and } \mu=(m+r+3)(n-r)c,$$

where c is some quantity which remains the same when $r-1$ is changed into $r+1$.

Substituting, we have

$$\begin{aligned}1&=c\{(m-r)(m+r+1)-(n+r+1)(n-r)\}\\&=c\{m(m+1)-r-r^2-n(n+1)+r+r^2\}\\&=c\{m(m+1)-n(n+1)\}=c(m+n+1)(m-n),\end{aligned}$$

so that c is independent of r and $=\dfrac{1}{(m-n)(m+n+1)}$.

Hence $$\lambda = \frac{(n+2-r)(m+r+1)}{(m-n)(m+n+1)} \text{ and } \mu = \frac{(m+r+3)(n-r)}{(m-n)(m+n+1)};$$

or calling $$\psi(r) = \frac{(n+1-r)(m+2+r)}{(m-n)(m+n+1)},$$

we have $$\lambda = \psi(r-1) \text{ and } \mu = \psi(r+1);$$

$$\therefore f(r) = \psi(r-1)\,\phi(r-1) - \psi(r+1)\,\phi(r+1).$$

Hence we can at once sum the series

$$f(1) + f(3) + \&c. + f(n).$$

For
$$f(1) = \psi(0)\,\phi(0) - \psi(2)\,\phi(2),$$
$$f(3) = \psi(2)\,\phi(2) - \psi(4)\,\phi(4),$$
$$\&c. = \&c.,$$
$$f(n) = \psi(n-1)\,\phi(n-1) - \psi(n+1)\,\phi(n+1).$$

Hence $$f(1) + f(3) + \&c. + f(n) = \psi(0)\,\phi(0) - \psi(n+1)\,\phi(n+1).$$

In this case evidently $\psi(n+1)$ vanishes and $\psi(0) = \dfrac{(n+1)(m+2)}{(m-n)(m+n+1)}$;

$$\therefore f(1) + f(3) + \&c. + f(n) = \frac{(n+1)(m+2)}{(m-n)(m+n+1)}\,\phi(0)$$

$$= \frac{(n+1)(m+2)}{(m-n)(m+n+1)}\,\frac{1\,.\,3\,.\,5\,\ldots\,(m-1)}{2\,.\,4\,.\,6\,\ldots\,(m+2)}\,\frac{1\,.\,3\,.\,5\,\ldots\,n}{2\,.\,4\,.\,6\,\ldots\,(n+1)}$$

$$= \frac{1}{(m-n)(m+n+1)}\,\frac{1\,.\,3\,.\,5\,\ldots\,(m-1)}{2\,.\,4\,.\,6\,\ldots\,m}\,\frac{1\,.\,3\,.\,5\,\ldots\,n}{2\,.\,4\,.\,6\,\ldots\,(n-1)}$$

the sum required, whence

$$\int_0^1 P_m P_n\,d\mu = (-1)^{\frac{m+n+1}{2}}\,\frac{1}{(m-n)(m+n+1)}\,\frac{1.3.5\ldots(m-1)}{2\,.\,4\,.\,6\,\ldots\,m}\,\frac{1\,.\,3\,.\,5\,\ldots\,n}{2\,.\,4\,.\,6\ldots(n-1)}\,\ldots\,(9).$$

Next, suppose m to be odd and n even, m being still the greater. Assume $f(r)$ and $\phi(r)$ to be of the same *forms* as before in m, n and r. Then since m and n are here changed as regards being even and odd,

$f(r)$ will now be unmeaning unless r be *even*,

and $\phi(r)$ will be unmeaning unless r be *odd*.

As before, it may be shewn that if

$$\psi(r) = \frac{(n+1-r)(m+2+r)}{(m-n)(m+n+1)}, \text{ the same function as before,}$$

then $$f(r) = \psi(r-1)\,\phi(r-1) - \psi(r+1)\,\phi(r+1).$$

But in this case we have

$$\int_0^1 P_m P_n d\mu = (-1)^{\frac{m+n-1}{2}} \{f(0) + f(2) + \&c. + f(n)\}$$

$$= (-1)^{\frac{m+n-1}{2}} \{\psi(-1)\,\phi(-1) - \psi(n+1)\,\phi(n+1)\}, \text{ as before;}$$

but $\psi(n+1)$ vanishes and $\psi(-1) = \dfrac{(n+2)(m+1)}{(m-n)(m+n+1)}$,

also $$\phi(-1) = \frac{1\,.\,3\,.\,5\,\ldots\,m}{2\,.\,4\,.\,6\,\ldots\,m+1}\,\frac{1\,.\,3\,.\,5\,\ldots\,(n-1)}{2\,.\,4\,.\,6\,\ldots\,(n+2)};$$

$$\therefore \int_0^1 P_m P_n d\mu = (-1)^{\frac{m+n-1}{2}} \frac{1}{(m-n)(m+n+1)}\,\frac{1\,.\,3\,.\,5\,\ldots\,m}{2\,.\,4\,.\,6\,\ldots\,(m-1)} \times \frac{1\,.\,3\,.\,5\,\ldots\,(n-1)}{2\,.\,4\,.\,6\,\ldots\,n} \quad \ldots(10).$$

13. We will now return to the consideration of

$$\int_0^1 S_m P_n d\mu \text{ and } \int_0^1 S_n P_m d\mu,$$

where m and n are either both even or both odd.

First suppose m and n to be even.

Then $$\int_0^1 S_m P_n d\mu = \frac{1}{2m+1}\int_0^1 P_{m+1} P_n d\mu - \frac{1}{2m+1}\int_0^1 P_{m-1} P_n d\mu;$$

$$\therefore \int_0^1 S_m P_n d\mu = \frac{(-1)^{\frac{m+n}{2}}}{2m+1}\,\frac{1\,.\,3\,.\,5\,\ldots\,(m-1)\,1\,.\,3\,.\,5\,\ldots\,(n-1)}{2\,.\,4\,.\,6\,\ldots\,(m-2)\,2\,.\,4\,.\,6\,\ldots\,n} \times \left[\frac{m+1}{m(m-n+1)(m+n+2)} + \frac{1}{(m-n-1)(m+n)}\right].$$

Now $\dfrac{m+1}{m(m-n+1)(m+n+2)}+\dfrac{1}{(m-n-1)(m+n)}$

$$=\frac{2m+1}{m(m-n+1)(m+n+2)}+\frac{1}{(m-n-1)(m+n)}-\frac{1}{(m-n+1)(m+n+2)}$$

$$=\frac{2m+1}{m(m-n+1)(m+n+2)}+\frac{4m+2}{(m-n-1)(m+n)(m-n+1)(m+n+2)}$$

$$=(2m+1)\left\{\frac{(m-n-1)(m+n)+2m}{m(m-n+1)(m+n+2)(m-n-1)(m+n)}\right\}$$

$$=\frac{(2m+1)(m-n)(m+n+1)}{m(m-n+1)(m+n+2)(m-n-1)(m+n)}.$$

Hence $$\int_0^1 S_m P_n d\mu=(-1)^{\frac{m+n}{2}}\frac{(m-n)(m+n+1)}{(m-n+1)(m-n-1)(m+n)(m+n+2)}$$

$$\times\frac{1\,.\,3\,.\,5\ldots(m-1)\,1\,.\,3\,.\,5\ldots(n-1)}{2\,.\,4\,.\,6\ldots m\,.\,2\,.\,4\,.\,6\ldots n}$$

$$=-\int_0^1 S_n P_m d\mu \ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Next suppose m and n to be both odd and m to be greater than n.

Then $$\int_0^1 S_n P_m d\mu=\frac{1}{2n+1}\left\{\int_0^1 P_m P_{n+1} d\mu-\int_0^1 P_m P_{n-1} d\mu\right\}$$

$$=\frac{(-1)^{\frac{m+n}{2}}}{(2n+1)}\frac{1\,.\,3\,.\,5\ldots m\,.\,1\,.\,3\,.\,5\ldots(n-2)}{2\,.\,4\,.\,6\ldots(m-1)\,2\,.\,4\,.\,6\ldots(n+1)}$$

$$\times\left\{\frac{n}{(m-n-1)(m+n+2)}+\frac{n+1}{(m-n+1)(m+n)}\right\}.$$

Now $\dfrac{n}{(m-n-1)(m+n+2)}+\dfrac{n+1}{(m-n+1)(m+n)}$

$$=\frac{(2n+1)\{m^2-n^2+m-n-2\}}{(m-n-1)(m+n+2)(m-n+1)(m+n)}.$$

Hence $$\int_0^1 S_n P_m d\mu=(-1)^{\frac{m+n}{2}}\frac{m(m+1)-n(n+1)-2}{(m-n-1)(m+n+2)(m-n+1)(m+n)}$$

$$\times\frac{1\,.\,3\,.\,5\ldots m\,.\,1\,.\,3\,.\,5\ldots(n-2)}{2\,.\,4\,.\,6\ldots(m-1)\,2\,.\,4\,.\,6\ldots(n+1)}\ldots\ldots(12).$$

In the same way it may be shewn that

$$\int_0^1 S_m P_n d\mu = (-1)^{\frac{m+n}{2}+1} \frac{m(m+1)-n(n+1)+2}{(m-n+1)(m-n-1)(m+n)(m+n+2)}$$

$$\times \frac{1 \,.\, 3 \,.\, 5 \ldots (m-2)}{2 \,.\, 4 \,.\, 6 \ldots (m+1)} \frac{1 \,.\, 3 \,.\, 5 \ldots n}{2 \,.\, 4 \,.\, 6 \ldots (n-1)},$$

which may also be found from $\int_0^1 S_n P_m d\mu$ by interchanging m and n.

Hence $\int_0^1 (S_m P_n + S_n P_m)\, d\mu = \Big[S_m S_n \Big]_0^1 = -[S_m S_n]_{\mu=0}$

$$= (-1)^{\frac{m+n}{2}} \frac{1 \,.\, 3 \,.\, 5 \ldots (m-2)}{2 \,.\, 4 \,.\, 6 \ldots (m+1)} \; \frac{1 \,.\, 3 \,.\, 5 \ldots (n-2)}{2 \,.\, 4 \,.\, 6 \ldots (n+1)}$$

$$\times \frac{m(m+1)[m(m+1)-n(n+1)-2]-n(n+1)[m(m+1)-n(n+1)+2]}{(m-n-1)(m-n+1)(m+n)(m+n+2)}.$$

If $m+1=n$ or if $n+1=m$ the numerator of the last fraction vanishes, hence $(m-n+1)$ and $(m-n-1)$ are factors of it, also $(m+n)$ is a factor and the remaining factor is $(m+n+2)$; hence the fraction $=1$,

and $\int_0^1 S_m P_n d\mu + \int_0^1 S_n P_m d\mu = \Big[S_m S_n \Big]_0^1$

$$= (-1)^{\frac{m+n}{2}} \frac{1 \,.\, 3 \,.\, 5 \ldots (m-2) \,.\, 1 \,.\, 3 \,.\, 5 \ldots (n-2)}{2 \,.\, 4 \,.\, 6 \ldots (m+1) \,.\, 2 \,.\, 4 \,.\, 6 \ldots (n+1)}.$$

14. We will take an example of the application of these last formulæ.

Let then $m=3$ and $n=1$.

Then by formulæ just found

$$\int_0^1 S_3 P_1 d\mu = -\frac{1}{48} \text{ and } \int_0^1 S_1 P_3 d\mu = \frac{1}{12}.$$

By actual formation of $S_1 P_3$ and $S_3 P_1$ and integration,

$$P_1 = \frac{1}{2} \frac{d}{d\mu} (\mu^2 - 1) = \mu, \qquad S_1 = \int P_1 d\mu = \frac{1}{2}(\mu^2 - 1),$$

$$P_3 = \frac{1}{2 \,.\, 4 \,.\, 6} \frac{d^3}{d\mu^3} (\mu^6 - 3\mu^4 + 3\mu^2 - 1) = \frac{5}{2}\mu^3 - \frac{3}{2}\mu,$$

$$S_3 = \frac{1}{2 \,.\, 4 \,.\, 6} \frac{d^2}{d\mu^2} (\mu^6 - 3\mu^4 + 3\mu^2 - 1) = \frac{5}{8}\mu^4 - \frac{3}{4}\mu^2 + \frac{1}{8}.$$

$$\therefore \int_0^1 S_3 P_1 d\mu = \int_0^1 \left(\frac{5}{48}\mu^6 - \frac{3}{16}\mu^4 + \frac{1}{16}\mu^2\right) d\mu = -\frac{1}{48}, \text{ as before,}$$

and $$\int_0^1 P_3 S_1 d\mu = \int_0^1 \left(\frac{5}{24}\mu^6 - \frac{1}{2}\mu^4 + \frac{3}{8}\mu^2\right) d\mu = \frac{1}{12}, \text{ as before.}$$

Hence our results are confirmed.

But we must be careful to note the paradoxical result that

$$1 \cdot 3 \cdot 5 \ldots (-1) = 1 \text{ and } 2 \cdot 4 \ldots (0) = 1.$$

If we call $1 \cdot 3 \cdot 5 \ldots m = f(m)$, the characteristic mark of f is that $mf(m-2) = f(m)$; applying this when $m=1$, we have $1 \times f(-1) = f(1) = 1$, $\therefore f(-1) = 1$. Similarly in the other case $n\phi(n-2) = \phi(n)$, make $n=2$, $\therefore 2\phi(0) = \phi(2) = 2$, $\therefore \phi(0) = 1$.

15. We have seen above that the general term of the expression $\int^m P_n d\mu^m$ will be

$$(-1)^r \frac{m!}{r!\,(m-r)!} \frac{2n+2m-4r+1}{(2n-2r+1)(2n-2r+3)\ldots(2n+2m-2r+1)} P_{n+m-2r},$$

r being taken from 0 to m and $0!$ being $=1$.

Now generally $$(2x+1)\int P_x d\mu = P_{x+1} - P_{x-1}.$$

Integrating each term of $\int^m P_n d\mu^m$ by means of this formula, we have

$$\int^{m+1} P_n d\mu^{m+1} = \ldots$$

$$+(-1)^r \frac{m!}{r!\,(m-r)!} \frac{P_{n+m-2r+1} - P_{n+m-2r-1}}{(2n-2r+1)(2n-2r+3)\ldots(2n+2m-2r+1)}$$

$$+(-1)^{r+1} \frac{m!}{(r+1)!\,(m-r-1)!} \frac{P_{n+m-2r-1} - P_{n+m-2r-3}}{(2n-2r-1)(2n-2r+1)\ldots(2n+2m-2r-1)}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The coefficient of $P_{n+m-2r-1}$ is

$$(-1)^{r+1} \frac{m!}{(r+1)!\,(m-r)!} \frac{(r+1)(2n-2r-1) + (m-r)(2n+2m-2r+1)}{(2n-2r-1)(2n-2r+1)\ldots(2n+2m-2r+1)}$$

$$=(-1)^{r+1}\frac{m!}{(r+1)!\,(m-r)!}$$
$$\times\frac{(r+1)(2n-2r-1+2m-2r)+(m-r)(2n+2m-2r+1-2r-2)}{(2n-2r-1)(2n-2r+1)\dots(2n+2m-2r+1)}$$
$$=(-1)^{r+1}\frac{(m+1)!}{(r+1)!\,(m-r)!}\,\frac{2n+2m-4r-1}{(2n-2r-1)(2n-2r+1)\dots(2n+2m-2r+1)},$$

which is the same in form as the expression for $\int^m P_n d\mu^m$, taking $r+1$ in place of r, and $m+1$ in place of m.

Hence the law is generally true and may be written

$$\int^m P_n d\mu^m = \Sigma\,(-1)^r\,\frac{m!}{r!\,(m-r)!}$$
$$\times\frac{1\,.\,3\,.\,5\,\dots(2n-2r-1)}{1\,.\,3\,.\,5\,\dots(2n+2m-2r+1)}\,(2n+2m-4r+1)\,P_{n+m-2r}.$$

16. Differentiating this expression m times we get

$$P_n=\Sigma\,(-1)^r\,\frac{m!}{r!\,(m-r)!}\,\frac{1\,.\,3\,.\,5\,\dots(2n-2r-1)}{1\,.\,3\,.\,5\,\dots(2n+2m-2r+1)}\,(2n+2m-4r+1)\,\frac{d^mP_{n+m-2r}}{d\mu^m}.$$

From the equation

$$(2x+1)\,P_x=\frac{dP_{x+1}}{d\mu}-\frac{dP_{x-1}}{d\mu},$$

we get

$$(2n+2m-4r+1)\,\frac{d^mP_{n+m-2r}}{d\mu^m}=\frac{d^{m+1}P_{n+m-2r+1}}{d\mu^{m+1}}-\frac{d^{m+1}P_{n+m-2r-1}}{d\mu^{m+1}}.$$

Substituting in the expression for P_n, we get

$$P_n=\dots+(-1)^{r-1}\,\frac{m!}{(r-1)!\,(m-r+1)!}\,\frac{1\,.\,3\,.\,5\,\dots(2n-2r+1)}{1\,.\,3\,.\,5\,\dots(2n+2m-2r+3)}$$
$$\times\left[\frac{d^{m+1}P_{n+m-2r+3}}{d\mu^{m+1}}-\frac{d^{m+1}P_{n+m-2r+1}}{d\mu^{m+1}}\right]$$
$$+(-1)^r\,\frac{m!}{r!\,(m-r)!}\,\frac{1\,.\,3\,.\,5\,\dots(2n-2r-1)}{1\,.\,3\,.\,5\,\dots(2n+2m-2r+1)}\left[\frac{d^{m+1}P_{n+m-2r+1}}{d\mu^{m+1}}-\frac{d^{m+1}P_{n+m-2r-1}}{d\mu^{m+1}}\right]$$
$$+\dots\dots\dots$$
$$=\Sigma\,(-1)^r\,\frac{m!}{r!\,(m-r+1)!}\,\frac{1\,.\,3\,.\,5\,\dots(2n-2r-1)}{1\,.\,3\,.\,5\,\dots(2n+2m-2r+3)}$$
$$\times\left[r\,(2n-2r+1)+(m-r+1)(2n+2m-2r+3)\right]\frac{d^{m+1}P_{n+m-2r+1}}{d\mu^{m+1}}.$$

The factor

$$[r(2n-2r+1)+(m-r+1)(2n+2m-2r+3)]=(m+1)(2n+2m-4r+3).$$

Hence $P_n = \Sigma(-1)^r \dfrac{(m+1)!}{r!\,(m-r+1)!} \dfrac{1.3.5\ldots(2n-2r-1)}{1.3.5\ldots(2n+2m-2r+3)}$

$$\times(2n+2m-4r+3)\frac{d^{m+1}P_{n+m-2r+1}}{d\mu^{m+1}},$$

which is of the same form as before with $m+1$ written in place of m, so that the law of formation is generally true.

17. We have seen above (p. 249) that

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{m+1}\frac{d^{m+1}P_n}{d\mu^{m+1}}\right\}+(n-m)(n+m+1)(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}=0.$$

If $m=0$, $$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\}+n(n+1)P_n=0.$$

If $m=1$, $$\frac{d}{d\mu}\left\{(1-\mu^2)^2\frac{d^2P_n}{d\mu^2}\right\}+(n-1)(n+2)(1-\mu^2)\frac{dP_n}{d\mu}=0.$$

Hence by successive integration we get

$$(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}=(-1)^m(n-m+1)(n-m+2)\ldots(n+m)\int^m P_n d\mu^m.$$

Hence the general term in $(1-\mu^2)^m \dfrac{d^mP_n}{d\mu^m}$ or S^m (see p. 250) is

$$(-1)^{m+r}\frac{m!}{r!\,(m-r)!}\,\frac{(n+m)!}{(n-m)!}\,\frac{1.3.5\ldots(2n-2r-1)}{1.3.5\ldots(2n+2m-2r+1)}$$

$$\times(2n+2m-4r+1)P_{n+m-2r}\ \ldots\ldots(13).$$

18. Adopting the notation of Section I., Art. 8 (p. 248), we have

$$Q_n^mQ_{n_1}^m=(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}\,\frac{d^mP_{n_1}}{d\mu^m}.$$

Integrating by parts between the limits 1 and -1, the part outside the sign of integration vanishes and we get by the above equation in Art. 17,

$$\int_{-1}^1(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}\,\frac{d^mP_{n_1}}{d\mu^m}d\mu=(n+m)(n-m+1)\int_{-1}^1(1-\mu^2)^{m-1}\frac{d^{m-1}P_n}{d\mu^{m-1}}\,\frac{d^{m-1}P_{n_1}}{d\mu^{m-1}}d\mu.$$

Or $\int_{-1}^{1} Q_n^m Q_{n_1}^m d\mu = (n+m)(n-m+1)\int_{-1}^{1} Q_n^{m-1} Q_{n_1}^{m-1} d\mu$

$$= (n+m)(n+m-1)(n-m+2)(n-m+1)\int_{-1}^{1} Q_n^{m-2} Q_{n_1}^{m-2} d\mu$$

$$= (n+m)(n+m-1)(n+m-2)\ldots(n+1)\,n\ldots(n-m+1)\int_{-1}^{1} P_n P_{n_1} d\mu$$

$$= \frac{(n+m)!}{(n-m)!}\int_{-1}^{1} P_n P_{n_1} d\mu.$$

Hence if n and n_1 are not equal, $\int_{-1}^{1} Q_n^m Q_{n_1}^m d\mu = 0$.

But if $n = n_1$, then

$$\int_{-1}^{1} (Q_n^m)^2 d\mu = 2\,\frac{(n+m)!}{(n-m)!}\,\frac{1}{2n+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

19. The position of a point on the unit sphere may be determined by the coordinates

$$\mu, \quad (1-\mu^2)^{\frac{1}{2}}\cos\phi, \quad (1-\mu^2)^{\frac{1}{2}}\sin\phi;$$

$2\pi\delta\mu$ is the surface of an elementary zone and therefore $\delta\phi\delta\mu$ is an element of the surface at the point defined by μ and ϕ.

Any rational and integral function of the coordinates of the point may be expressed in terms of the form

$$M(1-\mu^2)^{\frac{r}{2}}\cos(r\phi+\alpha),$$

where M is a rational and integral function of μ.

Let $$q = \cos\gamma = \mu\mu_1 + (1-\mu^2)^{\frac{1}{2}}(1-\mu_1^2)^{\frac{1}{2}}\cos\phi, \quad *$$

then if $$V = (1-2h\cos\gamma + h^2)^{-\frac{1}{2}},$$

and if Q_n be the coefficient of h^n in its expansion so that Q_n is the same function of q that P_n is of μ; then since

$$P_{n+1} = \mu P_n + \frac{\mu^2-1}{n+1}\frac{dP_n}{d\mu},$$

we have $$Q_{n+1} = qQ_n + \frac{q^2-1}{n+1}\frac{dQ_n}{dq},$$

and similar relations for Q_n to those which have been found above for P_n.

* Note. For μ_1 read μ' in Articles 19—22 of this Section.

Suppose δq to be an increment of q corresponding to increments $\delta\mu$, $\delta\mu_1$ and $\delta\phi$, then

$$\delta q = \delta\mu \left\{\mu_1 - \frac{\mu}{(1-\mu^2)^{\frac{1}{2}}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi\right\} + \delta\mu_1 \left\{\mu - \frac{\mu_1}{(1-\mu_1^2)^{\frac{1}{2}}} (1-\mu^2)^{\frac{1}{2}} \cos\phi\right\}$$
$$- \delta\phi\, (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \sin\phi.$$

Now let $\qquad \delta\mu = \epsilon\mu_1 (1-\mu^2), \quad \delta\mu_1 = \epsilon\mu (1-\mu_1^2),$

and $\qquad \delta\phi = -\epsilon (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \sin\phi.$

Then

$$\begin{aligned}\delta q &= \epsilon\{\mu_1^2 (1-\mu^2) + \mu^2 (1-\mu_1^2) - 2\mu\mu_1 (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi + (1-\mu^2)(1-\mu_1^2)\sin^2\phi\} \\ &= \epsilon\{1 - \mu^2\mu_1^2 - 2\mu\mu_1 (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi - (1-\mu^2)(1-\mu_1^2)\cos^2\phi\} \\ &= \epsilon(1-q^2).\end{aligned}$$

Hence the increment of Q_n corresponding to these increments will be

$$\frac{dQ}{dq}\delta q = \epsilon(1-q^2)\frac{dQ_n}{dq},$$

but if Q_n be regarded as a function of μ, μ_1 and ϕ, the same increment will be represented by

$$\frac{dQ_n}{d\mu}\delta\mu + \frac{dQ_n}{d\mu_1}\delta\mu_1 + \frac{dQ_n}{d\phi}\delta\phi$$
$$= \epsilon\left\{\mu_1 (1-\mu^2)\frac{dQ_n}{d\mu} + \mu(1-\mu_1^2)\frac{dQ_n}{d\mu_1} - (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \sin\phi \frac{dQ_n}{d\phi}\right\}.$$

Hence equating the coefficients of ϵ we have

$$(1-q^2)\frac{dQ_n}{dq} = \mu_1 (1-\mu^2)\frac{dQ_n}{d\mu} + \mu(1-\mu_1^2)\frac{dQ_n}{d\mu_1} - (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \sin\phi\frac{dQ_n}{d\phi}.$$

Substituting in the above equation for Q_{n+1} we have

$$Q_{n+1} = (\mu\mu_1 + (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi)\, Q_n - \frac{1}{n+1}\mu_1 (1-\mu^2)\frac{dQ_n}{d\mu}$$
$$- \frac{1}{n+1}\mu(1-\mu_1^2)\frac{dQ_n}{d\mu_1} + \frac{1}{n+1}(1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \sin\phi\frac{dQ_n}{d\phi}.$$

This equation may be made use of to prove that the general term of the series for Q_n is

$$2\frac{(n-m)!}{(n+m)!}(1-\mu^2)^{\frac{m}{2}} (1-\mu_1^2)^{\frac{m}{2}} \frac{d^m P_n}{d\mu^m}\frac{d^m P_n'}{d\mu_1^m}\cos m\phi.$$

This expression may more readily be proved as follows:—

20. We see that, when $n=1$ and $m=1$,

$$Q_1 = q = \mu\mu_1 + (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi$$

$$= P_1 P_1' + \frac{dP_1}{d\mu} \frac{dP_1'}{d\mu_1} (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi$$

$$= P_1 P_1' + \frac{2}{n(n+1)} \frac{dP_1}{d\mu} \frac{dP_1'}{d\mu_1} (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi.$$

Now assume that

$$Q_n = P_n P_n' + \frac{2}{n(n+1)} \frac{dP_1}{d\mu} \frac{dP_1'}{d\mu_1} (1-\mu^2)^{\frac{1}{2}} (1-\mu_1^2)^{\frac{1}{2}} \cos\phi + \&c.$$

$$+ 2\frac{(n-m)!}{(n+m)!} (1-\mu^2)^{\frac{m}{2}} (1-\mu_1^2)^{\frac{m}{2}} \frac{d^m P_n}{d\mu^m} \frac{d^m P_n'}{d\mu_1^m} \cos m\phi + \&c. \quad \ldots\ldots(15).$$

From equation (3) of Section I. we have

$$Q_{n+1}(n+1) = (2n+1)\, qQ_n - nQ_{n-1}.$$

The coefficient of $2\cos m\phi$ in qQ_n is

$$\frac{(n-m)!}{(n+m)!} \mu\mu_1 (1-\mu^2)^{\frac{m}{2}} (1-\mu_1^2)^{\frac{m}{2}} \frac{d^m P_n}{d\mu^m} \frac{d^m P_n'}{d\mu_1^m}$$

$$+ \frac{1}{2} \frac{(n-m+1)!}{(n+m-1)!} (1-\mu^2)^{\frac{m}{2}} (1-\mu_1^2)^{\frac{m}{2}} \frac{d^{m-1} P_n}{d\mu^{m-1}} \frac{d^{m-1} P_n'}{d\mu_1^{m-1}}$$

$$+ \frac{1}{2} \frac{(n-m-1)!}{(n+m+1)!} (1-\mu^2)^{\frac{m}{2}+1} (1-\mu_1^2)^{\frac{m}{2}+1} \frac{d^{m+1} P_n}{d\mu^{m+1}} \frac{d^{m+1} P_n'}{d\mu_1^{m+1}}.$$

Substituting for $\mu \dfrac{d^m P_n}{d\mu^m}$, $\dfrac{d^{m-1} P_n}{d\mu^{m-1}}$ and $(1-\mu^2)\dfrac{d^{m+1} P_n}{d\mu^{m+1}}$ in terms of

$$\frac{d^m P_{n+1}}{d\mu^m} \quad \frac{d^m P_{n-1}}{d\mu^m},$$

and similarly for $\mu_1 \dfrac{d^m P_n'}{d\mu_1^m}$, &c. by means of equations (1), (5) and (6) given above. Hence the coefficient of $2\cos m\phi$ in qQ_n becomes

$$\frac{1}{(2n+1)^2} \frac{(n-m-1)!}{(n+m+1)!} (1-\mu^2)^{\frac{m}{2}} (1-\mu_1^2)^{\frac{m}{2}} \left\{ \frac{1}{2} \left[-(n-m)(n-m+1) \frac{d^m P_{n+1}}{d\mu^m} \right.\right.$$

$$\left. + (n+m)(n+m+1) \frac{d^m P_{n-1}}{d\mu^m} \right]$$

$$\times \left[-(n-m)(n-m+1) \frac{d^m P'_{n+1}}{d\mu_1^m} + (n+m)(n+m+1) \frac{d^m P'_{n-1}}{d\mu_1^m} \right]$$

$$+(n-m)(n+m+1)\left[(n-m+1)\frac{d^m P_{n+1}}{d\mu^m}+(n+m)\frac{d^m P_{n-1}}{d\mu^m}\right]$$
$$\times\left[(n-m+1)\frac{d^m P'_{n+1}}{d\mu_1^m}+(n+m)\frac{d^m P'_{n-1}}{d\mu_1^m}\right]$$
$$+\frac{1}{2}(n-m)(n-m+1)(n+m)(n+m+1)\left[\frac{d^m P_{n+1}}{d\mu^m}-\frac{d^m P_{n-1}}{d\mu^m}\right]$$
$$\times\left[\frac{d^m P'_{n+1}}{d\mu_1^m}-\frac{d^m P'_{n-1}}{d\mu_1^m}\right]\Bigg\}.$$

The coefficient of $\dfrac{d^m P_{n+1}}{d\mu^m}\,\dfrac{d^m P'_{n+1}}{d\mu_1^m}$ within the large brackets is

$$\frac{1}{2}(n-m)^2(n-m+1)^2+(n-m)(n-m+1)^2(n+m+1)$$
$$+\frac{1}{2}(n-m)(n-m+1)(n+m)(n+m+1).$$

Unite half the middle term to each of the other two and this reduces to

$$\frac{1}{2}(n-m)(n-m+1)[(n-m+1)(n-m+n+m+1)$$
$$+(n+m+1)(n-m+1+n+m)]=(n-m)(n-m+1)(n+1)(2n+1).$$

Also it may readily be seen that the coefficient of

$$\frac{d^m P_{n+1}}{d\mu^m}\,\frac{d^m P'_{n-1}}{d\mu_1^m}+\frac{d^m P_{n-1}}{d\mu^m}\,\frac{d^m P'_{n+1}}{d\mu_1^m}$$

is equal to 0 identically.

And the coefficient of $\dfrac{d^m P_{n-1}}{d\mu^m}\,\dfrac{d^m P'_{n-1}}{d\mu_1^m}$ is

$$\frac{1}{2}(n+m)(n+m+1)[(n+m)(n+m+1+n-m)+(n-m)(n+m+n-m+1)]$$
$$=(n+m)(n+m+1)\,n\,(2n+1).$$

Hence the coefficient of $2\cos m\phi$ in $(2n+1)\,Q_n q$ is

$$(1-\mu^2)^{\frac{m}{2}}(1-\mu_1^2)^{\frac{m}{2}}\left\{(n+1)\frac{(n-m+1)!}{(n+m+1)!}\,\frac{d^m P_{n+1}}{d\mu^m}\,\frac{d^m P'_{n+1}}{d\mu_1^m}\right.$$
$$\left.+n\frac{(n-m-1)!}{(n+m-1)!}\,\frac{d^m P_{n-1}}{d\mu^m}\,\frac{d^m P'_{n-1}}{d\mu_1^m}\right\}.$$

And the coefficient of $2\cos m\phi$ in $-nQ_{n-1}$ is

$$(1-\mu^2)^{\frac{m}{2}}(1-\mu_1^2)^{\frac{m}{2}}\left\{-n\frac{(n-m-1)!}{(n+m-1)!}\frac{d^mP_{n-1}}{d\mu^m}\frac{d^mP'_{n-1}}{d\mu_1^m}\right\}.$$

Hence adding these last results and dividing by $(n+1)$, the coefficient of $2\cos m\phi$ in Q_{n+1} is

$$=\frac{(n-m+1)!}{(n+m+1)!}(1-\mu^2)^{\frac{m}{2}}(1-\mu_1^2)^{\frac{m}{2}}\frac{d^mP_{n+1}}{d\mu^m}\frac{d^mP'_{n+1}}{d\mu_1^m}.$$

Hence the same law holds good for Q_{n+1}, and since the expression assumed is evidently true when $n=0$ and when $n=1$, it is true generally.

21. (The same proof is applicable to the term independent of ϕ, i.e. when $m=0$.)

The term independent of ϕ in Q_nq is

$$\mu\mu_1P_nP_n'+\frac{(n-1)!}{(n+1)!}(1-\mu^2)(1-\mu_1^2)\frac{dP_n}{d\mu}\frac{dP_n'}{d\mu_1}.$$

Now $$\mu P_n-\frac{1-\mu^2}{n+1}\frac{dP_n}{d\mu}=P_{n+1},$$

and $$(2n+1)\mu P_n=(n+1)P_{n+1}+nP_{n-1};$$

hence $$(2n+1)\frac{1-\mu^2}{n+1}\frac{dP_n}{d\mu}=n(P_{n-1}-P_{n+1});$$

$$\therefore\ (2n+1)^2\mu\mu_1P_nP_n'=[(n+1)P_{n+1}+nP_{n-1}][(n+1)P'_{n+1}+nP'_{n-1}]$$

and $$\frac{(2n+1)^2}{(n+1)^2}(1-\mu^2)(1-\mu_1^2)\frac{dP_n}{d\mu}\frac{dP_n'}{d\mu_1}=n^2(-P_{n+1}+P_{n-1})(-P'_{n+1}+P'_{n-1});$$

therefore the term independent of ϕ in $(2n+1)^2Q_nq$ is

$$\begin{aligned}&[(n+1)P_{n+1}+nP_{n-1}][(n+1)P'_{n+1}+nP'_{n-1}]\\&\qquad+\frac{(n-1)!}{(n+1)!}n^2(n+1)^2[-P_{n+1}+P_{n-1}][-P'_{n+1}+P'_{n-1}]\\&=[(n+1)^2+n(n+1)]P_{n+1}P'_{n+1}+[n^2+n(n+1)]P_{n-1}P'_{n-1}\\&=(2n+1)[(n+1)P_{n+1}P'_{n+1}+nP_{n-1}P'_{n-1}];\end{aligned}$$

therefore the term independent of ϕ in

$$(n+1)Q_{n+1}\ \text{or}\ [(2n+1)Q_nq-nQ_{n-1}]\ \text{is}\ (n+1)P_{n+1}P'_{n+1}.$$

Hence the first term of Q_{n+1} is $P_{n+1}P'_{n+1}$, and the law is true generally.

The last term of Q_n, when $m=n$, is

$$\frac{1}{2n!}(1-\mu^2)^{\frac{n}{2}}(1-\mu_1^2)^{\frac{n}{2}}\frac{d^nP_n}{d\mu^n}\frac{d^nP_n'}{d\mu_1^n}2\cos n\phi.$$

Also
$$\frac{d^nP_n}{d\mu^n}=\frac{d^nP_n'}{d\mu_1^n}=\frac{1}{2^n}\frac{1}{n!}\frac{d^{2n}(\mu^2-1)^n}{d\mu^{2n}}$$

$$=\frac{1}{2^n}\frac{1}{n!}2n!=1.3.5\ldots(2n-1).$$

And the last term becomes

$$\frac{2n!}{2^{2n}(n!)^2}(1-\mu^2)^{\frac{n}{2}}(1-\mu_1^2)^{\frac{n}{2}}2\cos n\phi$$

$$=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}(1-\mu^2)^{\frac{n}{2}}(1-\mu_1^2)^{\frac{n}{2}}2\cos n\phi.$$

22. Since $\int^m P_n d\mu^m=\frac{d^{-m}P_n}{d\mu^{-m}}=(-1)^m\frac{(n-m)!}{(n+m)!}(1-\mu^2)^m\frac{d^mP_n}{d\mu^m}$,

where $\int^m P_n d\mu^m$ is so taken that at each successive integration the result is divisible by $1-\mu^2$ without remainder, we have

$$\frac{d^{-m}P_n}{d\mu^{-m}}\frac{d^{-m}P_n'}{d\mu_1^{-m}}=\left\{\frac{(n-m)!}{(n+m)!}\right\}^2(1-\mu^2)^m(1-\mu_1^2)^m\frac{d^mP_n}{d\mu^m}\frac{d^mP_n'}{d\mu_1^m},$$

or
$$\frac{(n+m)!}{(n-m)!}(1-\mu^2)^{-\frac{m}{2}}(1-\mu_1^2)^{-\frac{m}{2}}\frac{d^{-m}P_n}{d\mu^{-m}}\frac{d^{-m}P_n'}{d\mu_1^{-m}}$$

$$=\frac{(n-m)!}{(n+m)!}(1-\mu^2)^{\frac{m}{2}}(1-\mu_1^2)^{\frac{m}{2}}\frac{d^mP_n}{d\mu^m}\frac{d^mP_n'}{d\mu_1^m}\ \ldots\ldots(16).$$

Hence Q_n may be put under the symmetrical form

$$Q_n=\Sigma\frac{(n+m)!}{(n-m)!}(1-\mu^2)^{-\frac{m}{2}}(1-\mu_1^2)^{-\frac{m}{2}}\frac{d^{-m}P_n}{d\mu^{-m}}\frac{d^{-m}P_n'}{d\mu_1^{-m}}\cos(-m\phi)+P_nP_n'$$
$$+\Sigma\frac{(n-m)!}{(n+m)!}(1-\mu^2)^{\frac{m}{2}}(1-\mu_1^2)^{\frac{m}{2}}\frac{d^mP_n}{d\mu^m}\frac{d^mP_n'}{d\mu_1^m}\cos m\phi.$$

We observe that $0!=1$ and $(-1)!=\frac{0!}{0}=\infty$, so that there is no term before that involving $\frac{2n!}{0!}$.

Also there is no term after that involving $\frac{0!}{2n!}$ since $\frac{d^{n+1}P_n}{d\mu^{n+1}}=0$.

23. If now we put $\mu=\mu_1$, and if we put

$$2\cos\phi=x+\frac{1}{x} \text{ and } 2\cos m\phi=x^m+\frac{1}{x^m},$$

the value of Q_n becomes

$$Q_n=\frac{0\,!}{2n\,!}(1-\mu^2)^n\left(\frac{d^nP_n}{d\mu^n}\right)^2\left(x^n+\frac{1}{x^n}\right)+\dots$$

$$+\frac{(n-m)\,!}{(n+m)\,!}(1-\mu^2)^m\left(\frac{d^mP_n}{d\mu^m}\right)^2\left(x^m+\frac{1}{x^m}\right)+\dots$$

$$+(P_n)^2.$$

The value of Q_n may also be put under the form

$$Q_n=(-1)^n(1-\mu^2)^{-\frac{n}{2}}(1-\mu_1^2)^{\frac{n}{2}}\frac{d^{-n}P_n}{d\mu^{-n}}\frac{d^nP_n'}{d\mu_1^n}\frac{1}{x^n}+\dots$$

$$+(-1)^m(1-\mu^2)^{-\frac{m}{2}}(1-\mu_1^2)^{\frac{m}{2}}\frac{d^{-m}P_n}{d\mu^{-m}}\frac{d^mP_n'}{d\mu_1^m}\frac{1}{x^m}+\dots$$

$$+P_nP_n'+\dots$$

$$+(-1)^m(1-\mu^2)^{\frac{m}{2}}(1-\mu_1^2)^{-\frac{m}{2}}\frac{d^mP_n}{d\mu^m}\frac{d^{-m}P_n'}{d\mu_1^{-m}}x^m+\dots$$

$$+(-1)^n(1-\mu^2)^{\frac{n}{2}}(1-\mu_1^2)^{-\frac{n}{2}}\frac{d^nP_n}{d\mu^n}\frac{d^{-n}P_n'}{d\mu_1^{-n}}x^n.$$

If now we put $\mu=\mu_1$, the value of Q_n becomes

$$Q_n=(-1)^n\frac{d^{-n}P_n}{d\mu^{-n}}\frac{d^nP_n}{d\mu^n}\frac{1}{x^n}+\dots+(-1)^m\frac{d^{-m}P_n}{d\mu^{-m}}\frac{d^mP_n}{d\mu^m}\frac{1}{x^m}$$

$$+\dots+(P_n)^2+\dots+(-1)^m\frac{d^mP_n}{d\mu^m}\frac{d^{-m}P_n}{d\mu^{-m}}x^m+\dots+(-1)^n\frac{d^nP_n}{d\mu^n}\frac{d^{-n}P_n}{d\mu^{-n}}x^n.$$

24. When $\mu=\mu_1$,

$$\cos\gamma=\mu\mu_1+(1-\mu^2)^{\frac{1}{2}}(1-\mu_1^2)^{\frac{1}{2}}\cos\phi=\mu^2+(1-\mu^2)\cos\phi,$$

and
$$V=\{1-2h[\mu^2+(1-\mu^2)\cos\phi]+h^2\}^{-\frac{1}{2}}$$

$$=\{(1-h)^2+2h(1-\mu^2)(1-\cos\phi)\}^{-\frac{1}{2}}.$$

Expanding by the Binomial Theorem, we get

$$V=\frac{1}{1-h}\left\{1-\frac{1}{2}\frac{2h(1-\cos\phi)}{(1-h)^2}(1-\mu^2)+\frac{1\,.\,3}{2\,.\,4}\left[\frac{2h(1-\cos\phi)}{(1-h)^2}\right]^2(1-\mu^2)^2\right.$$

$$-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left[\frac{2h(1-\cos\phi)}{(1-h)^2}\right]^3(1-\mu^2)^3+\&c.,$$

the $(r+1)$th term being

$$\left.+(-1)^r\frac{1\,.\,3\,.\,5\ldots\overline{2r-1}}{2\,.\,4\,.\,6\ldots 2r}\left[\frac{2h(1-\cos\phi)}{(1-h)^2}\right]^r(1-\mu^2)^r+\&c.\right\}.$$

Now multiply by $d\mu$ and integrate from $\mu=-1$ to $\mu=1$, observing that

$$\int_{-1}^{1}(1-\mu^2)^r d\mu=2\int_0^1(1-\mu^2)^r d\mu=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2r+1}d\theta,\qquad [\text{if } \mu=\cos\theta]$$

or

$$\int_{-1}^{1}(1-\mu^2)^r d\mu=2\left\{\frac{2r\,\overline{2r-2}\ldots 2}{\overline{2r+1}\,\overline{2r-1}\ldots 3}\right\}.$$

Hence we have

$$\int_{-1}^{1}V d\mu=\frac{2}{1-h}\left\{1-\frac{1}{3}\left[\frac{2h(1-\cos\phi)}{(1-h)^2}\right]+\frac{1}{5}\left[\frac{2h(1-\cos\phi)}{(1-h)^2}\right]^2-\&c.,\right.$$

the $(r+1)$th term being

$$\left.+(-1)^r\frac{1}{2r+1}\left[\frac{2h(1-\cos\phi)}{(1-h)^2}\right]^r+\&c.\right\}.$$

Now if $$\frac{2h(1-\cos\phi)}{(1-h)^2}=\tan^2\theta \quad\text{or}\quad \tan\theta=\frac{h^{\frac{1}{2}}\,2\sin\frac{1}{2}\phi}{1-h},$$

we have $$\frac{2}{1-h}=\frac{\tan\theta}{h^{\frac{1}{2}}\sin\frac{1}{2}\phi},$$

and

$$\int_{-1}^{1}V d\mu=\frac{1}{h^{\frac{1}{2}}\sin\frac{1}{2}\phi}\left\{\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\&c.+(-1)^r\frac{1}{2r+1}\tan^{2r+1}\theta+\&c.\right\}$$

$$=\frac{1}{h^{\frac{1}{2}}\sin\frac{1}{2}\phi}\{\theta\},\ \text{as before found.}$$

Now $$\tan\theta\sqrt{(-1)} = \frac{h^{\frac{1}{2}}\,2\sqrt{(-1)}\sin\frac{1}{2}\phi}{1-h} = \frac{h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})}{1-h},\ \text{if } 2\cos\phi = x+\frac{1}{x};$$

$$\therefore \frac{\cos\theta-\sqrt{(-1)}\sin\theta}{\cos\theta+\sqrt{(-1)}\sin\theta} = \frac{1-\sqrt{(-1)}\tan\theta}{1+\sqrt{(-1)}\tan\theta} = \frac{1-h-h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})}{1-h+h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})},$$

or $$\epsilon^{-2\theta\sqrt{(-1)}} = \frac{(1-h^{\frac{1}{2}}x^{\frac{1}{2}})(1+h^{\frac{1}{2}}x^{-\frac{1}{2}})}{(1+h^{\frac{1}{2}}x^{\frac{1}{2}})(1-h^{\frac{1}{2}}x^{-\frac{1}{2}})}.$$

Take logarithms of both sides and change signs;

$$\therefore\ 2\theta\sqrt{(-1)} = 2\left\{h^{\frac{1}{2}}x^{\frac{1}{2}}+\frac{1}{3}h^{\frac{3}{2}}x^{\frac{3}{2}}+\frac{1}{5}h^{\frac{5}{2}}x^{\frac{5}{2}}+\text{\&c.}\right.$$
$$\left.-h^{\frac{1}{2}}x^{-\frac{1}{2}}-\frac{1}{3}h^{\frac{3}{2}}x^{-\frac{3}{2}}-\frac{1}{5}h^{\frac{5}{2}}x^{-\frac{5}{2}}-\text{\&c.}\right\},$$

or $$\theta = \frac{1}{\sqrt{(-1)}}\left\{h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})+\frac{1}{3}h^{\frac{3}{2}}(x^{\frac{3}{2}}-x^{-\frac{3}{2}})+\frac{1}{5}h^{\frac{5}{2}}(x^{\frac{5}{2}}-x^{-\frac{5}{2}})+\text{\&c.}\right\};$$

$$\therefore \int_{-1}^{1}V\,d\mu = \frac{\theta}{h^{\frac{1}{2}}\sin\frac{1}{2}\phi} = \frac{1}{h^{\frac{1}{2}}\sqrt{(-1)}\sin\frac{1}{2}\phi}\{h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})+\text{\&c., \&c., \&c.}\}$$

$$= \frac{2}{h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})}\left\{h^{\frac{1}{2}}(x^{\frac{1}{2}}-x^{-\frac{1}{2}})+\frac{1}{3}h^{\frac{3}{2}}(x^{\frac{3}{2}}-x^{-\frac{3}{2}})+\text{\&c.}\right\}$$

$$= 2\left\{1+\frac{1}{3}h\left(x+\frac{1}{x}+1\right)+\frac{1}{5}h^2\left(x^2+\frac{1}{x^2}+x+\frac{1}{x}+1\right)+\text{\&c.}\right\}$$

$$= 2\left\{1+\frac{1}{3}h\,[2\cos\phi+1]+\frac{1}{5}h^2[2\cos 2\phi+2\cos\phi+1]+\text{\&c.}\right\},$$

where the law is manifest.

SECTION IV.

ON THE PRODUCT OF ANY TWO LAPLACE'S COEFFICIENTS OF THE FORM $\frac{d^m P_n}{d\mu^m} \times \frac{d^p P_q}{d\mu^p}$.

1. We have already shewn (see Vol. I. p. 487) how to exhibit the product of two Legendre's coefficients, $P_n P_q$, by means of a series of Legendre's coefficients. In order to complete the theory, we must shew how to multiply together any two Laplace's coefficients, so as to exhibit the product as a sum of Laplace's coefficients. There can be little doubt that a method similar to that which has been already employed will be equally successful in the more general case.

The general form of two Laplace's coefficients, whose product we wish to express, may be denoted by

$$R_n^m = \frac{d^m P_n}{d\mu^m}(1-\mu^2)^{\frac{m}{2}} \cos m\lambda$$

and

$$R_q^p = \frac{d^p P_q}{d\mu^p}(1-\mu^2)^{\frac{p}{2}} \cos p\lambda,$$

where λ is the longitude. The product is of the form

$$R_n^m R_q^p = \frac{1}{2}\frac{d^m P_n}{d\mu^m}\frac{d^p P_q}{d\mu^p}(1-\mu^2)^{\frac{m+p}{2}}\{\cos(m+p)\lambda + \cos(m-p)\lambda\}$$

$$= \frac{1}{2}\frac{d^m P_n}{d\mu^m}\frac{d^p P_q}{d\mu^p}(1-\mu^2)^{\frac{m+p}{2}}\cos(m+p)\lambda$$

$$+ \frac{1}{2}\frac{d^m P_n}{d\mu^m}\frac{d^p P_q}{d\mu^p}(1-\mu^2)^p(1-\mu^2)^{\frac{m-p}{2}}\cos(m-p)\lambda.$$

Hence in order to solve our problem, we must find how to express $\frac{d^m P_n}{d\mu^m}\frac{d^p P_q}{d\mu^p}$ in terms of the form $\frac{d^{m+p} P}{d\mu^{m+p}}$ and also to express $\frac{d^m P_n}{d\mu^m}\frac{d^p P_q}{d\mu^p}(1-\mu^2)^p$ in terms of the form $\frac{d^{m-p} P}{d\mu^{m-p}}$, multiplied by constants.

We will now try how far *a priori* considerations will guide us to the form of the coefficient of $\frac{d^{m+p} P_{n+q-2r}}{d\mu^{m+p}}$ in the value of $\frac{d^m P_n}{d\mu^m}\frac{d^p P_q}{d\mu^p}$.

The highest index of P_x will be $n+q$, when $r=0$.

The coefficient of the corresponding term, where $p=q$ and $r=0$, is

$$\frac{1\,.\,3\,.\,5\ldots(2p-1)\,1\,.\,3\,.\,5\ldots(2n-1)}{1\,.\,3\,.\,5\ldots(2n+2p-1)}.$$

But in passing from the value of $\frac{d^p P_{q-1}}{d\mu^p}\frac{d^m P_n}{d\mu^m}$ to that of $\frac{d^p P_q}{d\mu^p}\frac{d^m P_n}{d\mu^m}$, the coefficient of the term with the highest index of P will be multiplied by

$$\frac{2q-1}{q-p}\,\frac{n+q-m-p}{2n+2q-1},$$

which equals

$$\frac{(q-p-1)!\;1\,.\,3\,.\,5\ldots(2q-1)\,(n+q-m-p)!\;1\,.\,3\,.\,5\ldots(2n+2q-3)}{(q-p)!\;1\,.\,3\,.\,5\ldots(2q-3)\,(n+q-m-p-1)!\;1\,.\,3\,.\,5\ldots(2n+2q-1)}.$$

The coefficient of the term which has the highest value of the subscribed index, viz. $n+q$, will be

$$\frac{(n+q-m-p)!\;1\,.\,3\,.\,5\ldots(2q-1)\,(1\,.\,3\,.\,5\ldots 2n-1)}{(n-m)!\;(q-p)!\;1\,.\,3\,.\,5\ldots(2n+2q-1)}.$$

The general term must reduce to this, when $r=0$.

Also when $p=0$ and $m=0$,

the form of the general term must reduce to

$$\frac{(n+q-r)!}{r!\,(n-r)!\,(q-r)!}\,\frac{1\,.\,3\,.\,5\ldots(2r-1)\,1\,.\,3\,.\,5\ldots(2q-2r-1)\,1\,.\,3\,.\,5\ldots(2n-2r-1)}{1\,.\,3\,.\,5\ldots(2n+2q-2r-1)}$$
$$\times\frac{(2n+2q-4r+1)}{(2n+2q-2r+1)},$$

i.e. to
$$\frac{A(r)\,A(n-r)\,A(q-r)}{A(n+q-r)}\,\frac{2n+2q-4r+1}{2n+2q-2r+1},$$

where
$$A(r)=\frac{1\,.\,3\,.\,5\,\ldots\,(2r-1)}{1\,.\,2\,.\,3\,\ldots\,r},$$

as in a previous paper (see Vol. I. p. 492).

Also if $p=q$, the coefficient of the general term reduces to

$$(-1)^r\frac{q!}{r!\,(q-r)!}\,\frac{1\,.\,3\,.\,5\,\ldots\,(2q-1)\,1\,.\,3\,.\,5\,\ldots\,(2n-2r-1)}{1\,.\,3\,.\,5\,\ldots\,(2n+2q-2r+1)}(2n+2q-4r+1).$$

Also if p is greater than q, or if m is greater than n, the whole expression vanishes. This seems to imply that $\frac{1}{(q-p)!}$ and $\frac{1}{(n-m)!}$ occur in every term.

If $m+p$ is greater than $n+q$, the values of the terms become indeterminate, since whatever their coefficients may be, they will be made to disappear by differentiation.

This would seem to imply that $(n+q-m-p)!$ is a factor.

The expression $A(r)=\frac{1\,.\,3\,.\,5\,\ldots\,(2r-1)}{r!}=\frac{2r!}{2^r\,(r!)^2}$.

Hence
$$2^r\,(r!)^2\,A(r)=2r!.$$

2. Let us now express $\frac{d^mP_n}{d\mu^m}$ or D^mP_n in terms of P_{n-m-2r}, adopting the notation D^m for $\left(\frac{d}{d\mu}\right)^m$ as an operator. We have

$$DP_n-DP_{n-2}=(2n-1)\,P_{n-1}.$$

Hence by successive additions we get

$$DP_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\&c.+3P_1 \text{ if } n \text{ be even,}$$

or
$$DP_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\&c.+\ P_0 \text{ if } n \text{ be odd.}$$

By differentiating and substituting for the first differential coefficients

$$\begin{aligned}D^2P_n=&(2n-1)\,\{(2n-3)\,P_{n-2}+(2n-\ 7)\,P_{n-4}+\&c.+P_0\}\\&+(2n-5)\,\{(2n-7)\,P_{n-4}+(2n-11)\,P_{n-6}+\&c.+P_0\}\\&+\&c.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+3P_0,\ \text{if } n \text{ be even.}\end{aligned}$$

The coefficient of P_0 is

$$(2n-1)+(2n-5)+\&c.+3 \text{ to } \frac{n}{2} \text{ terms} = \frac{n(n+1)}{2}.$$

The coefficient of P_2 is

$$5\{(2n-1)+(2n-5)+\&c.+7\} = 5\left\{\frac{n(n+1)}{2}-3\right\} = \frac{5(n+3)(n-2)}{2}.$$

The coefficient of P_4 is $\dfrac{9(n+5)(n-4)}{2}$, and so on.

The coefficient of P_{n-2} is $(2n-3)\dfrac{2(2n-1)}{2} = (2n-3)(2n-1)$.

Hence when n is even we have

$$2D^2P_n = 2(2n-1)(2n-3)P_{n-2} + 4(2n-3)(2n-7)P_{n-4} + \&c.$$
$$+ 2(r+1)(2n-2r-1)(2n-4r-3)P_{n-2-2r} + \&c. \text{ to } + n(n+1)P_0.$$

Similarly when n is odd we have

$$2D^2P_n = 2(2n-1)(2n-3)P_{n-2} + \&c. + 2(r+1)(2n-2r-1)(2n-4r-3)P_{n-2-2r}$$
$$+ \&c. \text{ to } (n-1)(n+2)3P_1.$$

Following the same method of expansion we get values for the successive differential coefficients,

$$2D^3P_n = \Sigma\{(r+1)(2n-2r-1)(2n-4r-3)DP_{n-2-2r}\}$$
$$= 2(2n-1)(2n-3)\{(2n-5)P_{n-3} + (2n-9)P_{n-5} + \&c. + P_0\}$$
$$+ 4(2n-3)(2n-7)\{(2n-9)P_{n-5} + \&c. + P_0\} + \&c.$$
$$+ 2(r+1)(2n-2r-1)(2n-4r-3)\{(2n-4r-5)P_{n-2r-3} + \&c. + P_0\}$$
$$+ \&c. + (n-1)(n+2)3P_0, \text{ when } n \text{ is odd.}$$

The coefficients of the successive terms when collected give the law of formation as follows:

The coefficient of $(2n-4r-1)P_{n-2r-1}$ is

$$r(r+1)(2n-2r+1)(2n-2r-1),$$

and the last term is that when $n-2r-1=0$ or 1, so that the coefficient of P_0 is $\dfrac{(n-1)(n+1)(n+2)n}{2\,.\,2}$ when n is odd, and the coefficient of P_1 is $\dfrac{(n-2)n(n+3)(n+1)3}{2\,.\,2}$ when n is even.

Hence

$$2D^3 P_n = \Sigma \{(r+1)(r+2)(2n-2r-1)(2n-2r-3)(2n-4r-5) P_{n-2r-3}\}.$$

Continuing the same method of reasoning and applying the method of induction to prove the law for successive terms, we get

$$3!\, D^4 P_n = \Sigma \{(r+1)(r+2)(r+3)(2n-2r-1)(2n-2r-3) \\ \times (2n-2r-5)(2n-4r-7) P_{n-2r-4}\},$$

writing down only the $(r+1)$th term.

When n is odd, the last term is when $n-2r-4=1$, so that the term is

$$\frac{n-3}{2}\frac{n-1}{2}\frac{n+1}{2}(n+4)(n+2)\,n\,.\,3P_1.$$

When n is even, the last term is when $n-2r-4=0$, so that the last term is

$$\frac{n-2}{2}\frac{n}{2}\frac{n+2}{2}(n+3)(n+1)(n-1)P_0.$$

Hence we get the $(r+1)$th term in the expression for

$$(m-1)!\, D^m P_n = \Sigma \{(r+1)(r+2)\dots(r+m-1)(2n-2r-1)(2n-2r-3)\dots \\ \times (2n-2r-2m+3)(2n-4r-2m+1) P_{n-m-2r}\},$$

the last value for $n-m-2r$ being 0 or 1, according as $n-m$ is even or odd.

Hence

$$D^m P_n = \Sigma \left\{\frac{(r+m-1)!}{r!\,(m-1)!}\frac{1\,.\,3\,.\,5\dots(2n-2r-1)}{1\,.\,3\,.\,5\dots(2n-2m-2r+1)}(2n-2m-4r+1) P_{n-m-2r}\right\}.$$

Putting $n+1$ for n and $m+1$ for m we get

$$D^{m+1} P_{n+1} = \Sigma \left\{\frac{(m+r)!}{r!\,m!}\frac{1\,.\,3\,.\,5\dots(2n-2r+1)}{1\,.\,3\,.\,5\dots(2n-2m-2r+1)}(2n-2m-4r+1) P_{n-m-2r}\right\}.$$

3. We have seen above that $(2n+1) D^m P_n = D^{m+1} P_{n+1} - D^{m+1} P_{n-1}$, and also that $(2n+1)\mu D^m P_n = (n-m+1) D^m P_{n+1} + (n+m) D^m P_{n-1}$.

Hence substituting the values given from the above series for $D^{m+1}P_{n+1}$ and $D^{m+1}P_{n-1}$, and taking the term involving P_{n-m-2r} in the series for $\mu D^{m+1}P_n$, we get

$$(2n+1)\mu D^{m+1} P_n = (n-m) D^{m+1} P_{n+1} + (n+m+1) D^{m+1} P_{n-1},$$

$$(2n+1)\mu D^{m+1} P_n = \Sigma \left\{\frac{(m+r-1)!}{m!\,(r-1)!}\frac{1\,.\,3\,.\,5\dots(2n-2r-1)}{1\,.\,3\,.\,5\dots(2n-2m-2r-1)}\right. \\ \left.\times\left[\frac{m+r}{r}\frac{(2n-2r+1)(n-m)}{(2n-2m-2r+1)} + (n+m+1)\right](2n-2m-4r+1) P_{n-m-2r}\right\}.$$

The quantity in brackets $=(2n+1)\left[\dfrac{(n-m)\,m+r\,(2n-2m-2r+1)}{r\,(2n-2m-2r+1)}\right]$;

hence

$$\mu D^{m+1}P_n=\Sigma\left\{\frac{(m+r-1)!}{m!\,r!}\,\frac{1\,.\,3\,.\,5\,\ldots\,(2n-2r-1)}{1\,.\,3\,.\,5\,\ldots\,(2n-2m-2r+1)}\right.$$
$$\left.\times(2n-2m-4r+1)\,P_{n-m-2r}\{(n-m)\,(m+2r)-r\,(2r-1)\}\right\}.$$

4. To express the value of $(1-\mu^2)^p D^{m+p}P_n$.

We have $$(2n+1)\,D^mP_n=D^{m+1}P_{n+1}-D^{m+1}P_{n-1}.$$

Also $$(2n+1)\,\mu D^{m+1}P_n=(n-m)\,D^{m+1}P_{n+1}+(n+m+1)\,D^{m+1}P_{n-1}.$$

From the fundamental differential equation we get

$$(1-\mu^2)\,D^{m+2}P_n-2\,(m+1)\,\mu D^{m+1}P_n+(n-m)\,(n+m+1)\,D^mP_n=0.$$

Hence

$$(2n+1)\,(1-\mu^2)\,D^{m+2}P_n+(n-m)\,(n-m-1)\,D^{m+1}P_{n+1}$$
$$-(n+m+1)\,(n+m+2)\,D^{m+1}P_{n-1}=0.$$

Or

$$(1-\mu^2)\,D^{m+2}P_n=-\frac{(n-m)\,(n-m-1)}{2n+1}\,D^{m+1}P_{n+1}+\frac{(n+m+1)\,(n+m+2)}{2n+1}\,D^{m+1}P_{n-1}.$$

Multiply by $(1-\mu^2)$ and repeat the above process on the right side of the equation, then

$$(1-\mu^2)^2\,D^{m+2}P_n=\frac{(n-m+2)\,(n-m+1)\,(n-m)\,(n-m-1)}{(2n+1)\,(2n+3)}\,D^mP_{n+2}$$
$$-2\,\frac{(n-m)\,(n-m-1)\,(n+m+1)\,(n+m+2)}{(2n-1)\,(2n+3)}\,D^mP_n$$
$$+\frac{(n+m+2)\,(n+m+1)\,(n+m)\,(n+m-1)}{(2n-1)\,(2n+1)}\,D^mP_{n-2}.$$

By repeating the above process we get

$$(1-\mu^2)^3\,D^{m+3}P_n=-\frac{(n-m+3)\,(n-m+2)\,\ldots\,(n-m-2)}{(2n+1)\,(2n+3)\,(2n+5)}\,D^mP_{n+3}$$
$$+\frac{(n-m+1)(n-m)\ldots(n-m-2)\,(n+m+3)\,(n+m+2)}{(2n+1)\,(2n+3)}\left[\frac{1}{2n+5}+\frac{2}{2n-1}\right]D^mP_{n+1}$$

$$-\frac{(n-m-1)(n-m-2)(n+m+3)(n+m+2)(n+m+1)(n+m)}{(2n-1)(2n+1)} \times \left[\frac{2}{2n+3}+\frac{1}{2n-3}\right] D^m P_{n-1}$$

$$+\frac{(n+m+3)(n+m+2)\ldots(n+m-2)}{(2n-3)(2n-1)(2n+1)} D^m P_{n-3}.$$

Hence $(1-\mu^2)^3 D^{m+3} P_n =$

$$-\frac{(n-m+3)(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)}{(2n+1)(2n+3)(2n+5)} D^m P_{n+3}$$

$$+3\frac{(n-m+1)(n-m)(n-m-1)(n-m-2)(n+m+3)(n+m+2)}{(2n-1)(2n+1)(2n+5)} D^m P_{n+1}$$

$$-3\frac{(n-m-1)(n-m-2)(n+m+3)(n+m+2)(n+m+1)(n+m)}{(2n-3)(2n+1)(2n+3)} D^m P_{n-1}$$

$$+\frac{(n+m+3)(n+m+2)(n+m+1)(n+m)(n+m-1)(n+m-2)}{(2n-3)(2n-1)(2n+1)} D^m P_{n-3}.$$

The law which is here observed is also found to hold for $(1-\mu^2)^4 D^{m+4} P_n$, and is true generally.

The general term of $(1-\mu^2)^p D^{m+p} P_n$ is

$$(-1)^{p+r}\frac{p(p-1)(p-2)\ldots(p-r+1)}{r!}$$

$$\times\frac{(n-m+p-2r)\ldots(n-m-p+1)(n+m+p)\ldots(n+m+p-2r+1)}{(2n-2r+1)(2n-2r+3)\ldots(2n+2p-2r+1)} \times(2n+2p-4r+1) D^m P_{n+p-2r}.$$

This may be expressed under the form

$$(-1)^{p+r}\frac{p!}{r!\,(p-r)!}\,\frac{(n-m+p-2r)!}{(n-m-p)!}\,\frac{(n+m+p)!}{(n+m+p-2r)!}$$

$$\times\frac{1\,.\,3\,.\,5\ldots(2n-2r-1)}{1\,.\,3\,.\,5\ldots(2n+2p-2r+1)}(2n+2p-4r+1) D^m P_{n+p-2r}.$$

As a test of the correctness of this result, when $m=0$, this reduces to the expression previously found for $(1-\mu^2)^p D^p P_n$.

Hence the general or $(r+1)$th term in $(\mu^2-1)^p D^{m+p} P_n$ is

$$(-1)^r \frac{p!}{r!\,(p-r)!} \cdot \frac{(n-m+p-2r)!\,(n+m+p)!\,1\,.\,3\,.\,5\,\ldots\,(2n-2r-1)}{(n-m-p)!\,(n+m+p-2r)!\,1\,.\,3\,.\,5\,\ldots\,(2n+2p-2r+1)} \times (2n+2p-4r+1)\,D^m P_{n+p-2r}$$

$$=(-1)^r \frac{p!\,2^p}{r!\,(p-r)!} \cdot \frac{(2n-2r)!\,(n-m+p-2r)!\,(n+m+p)!\,(n+p-r)!}{(n-r)!\,(n-m-p)!\,(n+m+p-2r)!\,(2n+2p-2r+1)!} \times (2n+2p-4r+1) D^m P_{n+p-2r}.$$

Hence putting $m-p$ for m, the general term in $(\mu^2-1)^p D^m P_n$ is

$$=(-1)^r \frac{p!\,2^p}{r!\,(p-r)!} \cdot \frac{(2n-2r)!\,(n-m+2p-2r)!\,(n+m)!\,(n+p-r)!}{(n-r)!\,(n-m)!\,(n+m-2r)!\,(2n+2p-2r+1)!} \times (2n+2p-4r+1)\,D^{m-p} P_{n+p-2r}.$$

5. From the expression for P_n we get

$$\int^p P_n\,d\mu^p = \Sigma(-1)^r \frac{p!}{r!\,(p-r)!} \frac{1\,.\,3\,.\,5\,\ldots\,(2n-2r-1)}{1\,.\,3\,.\,5\,\ldots\,(2n+2p-2r+1)} (2n+2p-4r+1)\,P_{n+p-2r}.$$

Now differentiating $m+p$ times we get

$$D^m P_n = \Sigma\,(-1)^r \frac{p!}{r!\,(p-r)!} \cdot \frac{1\,.\,3\,.\,5\,\ldots\,(2n-2r-1)}{1\,.\,3\,.\,5\,\ldots\,(2n+2p-2r+1)} \times (2n+2p-4r+1)\,D^{m+p} P_{n+p-2r}.$$

We have also $$D^p P_p = 1\,.\,3\,.\,5\,\ldots\,(2p-1),$$

and $$D^p P_{p+1} = 1\,.\,3\,.\,5\,\ldots\,(2p+1)\,\mu.$$

Now by formulae obtained in an earlier part of the work (see p. 255) we have

$$DP_{p+1} = \mu DP_p + (p+1)\,P_p,$$

and $$DP_{p-1} = \mu DP_p - pP_p;$$

hence we have $$DP_{p+1} - DP_{p-1} = (2p+1)\,P_p.$$

Putting $p+1$ for p in this equation we get

$$DP_{p+2} - DP_p = (2p+3)\,P_{p+1}.$$

Differentiating this equation successively we get

$$D^2 P_{p+2} - D^2 P_p = (2p+3)\,DP_{p+1},$$

and similar equations until we come to

$$D^p P_{p+2} - D^p P_p = (2p+3)\, D^{p-1} P_{p+1};$$

and differentiating once more (since $D^{p+1} P_p = 0$) we get

$$D^{p+1} P_{p+2} = (2p+3)\, D^p P_{p+1}.$$

We have also from the above equations,

$$(2p+1)\,\mu DP_p = pDP_{p+1} + (p+1)\, DP_{p-1},$$

or $(2n+2p-4r+1)\,\mu DP_{n+p-2r} = (n+p-2r)\, DP_{n+p-2r+1} + (n+p-2r+1)\, DP_{n+p-2r-1}$.

We have also from equation (6) in a previous part of the paper (p. 248),

$$(2n+2p-4r+1)\,\mu D^{m+p} P_{n+p-2r} = (n-m-2r+1)\, D^{m+p} P_{n+p-2r+1} + (n+m+2p-2r)\, D^{m+p} P_{n+p-2r-1}.$$

Making use of the above equations we get

$$D^m P_n \times D^p P_p = \Sigma\,(-1)^r \frac{p!}{r!\,(p-r)!}\, \frac{1\,.\,3\,.\,5 \ldots (2p-1)\; 1\,.\,3\,.\,5 \ldots (2n-2r-1)}{1\,.\,3\,.\,5 \ldots (2n+2p-2r+1)} \times (2n+2p-4r+1)\, D^{m+p} P_{n+p-2r},$$

and

$$D^m P_n \times D^p P_{p+1} = \Sigma\,(-1)^r \frac{p!}{r!\,(p-r)!}\, \frac{1\,.\,3\,.\,5 \ldots (2p+1)\; 1\,.\,3\,.\,5 \ldots (2n-2r-1)}{1\,.\,3\,.\,5 \ldots (2n+2p-2r+1)} \times (2n+2p-4r+1)\,\mu D^{m+p} P_{n+p-2r}.$$

Writing down the $(r+1)$th and the rth term of this series we have

$$(-1)^r \frac{p!}{r!\,(p-r+1)!}\, \frac{1\,.\,3\,.\,5 \ldots (2p+1)\; 1\,.\,3\,.\,5 \ldots (2n-2r-1)}{1\,.\,3\,.\,5 \ldots (2n+2p-2r+3)}$$
$$\times \left[(2n+2p-4r+1)\,\mu D^{m+p} P_{n+p-2r}\,(p-r+1)\,(2n+2p-2r+3) - (2n+2p-4r+5)\,\mu D^{m+p} P_{n+p-2r+2} \times r\,(2n-2r+1)\right].$$

But we have

$$(2n+2p-4r+1)\,\mu D^{m+p} P_{n+p-2r} = (n-m-2r+1)\, D^{m+p} P_{n+p-2r+} + (n+m+2p-2r)\, D^{m+p} P_{n+p-2r-1},$$

and

$$(2n+2p-4r+5)\,\mu D^{m+p} P_{n+p-2r+2} = (n-m-2r+3)\, D^{m+p} P_{n+v-2r+3} + (n+m+2p-2r+2)\, D^{m+p} P_{n+p-2r+1}.$$

Substituting these expressions, the coefficient of $D^{m+p}P_{n+p-2r+1}$ in the above square brackets becomes

$$(n-m-2r+1)(p-r+1)(2n+2p-2r+3)-(n+m+2p-2r+2)r(2n-2r+1)$$
$$=\{(2n+2p-4r+3)+2r\}(n-m-2r+1)(p-r+1)$$
$$-\{(2n+2p-4r+3)-2(p-r+1)\}(n+m+2p-2r+2)r$$
$$=(2n+2p-4r+3)\{n(p-2r+1)-m(p+1)-(p-r+1)(2r-1)\}$$
$$=(2n+2p-4r+3)\{(n-m-2r+1)(p+1)-r(2n-2r+1)\}$$
$$=(2n+2p-4r+3)\{(n-m-2r+1)(p-r+1)-r(n+m)\}.$$

Hence the coefficient of $(2n+2p-4r+3)D^{m+p}P_{n+p-2r+1}$ in $D^mP_n \times D^pP_{p+1}$ is

$$(-1)^r\frac{p!}{r!\,(p-r+1)!}\,\frac{1.3.5\ldots(2p+1)\;1.3.5\ldots(2n-2r-1)}{1.3.5\ldots(2n+2p-2r+3)}$$
$$\times\{(n-m-2r+1)(p-r+1)-r(n+m)\}.$$

In the expression for $D^mP_n \times D^pP_{p+1}$, the coefficient of

$$D^{m+p}P_{n+p-2r+1}\times(2n+2p-4r+3)$$

expressed in factorials is

$$(-1)^r\frac{2\,(2p+1)!\times(2n-2r)!\,(n+p-r+1)!}{r!\,(p-r+1)!\,(n-r)!\,(2n+2p-2r+3)!}$$
$$\times[(n-m+1)(p+1)-r(2n+2p-2r+3)]$$

$$=(-1)^r\frac{(2p+1)!\,(2n-2r)!\,(n+p-r+1)!}{r!\,(p-r+1)!\,(n-r)!\,(2n+2p-2r+3)!}$$
$$\times[(n-m+1)\,2\,(p+1)-2r\,(2n+2p-2r+3)]$$

$$=(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}[(2n-2r+1)(2q-2r)-2q(n+m)].$$

Hence in the value of $D^mP_n \times D^pP_q$, when $q=p$, the coefficient of

$$(2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}$$

expressed in factorials is

$$(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}.$$

Similarly in the expression for $D^m P_n \times D^p P_q$, when $q = p + 1$, the coefficient of $(2n + 2q - 4r + 1) D^{m+p} P_{n+q-2r}$ is

$$(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}[(2n-2r+1)(2q-2r) - 2q(n+m)].$$

NOTE. In the last expression the quantity in the square brackets may be stated in either of the following forms:

$$[(n-m+1)\,2q - 2r(2n+2q-2r+1)] = [(n-m-2r+1)\,2q - 2r(2n-2r+1)],$$

or $$= [\{(n+q-2r)-(m+p)\}\,2q - 2r(2n-2r+1)],$$

or $$= [\{(n+q)-(m+p)\}\,2q - 2r\{2(n+q)-2r+1\}],$$

or $$= [(j-i)(j+i+1) - (n+m)(n-m+1)],$$

where $j = n + q - 2r$ and $i = m + p$.

6. From the equation

$$(2n+1)\,\mu D^m P_n = (n-m+1)\,D^m P_{n+1} + (n+m)\,D^m P_{n-1},$$

we get, by putting $n+p-2r$ for n and $m+p$ for m,

$$(2n+2p-4r+1)\,\mu D^{m+p} P_{n+p-2r} = (n-m-2r+1)\,D^{m+p} P_{n+p-2r+1} + (n+m+2p-2r)\,D^{m+p} P_{n+p-2r-1}.$$

If we put $n+p-2r+1$ for n and $m+p$ for m, we get

$$(2n+2p-4r+3)\,\mu D^{m+p} P_{n+p-2r+1} = (n-m-2r+2)\,D^{m+p} P_{n+p-2r+2} + (n+m+2p-2r+1)\,D^{m+p} P_{n+p-2r}.$$

The successive equations may be obtained from these two formulae by changing r into $r-1$, $r-2$, &c. in the two formulae alternately. Thus putting $r-1$ for r in the last formula, we get

$$(2n+2p-4r+7)\,\mu D^{m+p} P_{n+p-2r+3} = (n-m-2r+4)\,D^{m+p} P_{n+p-2r+4} + (n+m+2p-2r+3)\,D^{m+p} P_{n+p-2r+2}.$$

Again, putting $p+1$ for n and p for m in the above equation, transposing and multiplying the result by $D^m P_n$, we get

$$2D^m P_n \times D^p P_{p+2} = (2p+3)\,\mu D^m P_n \times D^p P_{p+1} - (2p+1)\,D^m P_n \times D^p P_p$$
$$= \mu D^m P_n \times D^{p+1} P_{p+2} - D^m P_n \times D^{p+1} P_{p+1}.$$

From the series obtained for $D^mP_n \times D^pP_{p+1}$ by means of two of the above equations and by putting $p+1$ for p, we may get the value of the coefficient of $D^{m+p}P_{n+p-2r+2}$ in the expression for $\mu D^mP_n \times D^{p+1}P_{p+2}$.

We can also get the coefficient of $D^{m+p}P_{n+p-2r+2}$ in the expression for $D^mP_n \times D^{p+1}P_{p+1}$, and hence we may obtain the value of $2D^mP_n \times D^pP_{p+2}$ in the form of a series.

Our object is to find the law of formation of the coefficients of the successive terms.

Following the same process of successive substitution we obtain in the same manner, when $q=p+2$,

$$2 \times D^mP_n \times D^pP_q = \Sigma(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!} \times (2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}$$

$$\times\big[(2n-2r+1)(2n-2r+2)(2q-2r)(2q-2r-1)$$

$$-2(2n-2r+1)(2q-2r)(2q-1)(n+m)+(2q-1)\,2q\,(n+m)(n+m-1)\big].$$

The quantity in square brackets may be expressed in the form

$$\left[\frac{(2n-2r+2)!\,(2q-2r)!}{(2n-2r)!\,(2q-2r-2)!}-2\frac{(2n-2r+1)!\,(2q-2r)!\,(2q-1)!\,(n+m)!}{(2n-2r)!\,(2q-2r-1)!\,(2q-2)!\,(n+m-1)!}+\frac{2q!\,(n+m)!}{(2q-2)!\,(n+m-2)!}\right].$$

Here the law of formation is clear.

7. Applying the same process to the equation

$$3D^mP_n \times D^pP_{p+3} = (2p+5)\,\mu D^mP_n \times D^pP_{p+2} - (2p+2)\,D^mP_n \times D^pP_{p+1},$$

it appears that, when $q=p+3$, we have the corresponding coefficient in

$$3!\,D^mP_n \times D^pP_q = \Sigma(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left[\frac{(2n-2r+3)!\,(2q-2r)!}{(2q-2r-3)!\,(2n-2r)!}-3\frac{(2n-2r+2)!\,(2q-2r)!\,(2q-2)!\,(n+m)!}{(2q-2r-2)!\,(2n-2r)!\,(2q-3)!\,(n+m-1)!}\right.$$

$$\left.+3\frac{(2n-2r+1)!\,(2q-2r)!\,(2q-1)!\,(n+m)!}{(2q-2r-1)!\,(2n-2r)!\,(2q-3)!\,(n+m-2)!}-\frac{2q!\,(n+m)!}{(2q-3)!\,(n+m-3)!}\right].$$

Each of the above quantities is included in the expression

$$(q-p)!\, D^m P_n \times D^p P_q = \Sigma\, (-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!} \times (2n+2q-4r+1)\, D^{m+p} P_{n+q-2r}$$

$$\times \Bigg[\frac{(2n-2r+q-p)!\,(2q-2r)!}{(2q-2r-(q-p))!\,(2n-2r)!}$$

$$-(q-p)\frac{(2n-2r+q-p-1)!\,(2q-2r)!\,(q+p+1)!\,(n+m)!}{(2q-2r-(q-p-1))!\,(2n-2r)!\,(q+p)!\,(n+m-1)!} + \text{\&c.}$$

$$+(-1)^s \frac{(q-p)!}{s!\,(q-p-s)!} \frac{(2n-2r+q-p-s)!\,(2q-2r)!\,(q+p+s)!\,(n+m)!}{(2q-2r-(q-p-s))!\,(2n-2r)!\,(q+p)!\,(n+m-s)!} + \text{\&c.}\Bigg].$$

From which it appears that the general term in the expansion of $D^m P_n \times D^p P_q$ is

$$\frac{(-1)^{r+s}(q+p)!\,(2n-2r)!\,(n+q-r)!\,(q-p)!\,(2q-2r)!\,(n+m)!\,(2n-2r+q-p-s)!\,(q+p+s)!}{(q-p)!\,r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,s!\,(q-p-s)!\,(2n-2r)!\,(2q-2r-(q-p-s))!\,(q+p)!\,(n+m-s)!} \times (2n+2q-4r+1)\, D^{m+p} P_{n+q-2r},$$

where s takes all values from 0 to $(q-p)$.

Cancelling common terms in numerator and denominator we may reduce this to

$$(-1)^{r+s} \frac{(n+q-r)!\,(2q-2r)!\,(n+m)!\,(2n-2r+q-p-s)!\,(q+p+s)!}{r!\,s!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(q-p-s)!\,(q+p-2r+s)!\,(n+m-s)!} \times (2n+2q-4r+1)\, D^{m+p} P_{n+q-2r}.$$

8. Now let $m=0$ and $p=0$, so as to reduce to the case of the product $P_n \times P_q$.

Then the coefficient of $(2n+2q-4r+1)\, P_{n+q-2r}$ will become

$$(-1)^r \frac{(n+q-r)!\,(2q-2r)!\,n!}{r!(q-r)!(n-r)!(2n+2q-2r+1)!} \times \Sigma\,(-1)^s \frac{(q+s)!\,(2n-2r+q-s)!}{s!\,(q-s)!\,(n-s)!\,(q-2r+s)!}.$$

But by a former investigation (see p. 374) the coefficient* in this case is

$$\frac{A\,(n-r)\,A\,(r)\,A\,(q-r)}{A\,(n+q-r)\,(2n+2q-2r+1)},$$

where $$A\,(r) \times 2^r \times (r!)^2 = 2r!.$$

Hence this coefficient

$$=\frac{(2n-2r)!\,2r!\,(2q-2r)!\,((n+q-r)!)^2 \times 2^{n+q-r}}{(r!\,(n-r)!\,(q-r)!)^2\,(2n+2q-2r+1)! \times 2^{n+q-r}}.$$

Comparing these expressions we see that

$$\Sigma(-1)^s \frac{(q+s)!\,(2n-2r+q-s)!}{s!\,(q-s)!\,(n-s)!\,(q-2r+s)!} = (-1)^r \frac{2r!\,(2n-2r)!\,(n+q-r)!}{n!\,r!\,(n-r)!\,(q-r)!},$$

which is a remarkable expression.

It will be well to test this formula numerically.

For instance let $n=3$, $q=3$, $r=0$.

Then the series is

$$\frac{1.2.3.1.2.3.4.5.6.7.8.9}{1.2.3.1.2.3.1.2.3} - \frac{1.2.3.4.1.2.3.4.5.6.7.8}{1.1.2.1.2.1.2.3.4}$$

$$+\frac{1.2.3.4.5.1.2.3.4.5.6.7}{1.2.1.1.1.2.3.4.5} - \frac{1.2.3.4.5.6.1.2.3.4.5.6}{1.2.3.1.2.3.4.5.6}$$

$$=20.7.8.9-30.6.7.8+60.6.7-120=2400.$$

Also the other expression

$$=\frac{1.2.3.4.5.6.1.2.3.4.5.6}{1.2.3.1.2.3.1.2.3}$$

$$=20.120=2400, \text{ which agrees.}$$

Next let $r=1$.

Then the series is

$$\frac{1.2.3.1.2.3.4.5.6.7}{1.2.3.1.2.3.1} - \frac{1.2.3.4.1.2.3.4.5.6}{1.1.2.1.2.1.2}$$

$$+\frac{1.2.3.4.5.1.2.3.4.5}{1.2.1.1.1.2.3} - \frac{1.2.3.4.5.6.1.2.3.4}{1.2.3.1.2.3.4}$$

$$=20.6.7-108.20+60.20-120$$

$$=20[42-108+60-6]=-240,$$

and the other expression $= -\dfrac{1.2.1.2.3.4.1.2.3.4.5}{1.2.3.1.1.2.1.2} = -12.4.5 = -240$, which agrees.

Next let $r=2$.

Then $s=0$ gives zero since $f(-1)$ is infinity; and the series is

$$-\frac{1.2.3.4.1.2.3.4}{1.1.2.1.2} + \frac{1.2.3.4.5.1.2.3}{1.2.1.1.1} - \frac{1.2.3.4.5.6.1.2}{1.2.3.1.2}$$

$$=-144+360-120$$

$$=96;$$

and the other expression is

$$\frac{1.2.3.4.1.2.1.2.3.4}{1.2.3.1.2.1.1} = 96, \text{ which agrees.}$$

Next let $r = 3$.

Then $s = 0, 1, 2$ give zero results; also when $s = 3$ the series is reduced to the single term

$$-\frac{1.2.3.4.5.6}{1.2.3} = -120,$$

and the other expression is $-\dfrac{1.2.3.4.5.6.1.2.3}{1.2.3.1.2.3} = -120$, which agrees.

Hence there can be no doubt of the accuracy of this result, which is very curious.

9. We may obtain the first and last terms in the value of

$$D^m P_n \times D^p P_q$$

in a more convenient form.

The first and last terms in the value of $D^m P_n$ in $D^{m+p} P$ will be

$$D^m P_n = \frac{1.3.5 \ldots (2n-1)}{1.3.5 \ldots (2n+2p-1)} D^{m+p} P_{n+p} + \&c.$$

$$+ (-1)^p \frac{1.3.5 \ldots (2n-2p-1)}{1.3.5 \ldots (2n+1)} (2n-2p+1) D^{m+p} P_{n-p}.$$

Hence in $D^m P_n \times D^p P_p$ the first and last terms are

$$\frac{1.3.5 \ldots (2p-1)\, 1.3.5 \ldots (2n-1)}{1.3.5 \ldots (2n+2p-1)} D^{m+p} P_{n+p}$$

and $(-1)^p \dfrac{1.3.5 \ldots (2p-1)\, 1.3.5 \ldots (2n-2p-1)}{1.3.5 \ldots (2n+1)} (2n-2p+1) D^{m+p} P_{n-p}$.

Multiplying by $(2p+1)\mu$ we get the value of $D^m P_n \times D^p P_{p+1}$

$$= \frac{1.3.5 \ldots (2p+1)\, 1.3.5 \ldots (2n-1)(n-m+1)}{1.3.5 \ldots (2n+2p-1)(2n+2p+1)} D^{m+p} P_{n+p+1} + \&c.$$

$$-(-1)^p \frac{1.3.5 \ldots (2p+1)\, 1.3.5 \ldots (2n-2p+1)(n+m)}{1.3.5 \ldots (2n+1)(2n-2p+1)} D^{m+p} P_{n-p-1}.$$

Now $2D^mP_n \times D^pP_{p+2} = (2p+3)\,\mu D^pP_{p+1} +$ terms which do not affect the result wanted.

Hence $2D^mP_n \times D^pP_{p+2}$

$$= \frac{1.3.5 \ldots (2p+3)\; 1.3.5 \ldots (2n-1)}{1.3.5 \ldots (2n+2p+1)\,(2n+2p+3)} (n-m+1)(n-m+2)\, D^{m+p}P_{n+p+2} + \&c.$$

$$+(-1)^p \frac{1.3.5 \ldots (2p+3)\; 1.3.5 \ldots (2n-2p-1)}{1.3.5 \ldots (2n+1)\,(2n-2p-1)} (n+m)(n+m-1)\, D^{m+p}P_{n-p-2}.$$

Again $3!\,D^mP_n \times D^pP_{p+3} = (2p+5)\,\mu D^mP_n \times D^pP_{p+2} +$ terms not required

$$= \frac{1.3.5 \ldots (2p+5)\; 1.3.5 \ldots (2n-1)}{1.3.5 \ldots (2n+2p+3)\,(2n+2p+5)}$$

$$\times (n-m+1)(n-m+2)(n-m+3)\, D^{m+p}P_{n+p+3} + \&c.$$

$$-(-1)^p \frac{1.3.5 \ldots (2p+5)\; 1.3.5 \ldots (2n-2p-3)}{1.3.5 \ldots (2n+1)\,(2n-2p-3)}$$

$$\times (n+m)(n+m-1)(n+m-2)\, D^{m+p}P_{n-p-3}.$$

This may be conveniently written in the form $(q-p)!\,D^mP_n \times D^pP_q$

$$= \frac{1.3.5 \ldots (2q-1)\; 1.3.5 \ldots (2n-1)\,(n-m+q-p)!}{1.3.5 \ldots (2n+2q-1)\,(n-m)!} D^{m+p}P_{n+q} + \&c.$$

$$+(-1)^q \frac{1.3.5 \ldots (2q-1)\; 1.3.5 \ldots (2n-2q+1)\,(n+m)!}{1.3.5 \ldots (2n+1)\,(n+m-q+p)!} D^{m+p}P_{n-q}.$$

Putting it in this form we see that this expression is general and includes the previous results for $q-p=2$ and $q-p=1$ and $q=p$.

Hence it is true for all values of $q-p$.

Expressing this result in factorials we get

$$(q-p)!\,.\,D^mP_n \times D^pP_q = \frac{2q!\,2n!\,(n+q)!\,(n-m+q-p)!}{q!\,n!\,(2n+2q)!\,(n-m)!} D^{m+p}P_{n+q} + \&c.$$

$$+(-1)^q \frac{2q!\,(2n-2q+1)!\,n!\,(n+m)!}{q!\,(n-q)!\,(2n+1)!\,(n+m-q+p)!} D^{m+p}P_{n-q}.$$

10. The following is a simple method of determining the coefficient of $D^{m+p}P_{n+q}$ in the product $D^mP_n \times D^pP_q$.

The coefficient of μ^{n-m} in D^mP_n is $\dfrac{1.3.5 \ldots (2n-1)}{(n-m)!}$.

Hence the coefficient of $\mu^{n-m+q-p}$ in the product $D^m P_n \times D^p P_q$

$$\text{is}\quad \frac{1.3.5\ldots(2n-1)\,1.3.5\ldots(2q-1)}{(n-m)!\,(q-p)!}.$$

Similarly the coefficient of $\mu^{n-m+q-p}$ in $D^{m+p}P_{n+q}$

$$\text{is}\quad \frac{1.3.5\ldots(2n+2q-1)}{(n+q-m-p)!}.$$

Hence the coefficient of $D^{m+p}P_{n+q}$ in the value of $D^m P_n \times D^p P_q$

$$\text{is}\quad \frac{1.3.5\ldots(2n-1)\,1.3.5\ldots(2q-1)}{1.3.5\ldots(2n+2q-1)}\;\frac{(n+q-m-p)!}{(n-m)!\,(q-p)!}.$$

Also the coefficient of $\mu^{n-m+q+p}$ in $(\mu^2-1)^p D^m P_n \times D^p P_q$

$$\text{is}\quad \frac{1.3.5\ldots(2n-1)\,1.3.5\ldots(2q-1)}{(n-m)!\,(q-p)!},$$

but the coefficient of the same power of μ in $D^{m-p}P_{n+q}$

$$\text{is}\quad \frac{1.3.5\ldots(2n+2q-1)}{(n+q-m+p)!},$$

therefore the coefficient of $D^{m-p}P_{n+q}$ in the value of $(\mu^2-1)^p D^m P_n \times D^p P_q$

$$\text{is}\quad \frac{1.3.5\ldots(2n-1)\,1.3.5\ldots(2q-1)\,(n+q-m+p)!}{1.3.5\ldots(2n+2q-1)\,(n-m)!\,(q-p)!},$$

supposing m to be not less than p.

NOTE. In the value of $2D^m P_n . D^p P_q$ given in Art. 6 above (when $q=p+2$) the quantity contained in brackets [ ] may also be expressed in either of the following equivalent forms:

$$\begin{aligned}&[2q(2q-1)(n-m-2r+1)(n-m-2r+2)\\ &\qquad -2(2q-1)(n-m-2r+2)\,2r(2n-2r+1)\\ &\qquad +2r(2r-1)(2n-2r+1)(2n-2r+2)],\end{aligned}$$

or

$$\begin{aligned}&[(n-m-2r+2)(n-m-2r+1)(2q-2r)(2q-2r-1)\\ &\qquad -2(n-m-2r+2)(2q-2r)\,2r(n+m)\\ &\qquad +2r(2r-1)(n+m)(n+m-1)],\end{aligned}$$

or

$$\begin{aligned}&[(n-m-2r+2)(n-m-2r+1)(n+m+2q-2r)(n+m+2q-2r-1)\\ &\qquad -2(n-m-2r+2)(n+m+2q-2r-1)(n+m)(n-m+1)\\ &\qquad +(n+m)(n+m-1)(n-m+1)(n-m+2)].\end{aligned}$$

Hence using this last expression the coefficient of

$$(2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}$$

in $2D^mP_n \times D^pP_q$ will be

$$(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left[\frac{(n-m+q-p-2r)!\,(n+m+2q-2r)!}{(n-m-2r)!\,(n+m+2q-2r-2)!}\right.$$

$$-2\frac{(n-m+q-p-2r)!\,(n+m+2q-2r-1)!\,(n+m)!\,(n-m+1)!}{(n-m-2r+1)!\,(n+m+2q-2r-2)!\,(n+m-1)!\,(n-m)!}$$

$$\left.+\frac{(n+m)!\,(n-m+2)!}{(n+m-2)!\,(n-m)!}\right]$$

$$=(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!\,(n-m+q-p-2r)!\,(n+m)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(n+m+q+p-2r)!\,(n-m)!}$$

$$\times\left[\frac{(n+m+2q-2r)!\,(n-m)!}{(n-m-2r)!\,(n+m)!}-2\frac{(n+m+2q-2r-1)!\,(n-m+1)!}{(n-m-2r+1)!\,(n+m-1)!}\right.$$

$$\left.+\frac{(n+m+q+p-2r)!\,(n-m+2)!}{(n-m+q-p-2r)!\,(n+m-2)!}\right].$$

Taking the first of the expressions in the above note, the coefficient of $(2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}$ in $2D^mP_n\,.\,D^pP_q$ will be

$$(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!\,2r!\,(n-m+q-p-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(q+p)!\,(2n-2r)!}$$

$$\times\left[\frac{2q!\,(2n-2r)!}{2r!\,(n-m-2r)!}-2\frac{(2q-1)!\,(2n-2r+1)!}{(2r-1)!\,(n-m-2r+1)!}+\frac{(2q-2)!\,(2n-2r+2)!}{(2r-2)!\,(n-m-2r+2)!}\right]$$

$$=(-1)^r \frac{2r!\,(n+q-r)!\,(n-m+q-p-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left[\frac{2q!\,(2n-2r)!}{2r!\,(n-m-2r)!}-2\frac{(2q-1)!\,(2n-2r+1)!}{(2r-1)!\,(n-m-2r+1)!}\right.$$

$$\left.+\frac{(2q-2)!\,(2n-2r+2)!}{(2r-2)!\,(n-m-2r+2)!}\right].$$

The general term arising from this square bracket is

$$(-1)^s \frac{(q-p)!}{s!\,(q-p-s)!}\,\frac{(2q-s)!\,(2n-2r+s)!}{(2r-s)!\,(n-m-2r+s)!},$$

where s has all values from 0 to $q-p$.

11. Again from the expression for $(\mu^2-1)^p D^m P_n$ we may find the value of $(\mu^2-1)^p D^m P_n \times D^p P_q$ in terms of the form $D^{m-p} P$, multiplied by constants.

Thus from Art. 4 (p. 379) we have $(\mu^2-1)^p D^m P_n \times D^p P_p$

$$=\Sigma(-1)^r \frac{p!\,2^p.1.3.5\ldots(2p-1)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r)!\,(n+p-r)!}{r!\,(p-r)!\,(n-r)!\,(n-m)!\,(n+m-2r)!\,(2n+2p-2r+1)!}$$
$$\times(2n+2p-4r+1)\,D^{m-p}P_{n+p-2r}$$

$$=\Sigma(-1)^r \frac{2p!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r)!\,(n+p-r)!}{r!\,(p-r)!\,(n-r)!\,(n-m)!\,(n+m-2r)!\,(2n+2p-2r+1)!}$$
$$\times(2n+2p-4r+1)\,D^{m-p}P_{n+p-2r}.$$

Now multiply by $(2p+1)\mu$ and we get

$$\times(\mu^2-1)^p D^m P_n \times D^p P_{p+1} = \text{terms of the form}$$

$$(-1)^{r-1}\frac{(2p+1)!\,(2n-2r+2)!\,(n+m)!\,(n-m+2p-2r+2)!\,(n+p-r+1)!}{(r-1)!\,(p-r+1)!\,(n-r+1)!\,(n-m)!\,(n+m-2r+2)!\,(2n+2p-2r+3)!}$$
$$\times\left[(n-m+2p-2r+3)\,D^{m-p}P_{n+p-2r+3}+(n+m-2r+2)\,D^{m-p}P_{n+p-2r+1}\right]$$
$$+(-1)^r\frac{(2p+1)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r)!\,(n+p-r)!}{r!\,(p-r)!\,(n-r)!\,(n-m)!\,(n+m-2r)!\,(2n+2p-2r+1)!}$$
$$\times\left[(n-m+2p-2r+1)\,D^{m-p}P_{n+p-2r+1}+(n+m-2r)\,D^{m-p}P_{n+p-2r-1}\right].$$

Taking only the coefficient of $D^{m-p}P_{n+p-2r+1}$ in the result we have

$$(-1)^r\frac{(2p+1)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r+1)!\,(n+p-r+1)!}{r!\,(p-r+1)!\,(n-r)!\,(n-m)!\,(n+m-2r+1)!\,(2n+2p-2r+3)!}$$
$$\times\left[-2r\,(2n-2r+1)\,(n-m+2p-2r+2)\right.$$
$$\left.+(p-r+1)(n+m-2r+1)(2n+2p-2r+3)\,2\right].$$

The quantity in square brackets is

$$\{(2n+2p-4r+3)+2r\}\,(2p-2r+2)\,(n+m-2r+1)$$
$$-\{(2n+2p-4r+3)-(n+m-2r+1)\}\,(2n-2r+1)\,2r$$
$$=(2n+2p-4r+3)\,\{(n+m-2r+1)\,(2p-2r+2)-2r\,(2n-2r+1)\}$$
$$+(n+m-2r+1)\,2r\,(2n+2p-4r+3)$$
$$=(2n+2p-4r+3)\,\{(n+m-2r+1)\,(2p+2)-2r\,(2n-2r+1)\}.$$

Putting q for $p+1$ we see that the coefficient of

$$(2n+2q-4r+1)\,D^{m-p}P_{n+q-2r}$$

in $(\mu^2-1)^p D^m P_n \times D^p P_q$ (when $q=p+1$) is

$$(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r+1)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(n-m)!\,(n+m-2r+1)!\,(2n+2q-2r+1)!}$$
$$\times\left[(n+m-2r+1)\,2q-2r\,(2n-2r+1)\right].$$

NOTE. The quantity in square brackets in this expression

$$=\left[(2n-2r+1)\,(2q-2r)-2q\,(n-m)\right],$$

or $$\left[(n+m+1)\,2q-2r\,(2n+2q-2r+1)\right]$$

or $$\left[\{(n+q-2r)+(m-p)\}\,2q-2r\,(2n-2r+1)\right]$$

or $$\left[(n+q+m-p)\,2q-2r\,\{2\,(n+q)-2r+1\}\right]$$

or $$\left[(n+m+q-p-2r)\,(n-m+q+p-2r+1)-(n-m)\,(n+m+1)\right].$$

Hence the coefficient of $(2n+2q-4r+1)\,D^{m-p}P_{n+q-2r}$ in

$$(\mu^2-1)^p\,D^mP_n \times D^pP_q \text{ (when } q=p+1) \text{ is}$$

$$(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+m)!\,(n+q-r)!\,(n-m+q+p-2r)!}{r!\,(q-r)!\,(n-r)!\,(n-m)!\,(2n+2q-2r+1)!\,(n+m+q-p-2r)!}$$
$$\times\left[(2n-2r+1)\,(2q-2r)-(q+p+1)\,(n-m)\right]$$

$$=(-1)^r\frac{(n+m)!\,(n+q-r)!\,(n-m+q+p-2r)!\,(2q-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(n+m+q-p-2r)!}$$
$$\times\left[\frac{(2n-2r+1)!\,(q+p)!}{(2q-2r-1)!\,(n-m)!}-\frac{(2n-2r)!\,(q+p+1)!}{(2q-2r)!\,(n-m-1)!}\right].$$

This may also be put in the form

$$(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{2r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$
$$\times\left[\frac{(n-m+q+p-2r+1)!\,(n+m)!}{(n+m+q-p-2r-1)!\,(n-m)!}-\frac{(n-m+q+p-2r)!\,(n+m+1)!}{(n+m+q-p-2r)!\,(n-m-1)!}\right].$$

There are several other forms in which this system of factorials may be arranged.

12. Putting $r-1$ for r in the general term of the expression for $(\mu^2-1)^p D^m P_n \times D^p P_p$, we have

$$(-1)^{r-1} \frac{2p!\,(2n-2r+2)!\,(n+m)!\,(n-m+2p-2r+2)!\,(n+p-r+1)!}{(r-1)!(p-r+1)!(n-r+1)!(n-m)!(n+m-2r+2)!(2n+2p-2r+3)!} \times (2n+2p-4r+5)\, D^{m-p} P_{n+p-2r+2}.$$

Multiply by $-(2p+1)$ and we get

$$(-1)^{r} \frac{(2p+1)!\,(2n-2r+2)!\,(n+m)!\,(n-m+2p-2r+2)!\,(n+p-r+1)!}{(r-1)!\,(p-r+1)!\,(n-r+1)!\,(n-m)!\,(n+m-2r+2)!\,(2n+2p-2r+3)!} \times (2n+2p-4r+5)\, D^{m-p} P_{n+p-2r+2}.$$

Also writing down two terms of $(\mu^2-1)^p D^m P_n \times D^p P_{p+1}$ multiplied by

$$(2p+3)\,\mu,$$

we have

$$(-1)^{r-1} \frac{(2p+1)!\,(2n-2r+2)!\,(n+m)!\,(n-m+2p-2r+3)!\,(n+p-r+2)!}{(r-1)!\,(p-r+2)!\,(n-r+1)!\,(n-m)!\,(n+m-2r+3)!\,(2n+2p-2r+5)!}$$
$$\times \left[(2n-2r+3)(2p-2r+4)-2(p+1)(n-m)\right]$$
$$\times (2p+3)\left\{(n-m+2p-2r+4)\, D^{m-p} P_{n+p-2r+4} + (n+m-2r+3)\, D^{m-p} P_{n+p-2r+2}\right\}$$
$$+(-1)^{r} \frac{(2p+1)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r+1)!\,(n+p-r+1)!}{r!\,(p-r+1)!\,(n-r)!\,(n-m)!\,(n+m-2r+1)!\,(2n+2p-2r+3)!}$$
$$\times \left[(2n-2r+1)(2p-2r+2)-2(p+1)(n-m)\right]$$
$$\times (2p+3)\left\{(n-m+2p-2r+2)\, D^{m-p} P_{n+p-2r+2} + (n+m-2r+1)\, D^{m-p} P_{n+p-2r}\right\}.$$

Now take the coefficient of $D^{m-p} P_{n+p-2r+2}$ in the sum of these terms and we get the corresponding term in $2(\mu^2-1)^p D^m P_n \times D^p P_{p+2}$

$$=(-1)^{r} \frac{(2p+1)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r+2)!\,(n+p-r+2)!}{r!\,(p-r+2)!\,(n-r)!\,(n-m)!\,(n+m-2r+2)!\,(2n+2p-2r+5)!}$$
$$\times 2\left\{(2p+3)(p-r+2)(n+m-2r+2)(2n+2p-2r+5)\right.$$
$$\times \left[(n+m-2r+1)(2p+2)-2r(2n-2r+1)\right]$$
$$-(2n-2r+1)(n-m+2p-2r+3)r\left[(2n-2r+3)(2p-2r+4)-(2p+2)(n-m)\right]$$
$$\left.\times (2p+3)+2(2n-2r+1)\,r\,(p-r+2)(2n+2p-2r+5)(2n+2p-4r+5)\right\}.$$

The expression in large brackets may be arranged as follows, in order to separate out the factors $(2p+2)$ and $(2n+2p-4r+5)$, which are factors of this expression—

$$\begin{aligned}
&(2p+2)(2n+2p-4r+5)(2p+3)(p-r+2)(n+m-2r+2)(n+m-2r+1)\\
&+(2p+2)(2p+3)\,2r\,(p-r+2)(n+m-2r+2)(n+m-2r+1)\\
&-(2p+3)(2n+2p-4r+5)\,2r\,(p-r+2)(2n-2r+1)(n+m-2r+2)\\
&-(2p+3)\,2r\,(p-r+2)(2n-2r+1)\,2r\,(n+m-2r+2)\\
&-(2p+3)\,2r\,(p-r+2)(2n-2r+1)(2n-2r+3)\\
&\qquad\qquad\qquad\times\{(2n+2p-4r+5)-(n+m-2r+2)\}\\
&+(2p+2)(2p+3)\,r\,(2n-2r+1)(n-m)\{(2n+2p-4r+5)-(n+m-2r+2)\}\\
&+(2n+2p-4r+5)\,2r\,(p-r+2)(2n-2r+1)(2n+2p-2r+5)\\
&=(2p+2)(2n+2p-4r+5)\begin{bmatrix}(2p+3)(p-r+2)(n+m-2r+2)(n+m-2r+1)\\ +(2p+3)\,r\,(2n-2r+1)(n-m)\\ -2r(p-r+2)(2n-2r+1)(2n-2r+2)\\ -(2p+3)\,r\,(n+m-2r+2)(n-m)\end{bmatrix}\\
&=(2p+2)(2n+2p-4r+5)\begin{bmatrix}(2p+3)(p-r+2)(n+m-2r+2)(n+m-2r+1)\\ +(2p+3)\,r\,(n-m)(n-m-1)\\ -2r(p-r+2)(2n-2r+1)(2n-2r+2)\end{bmatrix};
\end{aligned}$$

also

$$=(2p+2)(2n+2p-4r+5)\begin{bmatrix}(2p+3)(p+2)(n+m-2r+2)(n+m-2r+1)\\ -(2p+3)(n+m-2r+2)\,2r\,(2n-2r+1)\\ +r\,(2r-1)(2n-2r+1)(2n-2r+2)\end{bmatrix}.$$

Hence the coefficient of

$$(2n+2p-4r+5)\,D^{m-p}P_{n+p-2r+2}$$

in the expansion of $\qquad 2\,(\mu^2-1)^p\,D^mP_n\times D^pP_{p+2}$

is $\quad (-1)^r\dfrac{(2p+2)!\,(2n-2r)!\,(n+m)!\,(n-m+2p-2r+2)!\,(n+p-r+2)!}{r!\,(p-r+2)!\,(n-r)!\,(n-m)!\,(n+m-2r+2)!\,(2n+2p-2r+5)!}$

$$\times\begin{bmatrix}(2p+3)(2p+4)(n+m-2r+2)(n+m-2r+1)\\ -2\,(2p+3)(n+m-2r+2)\,2r\,(2n-2r+1)\\ +2r\,(2r-1)(2n-2r+1)(2n-2r+2)\end{bmatrix}.$$

Or substituting $2q$ for $(2p+4)$ and $q+p$ for $(2p+2)$ we get:—the coefficient of

$$(2n+2q-4r+1)\,D^{m-p}P_{n+q-2r}$$

in the expansion of $\qquad 2\,(\mu^2-1)^p\,D^mP_n \times D^pP_q$

is $$(-1)^r\,\frac{(q+p)!\,(2n-2r)!\,(n+m)!\,(n-m+q+p-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(n-m)!\,(n+m-2r+q-p)!\,(2n+2q-2r+1)!}$$

$$\times\begin{bmatrix}2q\,(2q-1)\,(n+m-2r+1)\,(n+m-2r+2) \\ \quad -2\,(2q-1)\,(n+m-2r+2)\,2r\,(2n-2r+1) \\ \quad +2r\,(2r-1)\,(2n-2r+1)\,(2n-2r+2)\end{bmatrix}.$$

NOTE. It is readily shewn that the expression in this large bracket is equivalent to

$$\begin{aligned}[(2n-2r+2)\,&(2n-2r+1)\,(2q-2r)\,(2q-2r-1) \\ &-2\,(2n-2r+1)\,(2q-2r)\,(2q-1)\,(n-m) \\ &+2q\,(2q-1)\,(n-m)\,(n-m-1)],\end{aligned}$$

and also that it

$$=\begin{bmatrix}(n+m-2r+2)\,(n+m-2r+1)\,(2q-2r)\,(2q-2r-1) \\ \quad -2\,(n+m-2r+2)\,(2q-2r)\,2r\,(n-m) \\ \quad +2r\,(2r-1)\,(n-m)\,(n-m-1)\end{bmatrix}.$$

This expression is also

$$=\begin{bmatrix}(n+m-2r+2)\,(n+m-2r+1)\,(n-m+2q-2r)\,(n-m+2q-2r-1) \\ \quad -2\,(n+m-2r+2)\,(n-m+2q-2r-1)\,(n-m)\,(n+m+1) \\ \quad +(n-m)\,(n-m-1)\,(n+m+1)\,(n+m+2)\end{bmatrix}.$$

Substituting this last expression in the coefficient of

$$(2n+2q-4r+1)\,D^{m-p}P_{n+q-2r}$$

in the expansion of $\qquad 2\,(\mu^2-1)^pD^mP_n \times D^pP_q$,

we get $$(-1)^r\,\frac{(q+p)!\,(2n-2r)!\,(n+m)!\,(n+q-r)!\,(n-m+q+p-2r)!}{r!\,(q-r)!\,(n-r)!\,(n-m)!\,(2n+2q-2r+1)!\,(n+m+q-p-2r)!}$$

$$\times\begin{bmatrix}(n+m-2r+2)\,(n+m-2r+1)\,(n-m+2q-2r-1)\,(n-m+2q-2r) \\ \quad -2\,(n+m-2r+2)\,(n-m+2q-2r-1)\,(n-m)\,(n+m+1) \\ \quad +(n-m)\,(n-m-1)\,(n+m+1)\,(n+m+2)\end{bmatrix}.$$

And since $q=p+2$, this is

$$=(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left[\frac{(n+m)!\,(n-m+q+p-2r+2)!}{(n-m)!\,(n+m+q-p-2r-2)!}-2\frac{(n+m+1)!\,(n-m+q+p-2r+1)!}{(n-m-1)!\,(n+m+q-p-2r-1)!}\right.$$
$$\left.+\frac{(n+m+2)!\,(n-m+q+p-2r)!}{(n-m-2)!\,(n+m+q-p-2r)!}\right].$$

This last square bracket $\left[\quad\right]$ is equivalent to

$$\left[\frac{(n+m)!\,(n-m+q+p-2r+2)!}{(n-m)!\,(n+m-2r)!}-2\frac{(n+m+1)!\,(n-m+q+p-2r+1)!}{(n-m-1)!\,(n+m-2r+1)!}\right.$$
$$\left.+\frac{(n+m+2)!\,(n-m+q+p-2r)!}{(n-m-2)!\,(n+m-2r+2)!}\right].$$

NOTE. This square bracket differs from the corresponding square bracket in the value of $2D^mP_n \times D^pP_q$ only in the sign of m, and the law of formation of the terms is clearly seen.

13. Adopting the form for $D^mP_n \times D^pP_q$ as given in Arts. 6 and 7 above, we arrive at the conclusion that

$$(q-p)!\,D^mP_n \times D^pP_q = \Sigma\left\{(2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}\right.$$
$$\times(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!\,(2q-2r)!\,(n+m)!}{(q+p)!\,(2n-2r)!\,r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$
$$\left.\times\left[\Sigma(-1)^s\frac{(q-p)!\,(2n-2r+q-p-s)!\,(q+p+s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n+m-s)!}\right]\right\}$$

$$=\Sigma\left\{(2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}\times(-1)^r\frac{(n+q-r)!\,(2q-2r)!\,(n+m)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}\right.$$
$$\left.\times\left[\Sigma(-1)^s\frac{(q-p)!\,(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n+m-s)!}\right]\right\}.$$

From note to Art. 12 we see that another form of this coefficient to $(2n+2q-4r+1)\,D^{m+p}P_{n+q-2r}$ is

$$(-1)^r\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!\,(n+m)!\,(n-m+q-p-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(n-m)!\,(n+m+q+p-2r)!}$$
$$\times\left[\Sigma(-1)^s\frac{(q-p)!\,(n-m+s)!\,(n+m+2q-2r-s)!}{s!\,(q-p-s)!\,(n-m-2r+s)!\,(n+m-s)!}\right].$$

Similarly for the coefficient of $(2n+2q-4r+1)\,D^{m-p}P_{n+q-2r}$ in

$$(q-p)!\,(\mu^2-1)^p D^m P_n \times D^p P_q$$

we should get from Art. 12 either the expression

$$(-1)^r \frac{(n+q-r)!\,(2q-2r)!\,(n+m)!\,(n-m+q+p-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(n+m-2r+q-p)!}$$
$$\times\left[\Sigma\,(-1)^s \frac{(q-p)!\,(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n-m-s)!}\right],$$

or the equivalent expression

$$(-1)^r \frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$
$$\times\left[\Sigma\,(-1)^s \frac{(q-p)!\,(n+m+s)!\,(n-m+2q-2r-s)!}{s!\,(q-p-s)!\,(n+m-2r+s)!\,(n-m-s)!}\right].$$

14. Hence referring to Art. 1 we see that

$$2R_n^m R_q^p = 2Q_n^m \times Q_q^p \cos m\lambda \cos p\lambda$$
$$=\Sigma\,(-1)^r(2n+2q-4r+1)\,Q_{n+q-2r}^{m+p}\frac{(n+q-r)!\,(2q-2r)!\,(n+m)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(q-p)!}$$
$$\times\left[\Sigma\,(-1)^s \frac{(q-p)!\,(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n+m-s)!}\right]\cos(m+p)\lambda$$
$$+\Sigma\,(-1)^r(2n+2q-4r+1)\,Q_{n+q-2r}^{m-p}$$
$$\times\frac{(n+q-r)!\,(2q-2r)!\,(n+m)!\,(n-m+q+p-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(n+m+q-p-2r)!\,(q-p)!}$$
$$\times\left[\Sigma\,(-1)^s \frac{(q-p)!\,(2n-2r+q-p-s)!\,(q+p+s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n-m-s)!}\right]\cos(m-p)\lambda,$$

where r has all values from 0 to $2q$, and s has all values from 0 to $q-p$.

We also see that another form of this expression is

$$2R_n^m R_q^p = 2Q_n^m \times Q_q^p \cos m\lambda \cos p\lambda$$
$$=\Sigma\,(-1)^r(2n+2q-4r+1)\,Q_{n+q-2r}^{m+p}$$
$$\times\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!\,(n+m)!\,(n-m+q-p-2r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(q-p)!\,(n-m)!}$$
$$\times\frac{1}{(n+m+q+p-2r)!}$$
$$\times\left[\Sigma\,(-1)^s \frac{(q-p)!}{s!\,(q-p-s)!}\;\frac{(n+m+2q-2r-s)!\,(n-m+s)!}{(n-m-2r+s)!\,(n+m-s)!}\right]\cos(m+p)\lambda$$

$$+\Sigma(-1)^{r+p}(2n+2q-4r+1)\,Q_{n+q-2r}^{m-p}\,\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{(q-p)!\,r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left[\Sigma(-1)^s\frac{(q-p)!}{s!\,(q-p-s)!}\,\frac{(n-m+2q-2r-s)!\,(n+m+s)!}{(n+m-2r+s)!\,(n-m-s)!}\right]\cos(m-p)\lambda,$$

where s takes all values from 0 to $q-p$.

Simplifying these expressions we see that

$$2R_n^m R_q^p=\Sigma(-1)^r(2n+2q-4r+1)\frac{(n+q-r)!\,(2q-2r)!\,(n+m)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left\{R_{n+q-2r}^{m+p}\left[\Sigma(-1)^s\frac{(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n+m-s)!}\right]\right.$$

$$+R_{n+q-2r}^{m-p}\frac{(n-m+q+p-2r)!}{(n+m+q-p-2r)!}$$

$$\left.\times\left[\Sigma(-1)^s\frac{(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n-m-s)!}\right]\right\},$$

and also that

$$2R_n^m R_q^p=\Sigma(-1)^r(2n+2q-4r+1)\frac{(q+p)!\,(2n-2r)!\,(n+q-r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left\{R_{n+q-2r}^{m+p}\frac{(n+m)!\,(n-m+q-p-2r)!}{(n-m)!\,(n+m+q+p-2r)!}\right.$$

$$\times\left[\Sigma(-1)^s\frac{(n+m+2q-2r-s)!\,(n-m+s)!}{s!\,(q-p-s)!\,(n-m-2r+s)!\,(n+m-s)!}\right]$$

$$\left.+(-1)^p R_{n+q-2r}^{m-p}\left[\Sigma(-1)^s\frac{(n-m+2q-2r-s)!\,(n+m+s)!}{s!\,(q-p-s)!\,(n+m-2r+s)!\,(n-m-s)!}\right]\right\}.$$

15. From the relation between the functions Q_n^m and H_n^m, as given in Section I. (p. 248), we get

$$Q_{n+q-2r}^{m+p}=\frac{1\,.\,3\,.\,5\,\ldots\,(2n+2q-4r-1)}{(n-m+q-p-2r)!}H_{n+q-2r}^{m+p}$$

$$=\frac{(2n+2q-4r)!}{(n+q-2r)!\,(n-m+q-p-2r)!}\times\frac{H_{n+q-2r}^{m+p}}{2^{n+q-2r}},$$

and

$$Q_{n+q-2r}^{m-p}=\frac{1\,.\,3\,.\,5\,\ldots\,(2n+2q-4r-1)}{(n-m+q+p-2r)!}H_{n+q-2r}^{m-p}$$

$$=\frac{(2n+2q-4r)!}{(n+q-2r)!\,(n-m+q+p-2r)!}\times\frac{H_{n+q-2r}^{m-p}}{2^{n+q-2r}}.$$

Hence we get $2Q_n^m \times Q_q^p \cos m\lambda \cos p\lambda$

$$=\Sigma(-1)^r \frac{(2n+2q-4r+1)!\,(q+p)!\,(2n-2r)!\,(n+q-r)!\,(n+m)!}{(2n+2q-2r+1)!\,r!\,(q-r)!\,(n-r)!\,(n+q-2r)!\,(n-m)!\,(n+m+q+p-2r)!}$$

$$\times\left[\Sigma(-1)^s \frac{(n+m+2q-2r-s)!\,(n-m+s)!}{s!\,(q-p-s)!\,(n-m-2r+s)!\,(n+m-s)!}\right]\frac{H^{m+p}_{n+q-2r}}{2^{n+q-2r}}\cos(m+p)\lambda$$

$$+\Sigma(-1)^{r+p} \frac{(2n+2q-4r+1)!\,(q+p)!\,(2n-2r)!\,(n+q-r)!}{(2n+2q-2r+1)!\,r!\,(q-r)!\,(n-r)!\,(n+q-2r)!\,(n-m+q+p-2r)!}$$

$$\times\left[\Sigma(-1)^s \frac{(n-m+2q-2r-s)!\,(n+m+s)!}{s!\,(q-p-s)!\,(n+m-2r+s)!\,(n-m-s)!}\right]\frac{H^{m-p}_{n+q-2r}}{2^{n+q-2r}}\cos(m-p)\lambda,$$

where r has all values from 0 to $2q$, and s has all values from 0 to $q-p$.

16. From Art. 14 it appears that the product

$$2R_n^m R_q^p \quad\text{or}\quad Q_n^m Q_q^p[\cos(m+p)\lambda+\cos(m-p)\lambda]$$

when integrated with respect to λ between $\lambda=0$ and $\lambda=2\pi$ will vanish, hence

$$\int_0^{2\pi}\int_{-1}^{1} R_n^m R_q^p\, d\mu\, d\lambda=0.$$

Also if R_l^k be another Laplacian coefficient of the same form, then

$$\int_0^{2\pi}\int_{-1}^{1} R_l^k R_n^m R_q^p\, d\mu\, d\lambda=0,$$

except when $k=m+p$ or when $k=m-p$.

For R_n^m, R_q^p let us take the value of the Laplacian coefficient in its more general form, viz.: $Q_n^m\cos(m\lambda+\beta)$ and $Q_q^p\cos(p\lambda+\gamma)$,

where $$Q_n^m=(1-\mu^2)^{\frac{m}{2}} D^m P_n,$$

and $$Q_q^p=(1-\mu^2)^{\frac{p}{2}} D^p P_q,$$

then $$Q_l^k\cos(k\lambda+\alpha),\quad Q_n^m\cos(m\lambda+\beta),\quad Q_q^p\cos(p\lambda+\gamma)$$

represent any three such Laplace's coefficients.

Then since $2Q_n^m\cos(m\lambda+\beta)\times Q_q^p\cos(p\lambda+\gamma)$

$$=Q_n^m\times Q_q^p\{\cos[(m+p)\lambda+\beta+\gamma]+\cos[(m-p)\lambda+\beta-\gamma]\},$$

it is evident that the product of the three Laplace's coefficients when integrated with respect to λ between $\lambda = 0$ and $\lambda = 2\pi$ will vanish except when $k = m + p$ or $k = m - p$.

First let $k = m + p$ and $n + q - 2r = l$, and let K be the coefficient of Q^{m+p}_{n+q-2r} in the above product.

Then $$\int_0^{2\pi}\int_{-1}^{1} Q_l^k \cos(k\lambda + \alpha)\, Q_n^m \cos(m\lambda + \beta)\, Q_q^p \cos(p\lambda + \gamma)\, d\mu\, d\lambda$$

will be $$= \int_0^{2\pi}\int_{-1}^{1} K (Q_l^k)^2 \tfrac{1}{2} \cos(k\lambda + \beta + \gamma) \cos(k\lambda + \alpha)\, d\mu\, d\lambda$$

$$= \frac{2\pi}{4} \cos(\beta + \gamma - \alpha)\, K \int_{-1}^{1} (Q_l^k)^2\, d\mu = \pi \cos(\beta + \gamma - \alpha) \frac{K\,(l+k)!}{(2l+1)\,(l-k)!},$$

where $$K = (-1)^r (2l+1) \frac{(n+q-r)!\,(2q-2r)!\,(n+m)!}{r!\,(n-r)!\,(q-r)!\,(2n+2q-2r+1)!}$$

$$\times \left[\Sigma (-1)^s \frac{(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n+m-s)!}\right].$$

Another value determined above for

$$K = (-1)^r (2l+1) \frac{(q+p)!\,(2n-2r)!\,(l+r)!\,(n+m)!\,(l-k)!}{r!\,(q-r)!\,(n-r)!\,(n-m)!\,(2n+2q-2r+1)!\,(l+k)!}$$

$$\times \left[\Sigma (-1)^s \frac{(n-m+s)!\,(n+m+2q-2r-s)!}{s!\,(q-p-s)!\,(n-m-2r+s)!\,(n+m-s)!}\right].$$

Hence using this latter form for K we have

$$\int_0^{2\pi}\int_{-1}^{1} Q_l^k \cos(k\lambda + \alpha)\, Q_n^m \cos(m\lambda + \beta)\, Q_q^p \cos(p\lambda + \gamma)\, d\mu\, d\lambda$$

$$= \pi \cos(\beta + \gamma - \alpha)(-1)^r \frac{(q+p)!\,(2n-2r)!\,(l+r)!\,(n+m)!}{r!\,(q-r)!\,(n-r)!\,(n-m)!\,(n+q+l+1)!}$$

$$\times \left[\Sigma (-1)^s \frac{(l+k+q-p-s)!\,(n-m+s)!}{s!\,(q-p-s)!\,(n-m-2r+s)!\,(n+m-s)!}\right].$$

Now let $k = m - p$ and $n + q - 2r = l$, and let K' be the coefficient of Q^{m-p}_{n+q-2r} in the above product.

Then $$\int_0^{2\pi}\int_{-1}^{1} Q_l^k \cos(k\lambda+\alpha)\, Q_n^m \cos(m\lambda+\beta)\, Q_q^p \cos(p\lambda+\gamma)\, d\mu d\lambda$$

will be $$=\int_0^{2\pi}\int_{-1}^{1} K'(Q_l^k)^2\, \tfrac{1}{2}\cos(k\lambda+\beta-\gamma)\cos(k\lambda+\alpha)\, d\mu d\lambda$$

$$=\frac{2\pi}{4}\cos(\beta-\gamma-\alpha)\,K'\int_{-1}^{1}(Q_l^k)^2 d\mu=\pi\cos(\beta-\gamma-\alpha)\frac{K'\,(l+k)!}{(2l+1)\,(l-k)!},$$

where $$K'=(-1)^r(2l+1)\frac{(n+q-r)!\,(2q-2r)!\,(n+m)!\,(l-k)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!\,(l+k)!}$$

$$\times\left[\Sigma(-1)^s\frac{(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n-m-s)!}\right].$$

Another value determined above for

$$K'=(-1)^{r+p}(2l+1)\frac{(q+p)!\,(2n-2r)!\,(l+r)!}{r!\,(q-r)!\,(n-r)!\,(2n+2q-2r+1)!}$$

$$\times\left[\Sigma(-1)^s\frac{(n+m+s)!\,(n-m+2q-2r-s)!}{s!\,(q-p-s)!\,(n+m-2r+s)!\,(n-m-s)!}\right].$$

Hence using the first of these values of K' we have

$$\int_0^{2\pi}\int_{-1}^{1} Q_l^k\cos(k\lambda+\alpha)\,Q_n^m\cos(m\lambda+\beta)\,Q_q^p\cos(p\lambda+\gamma)\,d\mu d\lambda$$

$$=\pi\cos(\beta-\gamma-\alpha)\,(-1)^r\frac{(n+q-r)!\,(2q-2r)!\,(n+m)!}{r!\,(q-r)!\,(n-r)!\,(n+q+l+1)!}$$

$$\times\left[\Sigma(-1)^s\frac{(q+p+s)!\,(2n-2r+q-p-s)!}{s!\,(q-p-s)!\,(q+p-2r+s)!\,(n-m-s)!}\right].$$

SECTION V.

THE THEORY OF TERRESTRIAL MAGNETISM, GIVING THE EXPRESSIONS OF THE MAGNETIC FORCES ON THE EARTH'S SURFACE, THE EARTH BEING REGARDED AS A SPHERE.

1. LET V represent the magnetic potential, and let X, Y, Z be the magnetic forces in three directions at right angles to one another, X being the force towards the north perpendicular to the Earth's radius, Y the force perpendicular to the meridian towards the west, and Z the force towards the Earth's centre.

λ being the longitude and θ the colatitude, and r the distance from the Earth's centre,

$$\cos\theta = \mu \text{ and } \frac{d\mu}{d\theta} = -\sin\theta = -(1-\mu^2)^{\frac{1}{2}};$$

hence

$$X = -\frac{dV}{rd\theta} = \frac{(1-\mu^2)^{\frac{1}{2}}}{r}\frac{dV}{d\mu},$$

$$Y = -\frac{dV}{r\sin\theta d\lambda} = -\frac{(1-\mu^2)^{-\frac{1}{2}}}{r}\frac{dV}{d\lambda},$$

$$Z = -\frac{dV}{dr};$$

if east longitudes be considered positive. We may distinguish the two systems of values of V corresponding to magnetic forces whose origin is situated inside and outside the Earth's surface respectively by affecting them with the suffix n when the corresponding value of V involves a positive power of $\frac{1}{r}$, and with the negative suffix, $-n$, when the value of V involves a negative power of $\frac{1}{r}$.

Then $$V = \Sigma \frac{1}{r^{n+1}} \{H_n^m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\},$$

for the first class of terms:

and $$V = \Sigma r^n [H_n^m (g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda)],$$

for the second class of terms.

Let $$X_n^m = (1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu} \frac{1}{r^{n+2}} \text{ and } X_{-n}^m = r^{n-1} (1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu}.$$

Then by equation (22) of Section I. (see p. 257)

$$X_n^m = \frac{1}{r^{n+2}} [(n-m) H_n^{m+1} - m\mu (1-\mu^2)^{-\frac{1}{2}} H_n^m],$$

or by equation (16),

$$X_n^m = \frac{1}{r^{n+2}} \left[\frac{1}{2} (n-m) H_n^{m+1} - \frac{1}{2} (n+m) H_n^{m-1}\right];$$

hence for the first class of terms, i.e. for forces whose origin is situated in the interior of the Earth,

$$\begin{aligned} X &= \Sigma [X_n^m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)] \\ &= \Sigma \frac{1}{r^{n+2}} \left[\frac{1}{2} (n-m) H_n^{m+1} - \frac{1}{2} (n+m) H_n^{m-1}\right] (g_n^m \cos m\lambda + h_n^m \sin m\lambda), \\ Y &= \Sigma \frac{1}{r^{n+2}} [m H_n^m (1-\mu^2)^{-\frac{1}{2}} \qquad\qquad] (g_n^m \sin m\lambda - h_n^m \cos m\lambda), \\ Z &= \Sigma \frac{n+1}{r^{n+2}} H_n^m \qquad\qquad (g_n^m \cos m\lambda + h_n^m \sin m\lambda), \end{aligned}$$

where g_n^m, h_n^m are the Gaussian magnetic constants for positive integral values of m and n.

And for the second class of terms, i.e. for forces whose origin is outside the Earth, the corresponding terms are:

in the value of X

$$= \Sigma r^{n-1} \left[\frac{1}{2} (n-m) H_n^{m+1} - \frac{1}{2} (n+m) H_n^{m-1}\right] (g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda),$$

in the value of Y

$$= \Sigma r^{n-1} [m H_n^m (1-\mu^2)^{-\frac{1}{2}}] (g_{-n}^m \sin m\lambda - h_{-n}^m \cos m\lambda),$$

in the value of Z

$$= \Sigma r^{n-1} [-nH_n^m] (g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda),$$

where g_{-n}^m, h_{-n}^m are the magnetic constants for positive integral values of m and n.

2. Let $V_n^m = \frac{1}{r^{n+1}} H_n^m$ and $V_{-n}^m = r^n H_n^m$,

also let $\quad Y_n^m = \frac{1}{r^{n+2}} m H_n^m (1-\mu^2)^{-\frac{1}{2}}$ and $Y_{-n}^m = r^{n-1} m H_n^m (1-\mu^2)^{-\frac{1}{2}}$,

and let $\quad Z_n^m = \frac{1}{r^{n+2}} (n+1) H_n^m$ and $Z_{-n}^m = r^{n-1}(-nH_n^m)$.

Then taking both classes of terms together we have

$$V = \Sigma \{V_n^m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\} + \Sigma \{V_{-n}^m (g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda)\},$$
$$X = \Sigma \{X_n^m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\} + \Sigma \{X_{-n}^m (g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda)\},$$
$$Y = \Sigma \{Y_n^m (g_n^m \sin m\lambda - h_n^m \cos m\lambda)\} + \Sigma \{Y_{-n}^m (g_{-n}^m \sin m\lambda - h_{-n}^m \cos m\lambda)\},$$
$$Z = \Sigma \{Z_n^m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\} + \Sigma \{Z_{-n}^m (g_{-n}^m \cos m\lambda + h_{-n}^m \sin m\lambda)\}.$$

Collecting coefficients of $\cos m\lambda$ and $\sin m\lambda$,

the coefficient of $\cos m\lambda$ in V is $\Sigma (V_n^m g_n^m + V_{-n}^m g_{-n}^m)$,
.... X is $\Sigma (X_n^m g_n^m + X_{-n}^m g_{-n}^m)$,
.... Y is $-\Sigma (Y_n^m h_n^m + Y_{-n}^m h_{-n}^m)$,
.... Z is $\Sigma (Z_n^m g_n^m + Z_{-n}^m g_{-n}^m)$,
the coefficient of $\sin m\lambda$ in V is $\Sigma (V_n^m h_n^m + V_{-n}^m h_{-n}^m)$,
.... X is $\Sigma (X_n^m h_n^m + X_{-n}^m h_{-n}^m)$,
.... Y is $\Sigma (Y_n^m g_n^m + Y_{-n}^m g_{-n}^m)$,
.... Z is $\Sigma (Z_n^m h_n^m + Z_{-n}^m h_{-n}^m)$,

in which n takes all integral values for a given value of m.

The relations between the functions when the suffix is changed from n to $-n$ are

$$X_{-n}^m = r^{2n+1} X_n^m, \quad Y_{-n}^m = r^{2n+1} Y_n^m, \quad Z_{-n}^m = -\frac{n}{n+1} r^{2n+1} Z_n^m.$$

On the surface of a sphere of radius unity V_n^m and V_{-n}^m are each of them equal to H_n^m, i.e. to $G_n^m (1-\mu^2)^{\frac{m}{2}}$, and it will be convenient to express their values in terms of μ the cosine of the colatitude of a point on the surface of the sphere.

3. *Collection of the values of the quantities H_n^m or V_n^m.*

When $m=0$, $V_0^0=1$, $V_1^0=\mu$, $V_2^0=\mu^2-\frac{1}{3}$, $V_3^0=\mu^3-\frac{3}{5}\mu$,

$$V_4^0=\mu^4-\frac{6}{7}\mu^2+\frac{3}{35},\quad V_5^0=\mu^5-\frac{10}{9}\mu^3+\frac{5}{21}\mu,$$

$$V_6^0=\mu^6-\frac{15}{11}\mu^4+\frac{5}{11}\mu^2-\frac{5}{231},$$

$$V_7^0=\mu^7-\frac{21}{13}\mu^5+\frac{105}{143}\mu^3-\frac{35}{429}\mu,$$

$$V_8^0=\mu^8-\frac{28}{15}\mu^6+\frac{14}{13}\mu^4-\frac{28}{143}\mu^2+\frac{7}{1287},$$

$$V_9^0=\mu^9-\frac{36}{17}\mu^7+\frac{126}{85}\mu^5-\frac{84}{221}\mu^3+\frac{63}{2431}\mu,$$

$$V_{10}^0=\mu^{10}-\frac{45}{19}\mu^8+\frac{630}{323}\mu^6-\frac{210}{323}\mu^4+\frac{315}{4199}\mu^2-\frac{63}{46189}.$$

When $m=1$, $V_1^1=(1-\mu^2)^{\frac{1}{2}}$, $V_2^1=(1-\mu^2)^{\frac{1}{2}}(\mu)$,

$$V_3^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^2-\frac{1}{5}\right),\quad V_4^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^3-\frac{3}{7}\mu\right),$$

$$V_5^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^4-\frac{2}{3}\mu^2+\frac{1}{21}\right),$$

$$V_6^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^5-\frac{10}{11}\mu^3+\frac{5}{33}\mu\right),$$

$$V_7^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^6-\frac{15}{13}\mu^4+\frac{45}{143}\mu^2-\frac{5}{429}\right),$$

$$V_8^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^7-\frac{7}{5}\mu^5+\frac{7}{13}\mu^3-\frac{7}{143}\mu\right),$$

$$V_9^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^8-\frac{28}{17}\mu^6+\frac{14}{17}\mu^4-\frac{28}{221}\mu^2+\frac{7}{2431}\right),$$

$$V_{10}^1=(1-\mu^2)^{\frac{1}{2}}\left(\mu^9-\frac{36}{19}\mu^7+\frac{378}{323}\mu^5-\frac{84}{323}\mu^3+\frac{63}{4199}\mu\right).$$

When $m=2$, $V_2^2=(1-\mu^2)$, $V_3^2=(1-\mu^2)(\mu)$,

$$V_4^2=(1-\mu^2)\left(\mu^2-\frac{1}{7}\right), \qquad V_5^2=(1-\mu^2)\left(\mu^3-\frac{1}{3}\mu\right),$$

$$V_6^2=(1-\mu^2)\left(\mu^4-\frac{6}{11}\mu^2+\frac{1}{33}\right),$$

$$V_7^2=(1-\mu^2)\left(\mu^5-\frac{10}{13}\mu^3+\frac{15}{143}\mu\right),$$

$$V_8^2=(1-\mu^2)\left(\mu^6-\mu^4+\frac{3}{13}\mu^2-\frac{1}{143}\right),$$

$$V_9^2=(1-\mu^2)\left(\mu^7-\frac{21}{17}\mu^5+\frac{7}{17}\mu^3-\frac{7}{221}\mu\right),$$

$$V_{10}^2=(1-\mu^2)\left(\mu^8-\frac{28}{19}\mu^6+\frac{210}{323}\mu^4-\frac{28}{323}\mu^2+\frac{7}{4199}\right).$$

When $m=3$, $V_3^3=(1-\mu^2)^{\frac{3}{2}}$, $V_4^3=(1-\mu^2)^{\frac{3}{2}}(\mu)$,

$$V_5^3=(1-\mu^2)^{\frac{3}{2}}\left(\mu^2-\frac{1}{9}\right),$$

$$V_6^3=(1-\mu^2)^{\frac{3}{2}}\left(\mu^3-\frac{3}{11}\mu\right),$$

$$V_7^3=(1-\mu^2)^{\frac{3}{2}}\left(\mu^4-\frac{6}{13}\mu^2+\frac{3}{143}\right),$$

$$V_8^3=(1-\mu^2)^{\frac{3}{2}}\left(\mu^5-\frac{2}{3}\mu^3+\frac{1}{13}\mu\right),$$

$$V_9^3=(1-\mu^2)^{\frac{3}{2}}\left(\mu^6-\frac{15}{17}\mu^4+\frac{3}{17}\mu^2-\frac{1}{221}\right),$$

$$V_{10}^3=(1-\mu^2)^{\frac{3}{2}}\left(\mu^7-\frac{21}{19}\mu^5+\frac{105}{323}\mu^3-\frac{7}{323}\mu\right).$$

When $m=4$, $V_4^4=(1-\mu^2)^2$, $V_5^4=(1-\mu^2)^2(\mu)$,

$$V_6^4=(1-\mu^2)^2\left(\mu^2-\frac{1}{11}\right),$$

$$V_7^4=(1-\mu^2)^2\left(\mu^3-\frac{3}{13}\mu\right),$$

$$V_8^4=(1-\mu^2)^2\left(\mu^4-\frac{2}{5}\mu^2+\frac{1}{65}\right),$$

$$V_9^4=(1-\mu^2)^2\left(\mu^5-\frac{10}{17}\mu^3+\frac{1}{17}\mu\right),$$

$$V_{10}^4=(1-\mu^2)^2\left(\mu^6-\frac{15}{19}\mu^4+\frac{45}{323}\mu^2-\frac{1}{323}\right).$$

When $m=5$, $V_5^5=(1-\mu^2)^{\frac{5}{2}}$, $\quad V_6^5=(1-\mu^2)^{\frac{5}{2}}(\mu)$,

$$V_7^5=(1-\mu^2)^{\frac{5}{2}}\left(\mu^2-\frac{1}{13}\right),$$

$$V_8^5=(1-\mu^2)^{\frac{5}{2}}\left(\mu^3-\frac{1}{5}\mu\right),$$

$$V_9^5=(1-\mu^2)^{\frac{5}{2}}\left(\mu^4-\frac{6}{17}\mu^2+\frac{1}{85}\right),$$

$$V_{10}^5=(1-\mu^2)^{\frac{5}{2}}\left(\mu^5-\frac{10}{19}\mu^3+\frac{15}{323}\mu\right).$$

When $m=6$, $V_6^6=(1-\mu^2)^3$, $\quad V_7^6=(1-\mu^2)^3(\mu)$,

$$V_8^6=(1-\mu^2)^3\left(\mu^2-\frac{1}{15}\right),$$

$$V_9^6=(1-\mu^2)^3\left(\mu^3-\frac{3}{17}\mu\right),$$

$$V_{10}^6=(1-\mu^2)^3\left(\mu^4-\frac{6}{19}\mu^2+\frac{3}{323}\right).$$

When $m=7$, $V_7^7=(1-\mu^2)^{\frac{7}{2}}$, $\quad V_8^7=(1-\mu^2)^{\frac{7}{2}}(\mu)$,

$$V_9^7=(1-\mu^2)^{\frac{7}{2}}\left(\mu^2-\frac{1}{17}\right),$$

$$V_{10}^7=(1-\mu^2)^{\frac{7}{2}}\left(\mu^3-\frac{3}{19}\mu\right).$$

When $m=8$, $V_8^8=(1-\mu^2)^4$, $\quad V_9^8=(1-\mu^2)^4(\mu)$,

$$V_{10}^8=(1-\mu^2)^4\left(\mu^2-\frac{1}{19}\right).$$

When $m=9$, $V_9^9=(1-\mu^2)^{\frac{9}{2}}$, $\quad V_{10}^9=(1-\mu^2)^{\frac{9}{2}}(\mu)$.

When $m=10$, $V_{10}^{10}=(1-\mu^2)^5$.

4. On a sphere of radius unity, since $\cos\theta = \mu$,

$$X = -\frac{dV}{d\theta} = \frac{dV}{d\mu}(1-\mu^2)^{\frac{1}{2}},$$

and V_n^m becomes the same as H_n^m.

Collecting the values of the quantities X_n^m, we have:—

When $m=0$,

$$X_1^0 = (1-\mu^2)^{\frac{1}{2}},$$

$$X_2^0 = 2(1-\mu^2)^{\frac{1}{2}}(\mu),$$

$$X_3^0 = 3(1-\mu^2)^{\frac{1}{2}}\left(\mu^2 - \frac{1}{5}\right),$$

$$X_4^0 = 4(1-\mu^2)^{\frac{1}{2}}\left(\mu^3 - \frac{3}{7}\mu\right),$$

$$X_5^0 = 5(1-\mu^2)^{\frac{1}{2}}\left(\mu^4 - \frac{2}{3}\mu^2 + \frac{1}{21}\right),$$

$$X_6^0 = 6(1-\mu^2)^{\frac{1}{2}}\left(\mu^5 - \frac{10}{11}\mu^3 + \frac{5}{33}\mu\right),$$

$$X_7^0 = 7(1-\mu^2)^{\frac{1}{2}}\left(\mu^6 - \frac{15}{13}\mu^4 + \frac{45}{143}\mu^2 - \frac{5}{429}\right),$$

$$X_8^0 = 8(1-\mu^2)^{\frac{1}{2}}\left(\mu^7 - \frac{7}{5}\mu^5 + \frac{7}{13}\mu^3 - \frac{7}{143}\mu\right),$$

$$X_9^0 = 9(1-\mu^2)^{\frac{1}{2}}\left(\mu^8 - \frac{28}{17}\mu^6 + \frac{14}{17}\mu^4 - \frac{28}{221}\mu^2 + \frac{7}{2431}\right),$$

$$X_{10}^0 = 10(1-\mu^2)^{\frac{1}{2}}\left(\mu^9 - \frac{36}{19}\mu^7 + \frac{378}{323}\mu^5 - \frac{84}{323}\mu^3 + \frac{63}{4199}\mu\right).$$

When $m=1$,

$$X_1^1 = -\mu,$$

$$X_2^1 = -2\mu^2 + 1,$$

$$X_3^1 = -3\mu^3 + \frac{11}{5}\mu,$$

$$X_4^1 = -4\mu^4 + \frac{27}{7}\mu^2 - \frac{3}{7},$$

$$X_5^1 = -5\mu^5 + 6\mu^3 - \frac{29}{21}\mu,$$

$$X_6^1 = -6\mu^6 + \frac{95}{11}\mu^4 - \frac{100}{33}\mu^2 + \frac{5}{33},$$

$$X_7^1 = -7\mu^7 + \frac{153}{13}\mu^5 - \frac{795}{143}\mu^3 + \frac{275}{429}\mu,$$

$$X_8^1 = -8\mu^8 + \frac{77}{5}\mu^6 - \frac{119}{13}\mu^4 + \frac{245}{143}\mu^2 - \frac{7}{143},$$

$$X_9^1 = -9\mu^9 + \frac{332}{17}\mu^7 - \frac{238}{17}\mu^5 + \frac{812}{221}\mu^3 - \frac{623}{2431}\mu,$$

$$X_{10}^1 = -10\mu^{10} + \frac{459}{19}\mu^8 - \frac{6552}{323}\mu^6 + \frac{2226}{323}\mu^4 - \frac{3402}{4199}\mu^2 + \frac{63}{4199}.$$

When $m=2$, $X_2^2 = -2(1-\mu^2)^{\frac{1}{2}}(\mu) = (1-\mu^2)^{\frac{1}{2}}(-2\mu),$

$$X_3^2 = -3(1-\mu^2)^{\frac{1}{2}}\left(\mu^2 - \frac{1}{3}\right) = (1-\mu^2)^{\frac{1}{2}}(-3\mu^2+1),$$

$$X_4^2 = -4(1-\mu^2)^{\frac{1}{2}}\left(\mu^3 - \frac{4}{7}\mu\right) = (1-\mu^2)^{\frac{1}{2}}\left(-4\mu^3 + \frac{16}{7}\mu\right),$$

$$X_5^2 = -5(1-\mu^2)^{\frac{1}{2}}\left(\mu^4 - \frac{4}{5}\mu^2 + \frac{1}{15}\right) = (1-\mu^2)^{\frac{1}{2}}\left(-5\mu^4 + 4\mu^2 - \frac{1}{3}\right),$$

$$X_6^2 = (1-\mu^2)^{\frac{1}{2}}\left(-6\mu^5 + \frac{68}{11}\mu^3 - \frac{38}{33}\mu\right),$$

$$X_7^2 = (1-\mu^2)^{\frac{1}{2}}\left(-7\mu^6 + \frac{115}{13}\mu^4 - \frac{375}{143}\mu^2 + \frac{15}{143}\right),$$

$$X_8^2 = (1-\mu^2)^{\frac{1}{2}}\left(-8\mu^7 + 12\mu^5 - \frac{64}{13}\mu^3 + \frac{68}{143}\mu\right),$$

$$X_9^2 = (1-\mu^2)^{\frac{1}{2}}\left(-9\mu^8 + \frac{266}{17}\mu^6 - \frac{140}{17}\mu^4 + \frac{294}{221}\mu^2 - \frac{7}{221}\right),$$

$$X_{10}^2 = (1-\mu^2)^{\frac{1}{2}}\left(-10\mu^9 + \frac{376}{19}\mu^7 - \frac{4116}{323}\mu^5 + \frac{952}{323}\mu^3 - \frac{742}{4199}\mu\right).$$

When $m=3$,

$$X_3^3=(1-\mu^2)(-3\mu),$$

$$X_4^3=(1-\mu^2)(-4\mu^2+1),$$

$$X_5^3=(1-\mu^2)\left(-5\mu^3+\frac{7}{3}\mu\right),$$

$$X_6^3=(1-\mu^2)\left(-6\mu^4+\frac{45}{11}\mu^2-\frac{3}{11}\right),$$

$$X_7^3=(1-\mu^2)\left(-7\mu^5+\frac{82}{13}\mu^3-\frac{141}{143}\mu\right),$$

$$X_8^3=(1-\mu^2)\left(-8\mu^6+9\mu^4-\frac{30}{13}\mu^2+\frac{1}{13}\right),$$

$$X_9^3=(1-\mu^2)\left(-9\mu^7+\frac{207}{17}\mu^5-\frac{75}{17}\mu^3+\frac{81}{221}\mu\right),$$

$$X_{10}^3=(1-\mu^2)\left(-10\mu^8+\frac{301}{19}\mu^6-\frac{2415}{323}\mu^4+\frac{343}{323}\mu^2-\frac{7}{323}\right).$$

When $m=4$,

$$X_4^4=(1-\mu^2)^{\frac{3}{2}}(-4\mu),$$

$$X_5^4=(1-\mu^2)^{\frac{3}{2}}(-5\mu^2+1),$$

$$X_6^4=(1-\mu^2)^{\frac{3}{2}}\left(-6\mu^3+\frac{26}{11}\mu\right),$$

$$X_7^4=(1-\mu^2)^{\frac{3}{2}}\left(-7\mu^4+\frac{54}{13}\mu^2-\frac{3}{13}\right),$$

$$X_8^4=(1-\mu^2)^{\frac{3}{2}}\left(-8\mu^5+\frac{32}{5}\mu^3-\frac{56}{65}\mu\right),$$

$$X_9^4=(1-\mu^2)^{\frac{3}{2}}\left(-9\mu^6+\frac{155}{17}\mu^4-\frac{35}{17}\mu^2+\frac{1}{17}\right),$$

$$X_{10}^4=(1-\mu^2)^{\frac{3}{2}}\left(-10\mu^7+\frac{234}{19}\mu^5-\frac{1290}{323}\mu^3+\frac{94}{323}\mu\right).$$

When $m=5$,

$$X_5^5=(1-\mu^2)^2(-5\mu),$$

$$X_6^5=(1-\mu^2)^2(-6\mu^2+1),$$

$$X_7^5=(1-\mu^2)^2\left(-7\mu^3+\frac{31}{13}\mu\right),$$

$$X_8^5=(1-\mu^2)^2\left(-8\mu^4+\frac{21}{5}\mu^2-\frac{1}{5}\right),$$

$$X_9^5=(1-\mu^2)^2\left(-9\mu^5+\frac{110}{17}\mu^3-\frac{13}{17}\mu\right),$$

$$X_{10}^5=(1-\mu^2)^2\left(-10\mu^6+\frac{175}{19}\mu^4-\frac{600}{323}\mu^2+\frac{15}{323}\right).$$

When $m=6$, $X_6^6=(1-\mu^2)^{\frac{5}{2}}(-6\mu)$,

$$X_7^6=(1-\mu^2)^{\frac{5}{2}}(-7\mu^2+1),$$

$$X_8^6=(1-\mu^2)^{\frac{5}{2}}\left(-8\mu^3+\frac{12}{5}\mu\right),$$

$$X_9^6=(1-\mu^2)^{\frac{5}{2}}\left(-9\mu^4+\frac{72}{17}\mu^2-\frac{3}{17}\right),$$

$$X_{10}^6=(1-\mu^2)^{\frac{5}{2}}\left(-10\mu^5+\frac{124}{19}\mu^3-\frac{222}{323}\mu\right).$$

When $m=7$, $X_7^7=(1-\mu^2)^3(-7\mu)$,

$$X_8^7=(1-\mu^2)^3(-8\mu^2+1),$$

$$X_9^7=(1-\mu^2)^3\left(-9\mu^3+\frac{41}{17}\mu\right),$$

$$X_{10}^7=(1-\mu^2)^3\left(-10\mu^4+\frac{81}{19}\mu^2-\frac{3}{19}\right).$$

When $m=8$, $X_8^8=(1-\mu^2)^{\frac{7}{2}}(-8\mu)$,

$$X_9^8=(1-\mu^2)^{\frac{7}{2}}(-9\mu^2+1),$$

$$X_{10}^8=(1-\mu^2)^{\frac{7}{2}}\left(-10\mu^3+\frac{46}{19}\mu\right).$$

When $m=9$, $X_9^9=(1-\mu^2)^4(-9\mu)$,

$$X_{10}^9=(1-\mu^2)^4(-10\mu^2+1).$$

When $m=10$, $X_{10}^{10}=(1-\mu^2)^{\frac{9}{2}}(-10\mu)$.

5. To adapt the preceding investigations on Legendre's and Laplace's coefficients to the theory of Terrestrial Magnetism, there are certain relations of the functions H_n^m and the functions X_n^m, Y_n^m and Z_n^m which still remain to be developed and which will be found useful and will greatly facilitate the determination of the magnetic constants of Terrestrial Magnetism from the observed values of the magnetic elements at places regularly distributed over the Globe.

We proceed now to the development of this Theory of Terrestrial Magnetism.

Assuming, as in Section I., that $Q_n^m = (1-\mu^2)^{\frac{m}{2}} D^m P_n$, it has been proved in Section III. (see p. 363), that

$$\int_{-1}^{1} Q_n^m Q_{n_1}^m d\mu = 0,$$

except when $n = n_1$; and that when $n_1 = n$, we have

$$\int_{-1}^{1} (Q_n^m)^2 d\mu = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}.$$

Also from Section I. we have

$$H_n^m = \frac{(n-m)!}{1.3.5 \ldots (2n-1)} Q_n^m;$$

hence we have $\int_{-1}^{1} H_n^m H_{n_1}^m d\mu = 0$, except when $n = n_1$; and when $n_1 = n$, we have

$$\int_{-1}^{1} (H_n^m)^2 d\mu = \frac{2}{2n+1} \frac{(n-m)!\,(n+m)!}{[1.3.5 \ldots (2n-1)]^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

6. From equation (4) above (p. 246) we have

$$DP_{n+1} = (2n+1) P_n + DP_{n-1},$$

therefore by successive substitutions,

$$DP_{n+1} = (2n+1) P_n + (2n-3) P_{n-2} + (2n-7) P_{n-4} + \&c.,$$

the last term when n is even being P_0, and when n is odd being $3P_1$.

Putting n for $n+1$ and differentiating $m-1$ times, we get

$$D^m P_n = (2n-1) D^{m-1} P_{n-1} + (2n-5) D^{m-1} P_{n-3} + \&c. + \begin{bmatrix} D^{m-1} P_0 \text{ when } n \text{ is odd,} \\ 3D^{m-1} P_1 \text{ when } n \text{ is even.} \end{bmatrix}$$

Also from equation (3) above (p. 246) by differentiating m times we get

$$\mu D^m P_n + m D^{m-1} P_n = \frac{n+1}{2n+1} D^m P_{n+1} + \frac{n}{2n+1} D^m P_{n-1}.$$

Hence $\mu D^m P_n + m D^{m-1} P_n = \dfrac{n+1}{2n+1} \left\{ (2n+1) D^{m-1} P_n + (2n-3) D^{m-1} P_{n-2} + \text{\&c.} \right.$

$$\left. + \begin{bmatrix} D^{m-1} P_0 \text{ when } n \text{ is even,} \\ 3D^{m-1} P_1 \text{ when } n \text{ is odd.} \end{bmatrix} \right\}$$

$$+ \frac{n}{2n+1} \left\{ (2n-3) D^{m-1} P_{n-2} + (2n-7) D^{m-1} P_{n-4} + \text{\&c.} \right.$$

$$\left. + \begin{bmatrix} D^{m-1} P_0 \text{ when } n \text{ is even,} \\ 3D^{m-1} P_1 \text{ when } n \text{ is odd.} \end{bmatrix} \right\}$$

i.e. $\mu D^m P_n = (n-m+1) D^{m-1} P_n + (2n-3) D^{m-1} P_{n-2} + \text{\&c.}$

$$+ \begin{bmatrix} D^{m-1} P_0 \text{ when } n \text{ is even,} \\ 3D^{m-1} P_1 \text{ when } n \text{ is odd.} \end{bmatrix}$$

Also we have seen above (see p. 249) that

$$(1-\mu^2) D^{m+1} P_n = 2m\mu D^m P_n - (n+m)(n-m+1) D^{m-1} P_n.$$

$$\therefore (1-\mu^2) D^{m+1} P_n = -(n-m)(n-m+1) D^{m-1} P_n + 2m(2n-3) D^{m-1} P_{n-2} + \text{\&c.}$$
$$+ 2m(2n-4r+1) D^{m-1} P_{n-2r} + \text{\&c.}$$

Multiplying by $(1-\mu^2)^{\frac{m-1}{2}}$ and putting

$$Q_n^m \text{ for } (1-\mu^2)^{\frac{m}{2}} D^m P_n,$$

we get $Q_n^{m+1} = -(n-m)(n-m+1) Q_n^{m-1} + 2m(2n-3) Q_{n-2}^{m-1} + \text{\&c.}$

$$+ 2m(2n-4r+1) Q_{n-2r}^{m-1} + \text{\&c.};$$

and since $Q_n^{m+1} = \dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(n-m-1)!} H_n^{m+1}$, we get

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(n-m-1)!} H_n^{m+1} = -(n-m)(n-m+1) \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(n-m+1)!} H_n^{m-1}$$
$$+ 2m \left\{ (2n-3) \frac{1 \cdot 3 \cdot 5 \ldots (2n-5)}{(n-m-1)!} H_{n-2}^{m-1} + \text{\&c.} \right\}.$$

Hence $H_n^{m+1} = -H_n^{m-1} + 2m \left\{ \dfrac{1}{2n-1} H_{n-2}^{m-1} + \dfrac{(n-m-1)(n-m-2)}{(2n-1)(2n-3)(2n-5)} H_{n-4}^{m-1} + \text{\&c.} \right.$

$$\left. + \frac{(n-m-1)(n-m-2)\ldots(n-m-2r+2)}{(2n-1)(2n-3)\ldots(2n-4r+3)} H_{n-2r}^{m-1} + \text{\&c.} \right\} \ldots\ldots(2).$$

Now multiply by $H_{n-2r}^{m-1}\,d\mu$ and integrate from $\mu=-1$ to $\mu=1$, then

$$\int_{-1}^{1} H_n^{m+1} H_{n-2r}^{m-1}\,d\mu = 2m\frac{(n-m-1)(n-m-2)\dots(n-m-2r+2)}{(2n-1)(2n-3)\dots(2n-4r+3)}\int_{-1}^{1}(H_{n-2r}^{m-1})^2\,d\mu$$

$$= 2m\frac{(n-m-1)(n-m-2)\dots(n-m-2r+2)}{(2n-1)(2n-3)\dots(2n-4r+3)}$$

$$\times 2\frac{(n-m-2r+1)!\,(n+m-2r+1)!}{\{1.3.5\dots(2n-4r-1)\}^2(2n-4r+1)}$$

$$= 4m\frac{(n-m-1)!\,(n+m-2r-1)!}{1.3.5\dots(2n-1)\,1.3.5\dots(2n-4r-1)}\quad\dots\dots\dots\dots\dots(3).$$

Putting $n-2r=n_1$, and recollecting that n and n_1 are both even or both odd, and that when one of them is even and the other odd the integral evidently vanishes, we have

$$\int_{-1}^{1} H_n^{m+1} H_{n_1}^{m-1}\,d\mu = 4m\frac{(n-m-1)!\,(n_1+m-1)!}{1.3.5\dots(2n-1)\,1.3.5\dots(2n_1-1)},$$

where n_1 is less than n.

Writing n_1 for n in the above equation before integration and multiplying by $H_n^{m-1}\,d\mu$ and then integrating we get

$$\int_{-1}^{1} H_{n_1}^{m+1} H_n^{m-1}\,d\mu = 0,$$

since all the quantities n_1, n_1-2, &c. are less than n and so all the terms separately vanish.

Again multiplying the above equation before integration by $H_n^{m-1}\,d\mu$ we get

$$\int_{-1}^{1} H_n^{m+1} H_n^{m-1}\,d\mu = -\int_{-1}^{1}(H_n^{m-1})^2\,d\mu$$

$$= -2\frac{(n-m+1)!\,(n+m-1)!}{\{1.3.5\dots(2n-1)\}^2(2n+1)}.$$

7. We have seen above that

$$D^2P_n = (2n-1)(2n-3)P_{n-2} + 2(2n-3)(2n-7)P_{n-4}$$
$$+3(2n-5)(2n-11)P_{n-6} + \&c. + r(2n-2r+1)(2n-4r+1)P_{n-2r} + \&c.$$

Hence $D^{m+2}P_n = (2n-1)(2n-3)D^mP_{n-2} + 2(2n-3)(2n-7)D^mP_{n-4} + \&c.$
$$+r(2n-2r+1)(2n-4r+1)D^mP_{n-2r} + \&c.$$

$$\therefore\ (1-\mu^2)^{\frac{m}{2}} D^{m+2} P_n = (2n-1)(2n-3)(1-\mu^2)^{\frac{m}{2}} D^m P_{n-2}$$
$$+ 2(2n-3)(2n-7)(1-\mu^2)^{\frac{m}{2}} D^m P_{n-4} + \&c.$$
$$+ r(2n-2r+1)(2n-4r+1)(1-\mu^2)^{\frac{m}{2}} D^m P_{n-2r} + \&c.$$

$$\therefore\ \frac{Q_n^{m+2}}{1-\mu^2} = (2n-1)(2n-3)\,Q_{n-2}^m + 2(2n-3)(2n-7)\,Q_{n-4}^m + \&c.$$
$$+ r(2n-2r+1)(2n-4r+1)\,Q_{n-2r}^m + \&c.,$$

hence
$$\frac{Q_n^m}{1-\mu^2} = (2n-1)(2n-3)\,Q_{n-2}^{m-2} + 2(2n-3)(2n-7)\,Q_{n-4}^{m-2} + \&c.$$
$$+ r(2n-2r+1)(2n-4r+1)\,Q_{n-2r}^{m-2} + \&c.$$

From which we obtain

$$\frac{H_n^m}{1-\mu^2} = H_{n-2}^{m-2} + 2(2n-3)\frac{(n-m)(n-m-1)}{(2n-1)(2n-3)(2n-5)} H_{n-4}^{m-2} + \&c.$$
$$+ r(2n-2r+1)\frac{(n-m)(n-m-1)\ldots(n-m-2r+3)}{(2n-1)(2n-3)\ldots(2n-4r+3)} H_{n-2r}^{m-2} + \&c \ldots (4).$$

Multiply the value of $(1-\mu^2)^{\frac{m}{2}} D^{m+2} P_n$ obtained above by $(1-\mu^2)^{\frac{m}{2}} D^m P_{n-2r}$ and integrate from $\mu = -1$ to $\mu = 1$, then

$$\int_{-1}^{1} (1-\mu^2)^m D^{m+2} P_n \,.\, D^m P_{n-2r}\, d\mu$$
$$= r(2n-2r+1)(2n-4r+1)\int_{-1}^{1} (1-\mu^2)^m (D^m P_{n-2r})^2\, d\mu$$
$$= r(2n-2r+1)(2n-4r+1)\frac{2}{2n-4r+1}\frac{(n+m-2r)!}{(n-m-2r)!}$$
$$= 2r(2n-2r+1)\frac{(n+m-2r)!}{(n-m-2r)!}.$$

Putting $n-2r = n_1$, we get

$$\int_{-1}^{1} (1-\mu^2)^m D^{m+2} P_n \,.\, D^m P_{n_1}\, d\mu = (n-n_1)(n+n_1+1)\frac{(n_1+m)!}{(n_1-m)!}.$$

We have seen that

$$H_n^m = -H_n^{m-2} + 2(m-1)\left\{\frac{1}{2n-1} H_{n-2}^{m-2} + \frac{(n-m)(n-m-1)}{(2n-1)(2n-3)(2n-5)} H_{n-4}^{m-2} + \&c.\right.$$
$$\left. + \frac{(n-m)(n-m-1)\ldots(n-m-2r+3)}{(2n-1)(2n-3)\ldots(2n-4r+3)} H_{n-2r}^{m-2} + \&c.\right\},$$

and

$$\frac{H_n^m}{1-\mu^2} = H_{n-2}^{m-2} + 2(2n-3)\frac{(n-m)(n-m-1)}{(2n-1)(2n-3)(2n-5)} H_{n-4}^{m-2} + \&c.$$
$$+ r(2n-2r+1)\frac{(n-m)(n-m-1)\dots(n-m-2r+3)}{(2n-1)(2n-3)\dots(2n-4r+3)} H_{n-2r}^{m-2} + \&c.$$

Multiply this last equation by $H_n^m d\mu$ and integrate, then

$$\int_{-1}^{1}\frac{(H_n^m)^2}{1-\mu^2}d\mu = \int_{-1}^{1} H_n^m H_{n-2}^{m-2} d\mu + 2(2n-3)\frac{(n-m)(n-m-1)}{(2n-1)(2n-3)(2n-5)}\int_{-1}^{1} H_n^m H_{n-4}^{m-2} d\mu$$
$$+ r(2n-2r+1)\frac{(n-m)(n-m-1)\dots(n-m-2r+3)}{(2n-1)(2n-3)\dots(2n-4r+3)}\int_{-1}^{1} H_n^m H_{n-2r}^{m-2} d\mu \dots (5).$$

But $\displaystyle\int_{-1}^{1} H_n^m H_{n_1}^{m-2} d\mu = 4(m-1)\frac{(n-m)!\,(n_1+m-2)!}{1.3.5\dots(2n-1)\,1.3.5\dots(2n_1-1)}$,

n_1 being less than n, hence the value of the above definite integral $\displaystyle\int_{-1}^{1}\frac{(H_n^m)^2}{1-\mu^2}d\mu$ may be found by substituting $\overline{n-2}$, $\overline{n-4}$, &c. successively for n_1 in this equation.

8. We will now find a formula of reduction for $\displaystyle\int_{-1}^{1}(1-\mu^2)^{m+1}(D^m P_n)^2 d\mu$.

We have seen that $\displaystyle\int_{-1}^{1}(1-\mu^2)^m (D^m P_n)^2 d\mu = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$,

and
$$P_2 = \frac{3}{2}\left(\mu^2 - \frac{1}{3}\right),$$

also from a previous paper (see Vol. I. p. 488) we have

$$P_n P_2 = \frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)}P_n + \frac{3}{2}\frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2};$$

hence
$$\left(\mu^2 - \frac{1}{3}\right)P_n = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \frac{2}{3}\frac{n(n+1)}{(2n-1)(2n+3)}P_n$$
$$+ \frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2},$$

or
$$(\mu^2 - 1)P_n = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} - 2\frac{n^2+n-1}{(2n-1)(2n+3)}P_n$$
$$+ \frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2}.$$

Differentiating m times, we get

$$(\mu^2-1)D^m P_n + 2\mu m D^{m-1}P_n + m(m-1)D^{m-2}P_n$$

$$= \frac{(n+1)(n+2)}{(2n+1)(2n+3)} D^m P_{n+2} - 2\frac{n^2+n-1}{(2n-1)(2n+3)} D^m P_n + \frac{n(n-1)}{(2n-1)(2n+1)} D^m P_{n-2}.$$

But $\mu P_n = \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1}$

$$= \frac{n+1}{2n+1}\left\{\frac{1}{2n+3} DP_{n+2} - \frac{1}{2n+3} DP_n\right\} + \frac{n}{2n+1}\left\{\frac{1}{2n-1} DP_n - \frac{1}{2n-1} DP_{n-2}\right\}$$

$$= \frac{n+1}{(2n+1)(2n+3)} DP_{n+2} + \frac{1}{(2n-1)(2n+3)} DP_n - \frac{n}{(2n+1)(2n-1)} DP_{n-2}.$$

Differentiating $m-1$ times, we get

$$\mu D^{m-1}P_n + (m-1)D^{m-2}P_n = \frac{n+1}{(2n+1)(2n+3)} D^m P_{n+2} + \frac{1}{(2n-1)(2n+3)} D^m P_n$$
$$- \frac{n}{(2n+1)(2n-1)} D^m P_{n-2}.$$

Also $$P_n = \frac{1}{2n+1} DP_{n+1} - \frac{1}{(2n+1)} DP_{n-1}.$$

Differentiating we get

$$DP_n = \frac{1}{2n+1} D^2 P_{n+1} - \frac{1}{(2n+1)} D^2 P_{n-1};$$

$$\therefore\ P_n = \frac{1}{2n+1}\left\{\frac{1}{2n+3} D^2 P_{n+2} - \frac{1}{2n+3} D^2 P_n\right\}$$
$$- \frac{1}{2n+1}\left\{\frac{1}{2n-1} D^2 P_n - \frac{1}{2n-1} D^2 P_{n-2}\right\}$$

$$= \frac{1}{(2n+1)(2n+3)} D^2 P_{n+2} - \frac{2}{(2n-1)(2n+3)} D^2 P_n + \frac{1}{(2n-1)(2n+1)} D^2 P_{n-2}.$$

Differentiate this equation $m-2$ times, then multiply the result by $m(m-1)$ and subtract from $2m[\mu D^{m-1}P_n + (m-1)D^{m-2}P_n]$ and we get

$$2m\mu D^{m-1}P_n + m(m-1)D^{m-2}P_n$$

$$= \frac{2m(n+1) - m(m-1)}{(2n+1)(2n+3)} D^m P_{n+2}$$
$$+ \frac{2m + 2m(m-1)}{(2n-1)(2n+3)} D^m P_n - \frac{2mn + m(m-1)}{(2n-1)(2n+1)} D^m P_{n-2}$$

$$= \frac{m(2n+3) - m^2}{(2n+1)(2n+3)} D^m P_{n+2} + \frac{2m^2}{(2n-1)(2n+3)} D^m P_n - \frac{m(2n-1)+m^2}{(2n-1)(2n+1)} D^m P_{n-2};$$

$$\therefore\ (\mu^2-1)\,D^m P_n = \frac{(n+1-m)\,(n+2-m)}{(2n+1)\,(2n+3)}\,D^m P_{n+2} - 2\,\frac{n^2+n-1+m^2}{(2n-1)\,(2n+3)}\,D^m P_n + \frac{(n+m)\,(n-1+m)}{(2n-1)\,(2n+1)}\,D^m P_{n-2}.$$

Now multiply by $(1-\mu^2)^m D^m P_n$ and integrate between limits, then

$$\int_{-1}^{1}(1-\mu^2)^{m+1}(D^m P_n)^2\,d\mu = 2\,\frac{n^2+n+m^2-1}{(2n-1)\,(2n+3)}\int_{-1}^{1}(1-\mu^2)^m (D^m P_n)^2\,d\mu$$

$$= 4\,\frac{n^2+n+m^2-1}{(2n-1)\,(2n+3)\,(2n+1)}\,\frac{(n+m)\,!}{(n-m)\,!}\ \ldots\ldots(6).$$

9. From the above equation we derive the following definite integrals:

$$\int_{-1}^{1}(1-\mu^2)\,(H_n^m)^2\,d\mu = 4\,\frac{n^2+n+m^2-1}{(2n-1)\,(2n+1)\,(2n+3)}\,\frac{(n-m)\,!\,(n+m)\,!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2},$$

hence
$$\int_{-1}^{1}\mu^2\,(H_n^m)^2\,d\mu = \int_{-1}^{1}(H_n^m)^2\,d\mu\left\{1-\frac{2n^2+2n+2m^2-2}{(2n-1)\,(2n+3)}\right\}$$

$$= \int_{-1}^{1}(H_n^m)^2\,d\mu\,\frac{2n^2+2n-2m^2-1}{(2n-1)\,(2n+3)},$$

and
$$\int_{-1}^{1}\mu H_n^m\,\frac{dH_n^m}{d\mu}\,d\mu = \frac{1}{2}\Big[\mu\,(H_n^m)^2\Big]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}(H_n^m)^2\,d\mu$$

$$= -\frac{1}{2}\int_{-1}^{1}(H_n^m)^2\,d\mu = -\frac{(n-m)\,!\,(n+m)\,!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2\,(2n+1)}\ \ldots\ldots(7).$$

Also
$$\int\mu\,(1-\mu^2)\,H_n^m\,\frac{dH_n^m}{d\mu}\,d\mu = \frac{1}{2}\,\mu\,(1-\mu^2)\,(H_n^m)^2 - \frac{1}{2}\int(H_n^m)^2\,(1-3\mu^2)\,d\mu$$

$$= \frac{1}{2}\,\mu\,(1-\mu^2)\,(H_n^m)^2 + \int(H_n^m)^2\,d\mu - \frac{3}{2}\int(1-\mu^2)\,(H_n^m)^2\,d\mu,$$

hence
$$\int_{-1}^{1}\mu\,(1-\mu^2)\,H_n^m\,\frac{dH_n^m}{d\mu}\,d\mu$$

$$= 2\,\frac{(n-m)\,!\,(n+m)\,!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2}\left\{\frac{1}{2n+1}-\frac{3\,(n^2+n+m^2-1)}{(2n-1)\,(2n+1)\,(2n+3)}\right\}$$

$$= 2\,\frac{(n-m)\,!\,(n+m)\,!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2}\,\frac{n^2+n-3m^2}{(2n-1)\,(2n+1)\,(2n+3)}\ \ldots\ldots\ldots(8).$$

10. We have seen above that

$$(\mu^2-1)\, D^m P_n = \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} D^m P_{n+2}$$
$$-2\frac{n^2+n+m^2-1}{(2n-1)(2n+3)} D^m P_n + \frac{(n+m)(n+m-1)}{(2n-1)(2n+1)} D^m P_{n-2}.$$

Multiply by $(1-\mu^2)^m D^m P_{n_1}$ and integrate between limits, supposing that n_1 and n are both odd or both even. Then

$$\int_{-1}^{1} (1-\mu^2)\, Q_n^m Q_{n_1}^m d\mu = \int_{-1}^{1} (1-\mu^2)^{m+1} D^m P_n \,.\, D^m P_{n_1} d\mu,$$

which vanishes except when $n_1 = n$, or when $n_1 = n-2$, or when $n_1 = n+2$.

When $n_1 = n-2$

$$\int_{-1}^{1} (1-\mu^2)\, Q_n^m Q_{n_1}^m d\mu = -\frac{(n+m)(n+m-1)}{(2n-1)(2n+1)} \int_{-1}^{1} (Q_{n-2}^m)^2 d\mu$$

$$= -\frac{(n+m)(n+m-1)}{(2n-1)(2n+1)} \frac{2}{2n-3} \frac{(n+m-2)!}{(n-m-2)!} = -\frac{2(n+m)!}{(2n-3)(2n-1)(2n+1)(n_1-m)!},$$

and $$\int_{-1}^{1} (1-\mu^2)\, H_n^m H_{n-2}^m d\mu = -\frac{2}{2n+1} \frac{(n+m)!\,(n-m)!}{1\,.\,3\,.\,5 \ldots (2n-1)\; 1\,.\,3\,.\,5 \ldots (2n-1)}.$$

When $n_1 = n+2$ we get

$$\int_{-1}^{1} (1-\mu^2)\, Q_n^m Q_{n_1}^m d\mu = -\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} \int_{-1}^{1} (Q_{n+2}^m)^2 d\mu$$

$$= -\frac{2(n+m+2)!}{(2n+1)(2n+3)(2n+5)(n-m)!} = -\frac{2(n_1+m)!}{(2n+1)(2n+3)(2n+5)(n-m)!},$$

and $$\int_{-1}^{1} (1-\mu^2)\, H_{n+2}^m H_n^m d\mu = -\frac{2}{(2n+5)} \frac{(n+m+2)!\,(n-m+2)!}{1\,.\,3\,.\,5 \ldots (2n+3)\; 1\,.\,3\,.\,5 \ldots (2n+3)}.$$

Hence, when $n_1 = n-2$,

$$\int_{-1}^{1} \mu^2 H_n^m H_{n_1}^m d\mu = \frac{2}{2n+1} \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2} \ldots\ldots\ldots\ldots\ldots (9),$$

and when $n_1 = n+2$

$$\int_{-1}^{1} \mu^2 H_{n_1}^m H_n^m d\mu = \frac{2}{2n_1+1} \frac{(n_1+m)!\,(n_1-m)!}{\{1\,.\,3\,.\,5 \ldots (2n_1-1)\}^2} \ldots\ldots\ldots\ldots (10).$$

In all other cases this integral vanishes.

Hence $\int_{-1}^{1}(1-\mu^2)(Q_n^m)^2\,d\mu=\dfrac{2(n^2+n+m^2-1)}{(2n-1)(2n+3)}\int_{-1}^{1}(Q_n^m)^2\,d\mu$

$$=\frac{4(n^2+n+m^2-1)}{(2n-1)(2n+1)(2n+3)}\frac{(n+m)!}{(n-m)!},$$

and $$\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)(Q_n^m)^2\,d\mu=-\frac{4}{3}\frac{n^2+n-3m^2}{(2n-1)(2n+1)(2n+3)}\frac{(n+m)!}{(n-m)!};$$

also $$\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)Q_n^m Q_{n-2}^m\,d\mu=-\frac{2}{(2n-3)(2n-1)(2n+1)}\frac{(n+m)!}{(n-m-2)!}.$$

Similarly $$\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)Q_{n+2}^m Q_n^m\,d\mu=-\frac{2}{(2n+1)(2n+3)(2n+5)}\frac{(n+m+2)!}{(n-m)!};$$

in all other cases $\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)Q_n^m Q_{n_1}^m\,d\mu$ or $\int_{-1}^{1}\mu^2 Q_n^m Q_{n_1}^m\,d\mu$ vanishes.

The value of $\int_{-1}^{1}\mu^2 H_n^m H_{n_1}^m\,d\mu$ may be derived from the expression for $\int_{-1}^{1}\mu^2 Q_n^m Q_{n_1}^m\,d\mu$ by multiplying by the ratio

$$\frac{H_n^m H_{n_1}^m}{Q_n^m Q_{n_1}^m}=\frac{(n-m)!\,(n_1-m)!}{1\,.\,3\,.\,5\ldots(2n-1)\;1\,.\,3\,.\,5\ldots(2n_1-1)};$$

hence $$\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)(H_n^m)^2\,d\mu=-\frac{4}{3}\frac{n^2+n-3m^2}{(2n-1)(2n+1)(2n+3)}\frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2},$$

$$\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)H_n^m H_{n-2}^m\,d\mu=-\frac{2}{2n+1}\frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2},$$

and $$\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)H_n^m H_{n+2}^m\,d\mu=-\frac{2}{2n+5}\frac{(n+m+2)!\,(n-m+2)!}{\{1\,.\,3\,.\,5\ldots(2n+3)\}^2}.$$

11. From Art. 1 above (p. 402) we have

$$X_n^m=\frac{1}{r^{n+2}}\left[\frac{1}{2}(n-m)H_n^{m+1}-\frac{1}{2}(n+m)H_n^{m-1}\right],$$

$$Y_n^m=\frac{1}{r^{n+2}}mH_n^m(1-\mu^2)^{-\frac{1}{2}},$$

$$Z_n^m=\frac{n+1}{r^{n+2}}H_n^m.$$

Hence on a sphere of radius 1 we have

$$X_n^m = \left[\frac{1}{2}(n-m) H_n^{m+1} - \frac{1}{2}(n+m) H_n^{m-1}\right].$$

Also, as above,

$$X_n^m = m\mu(1-\mu^2)^{-\frac{1}{2}} H_n^m - (n+m) H_n^{m-1},$$

$$Y_n^m = mH_n^m (1-\mu^2)^{-\frac{1}{2}}, \text{ and } Z_n^m = (n+1) H_n^m.$$

Hence
$$\mu Y_n^m - X_n^m = (n+m) H_n^{m-1},$$

$$\mu Y_n^m + X_n^m = (n-m) H_n^{m+1},$$

and
$$(1-\mu^2)^{\frac{1}{2}} Y_n^m = mH_n^m.$$

Also we have

$$X_n^m = (1-\mu^2)^{\frac{1}{2}} \frac{dH_n^m}{d\mu}.$$

From these formulae we find

$$(X_n^m)^2 + (Y_n^m)^2 = (1-\mu^2)\left(\frac{dH_n^m}{d\mu}\right)^2 + \frac{m^2}{1-\mu^2}(H_n^m)^2,$$

and
$$\int_{-1}^{1}(1-\mu^2)\left(\frac{dH_n^m}{d\mu}\right)^2 d\mu = \int_{-1}^{1}(X_n^m)^2 d\mu.$$

We have also

$$(X_n^m)^2 + (Y_n^m)^2 = \frac{1}{2}(n+m)^2 (H_n^{m-1})^2 + \frac{1}{2}(n-m)^2 (H_n^{m+1})^2 + m^2 (H_n^m)^2;$$

$$\therefore \int_{-1}^{1}(X_n^m)^2 d\mu + \int_{-1}^{1}(Y_n^m)^2 d\mu = \frac{1}{2}(n+m)^2 \int_{-1}^{1}(H_n^{m-1})^2 d\mu$$
$$+\frac{1}{2}(n-m)^2 \int_{-1}^{1}(H_n^{m+1})^2 d\mu + m^2 \int_{-1}^{1}(H_n^m)^2 d\mu$$

$$= 2\frac{(n-m)!\,(n+m)!}{\{1 \,.\, 3 \,.\, 5 \ldots (2n-1)\}^2 (2n+1)}\left[\frac{1}{2}(n-m+1)(n+m)\right.$$
$$\left. + \frac{1}{2}(n-m)(n+m+1) + m^2\right]$$

$$= 2\frac{(n-m)!\,(n+m)!}{\{1 \,.\, 3 \,.\, 5 \ldots (2n-1)\}^2}\frac{n(n+1)}{2n+1} = n(n+1)\int_{-1}^{1}(H_n^m)^2 d\mu \quad \ldots\ldots\ldots\ldots(11).$$

And since $Z_n^m = (n+1) H_n^m$, we have

$$\int_{-1}^{1} (X_n^m)^2 d\mu + \int_{-1}^{1} (Y_n^m)^2 d\mu + \int_{-1}^{1} (Z_n^m)^2 d\mu = 2 \frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2} (n+1).$$

12. Also we have

$$\mu X_n^m Y_n^m = \frac{1}{4}(n-m)^2 (H_n^{m+1})^2 - \frac{1}{4}(n+m)^2 (H_n^{m-1})^2;$$

$$\therefore \int_{-1}^{1} \mu X_n^m Y_n^m d\mu = \frac{1}{4}(n-m)^2 \int_{-1}^{1} (H_n^{m+1})^2 d\mu - \frac{1}{4}(n+m)^2 \int_{-1}^{1} (H_n^{m-1})^2 d\mu$$

$$= \frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2 (2n+1)} \left[\frac{1}{2}(n-m)(n+m+1) - \frac{1}{2}(n-m+1)(n+m)\right]$$

$$= -\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2} \frac{m}{2n+1}.$$

Also since

$$\mu Y_{n_1}^m - X_{n_1}^m = (n_1 + m) H_{n_1}^{m-1},$$

$$\mu Y_{n_1}^m + X_{n_1}^m = (n_1 - m) H_{n_1}^{m+1},$$

and

$$(1-\mu^2)^{\frac{1}{2}} Y_{n_1}^m = m H_{n_1}^m,$$

we have

$$\frac{1}{2}(\mu Y_n^m - X_n^m)(\mu Y_{n_1}^m - X_{n_1}^m) + \frac{1}{2}(\mu Y_n^m + X_n^m)(\mu Y_{n_1}^m + X_{n_1}^m) + (1-\mu^2) Y_n^m Y_{n_1}^m$$

$$= X_n^m X_{n_1}^m + Y_n^m Y_{n_1}^m$$

$$= \frac{1}{2}(n+m)(n_1+m) H_n^{m-1} H_{n_1}^{m-1} + \frac{1}{2}(n-m)(n_1-m) H_n^{m+1} H_{n_1}^{m+1} + m^2 H_n^m H_{n_1}^m;$$

hence

$$\int_{-1}^{1} X_n^m X_{n_1}^m d\mu + \int_{-1}^{1} Y_n^m Y_{n_1}^m d\mu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

since all the integrals on the right-hand side of the equation vanish.

Also we have $\int_{-1}^{1} Z_n^m Z_{n_1}^m d\mu = 0$, since $\int_{-1}^{1} H_n^m H_{n_1}^m d\mu = 0$.

13. Also

$$\frac{1}{2}\int_{-1}^{1} (\mu Y_n^m + X_n^m)(\mu Y_{n_1}^m + X_{n_1}^m) d\mu - \frac{1}{2}\int_{-1}^{1} (\mu Y_n^m - X_n^m)(\mu Y_{n_1}^m - X_{n_1}^m) d\mu$$

$$= \frac{1}{2}(n-m)(n_1-m)\int_{-1}^{1} H_n^{m+1} H_{n_1}^{m+1} d\mu - \frac{1}{2}(n+m)(n_1+m)\int_{-1}^{1} H_n^{m-1} H_{n_1}^{m-1} d\mu,$$

or

$$\int_{-1}^{1} \mu (X_n^m Y_{n_1}^m + X_{n_1}^m Y_n^m) d\mu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Also we have

$$\int_{-1}^{1}\{\mu^2(Y_n^m)^2-(X_n^m)^2\}\,d\mu=(n+m)(n-m)\int_{-1}^{1}H_n^{m-1}H_n^{m+1}\,d\mu$$

$$=-(n+m)(n-m)\int_{-1}^{1}(H_n^{m-1})^2\,d\mu$$

$$=-2\frac{(n-m+1)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2(2n+1)}(n-m),$$

and

$$\int_{-1}^{1}(1-\mu^2)(Y_n^m)^2\,d\mu=m^2\int_{-1}^{1}(H_n^m)^2\,d\mu=2\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2(2n+1)}m^2.$$

Adding we get

$$\int_{-1}^{1}\{(Y_n^m)^2-(X_n^m)^2\}\,d\mu=2\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2(2n+1)}[m^2-(n-m+1)(n-m)]$$

$$=-2\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2(2n+1)}[n(n+1)-(2n+1)m]$$

$$=-2\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}\left[\frac{n(n+1)}{2n+1}-m\right].$$

Combining this with

$$\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu=2\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}\frac{n(n+1)}{2n+1},$$

we have

$$\int_{-1}^{1}(X_n^m)^2\,d\mu=2\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}\left[\frac{n(n+1)}{2n+1}-\frac{1}{2}m\right],$$

and

$$\int_{-1}^{1}(Y_n^m)^2\,d\mu=\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}m.$$

Also $\displaystyle\int_{-1}^{1}(\mu Y_n^m+X_n^m)(\mu Y_{n_1}^m-X_{n_1}^m)\,d\mu+\int_{-1}^{1}(\mu Y_n^m-X_n^m)(\mu Y_{n_1}^m+X_{n_1}^m)\,d\mu$

$$=(n-m)(n_1+m)\int_{-1}^{1}H_n^{m+1}H_{n_1}^{m-1}\,d\mu+(n+m)(n_1-m)\int_{-1}^{1}H_n^{m-1}H_{n_1}^{m+1}d\mu,$$

or $\displaystyle\int_{-1}^{1}\mu^2Y_n^mY_{n_1}^m\,d\mu-\int_{-1}^{1}X_n^mX_{n_1}^m\,d\mu$

$$=\frac{1}{2}(n-m)(n_1+m)\int_{-1}^{1}H_n^{m+1}H_{n_1}^{m-1}\,d\mu+\frac{1}{2}(n+m)(n_1-m)\int_{-1}^{1}H_n^{m-1}H_{n_1}^{m+1}\,d\mu$$

$$=2m\frac{(n-m)!\,(n_1+m)!}{1\,.\,3\,.\,5\,\ldots\,(2n-1)\;1\,.\,3\,.\,5\,\ldots\,(2n_1-1)}.$$

Also
$$\int_{-1}^{1} (1-\mu^2)\, Y_n^m Y_{n_1}^m \, d\mu = 0;$$
therefore
$$\int_{-1}^{1} Y_n^m Y_{n_1}^m \, d\mu - \int_{-1}^{1} X_n^m X_{n_1}^m \, d\mu = 2m \frac{(n-m)!\,(n_1+m)!}{1\,.\,3\,.\,5 \ldots (2n-1)\,1\,.\,3\,.\,5 \ldots (2n_1-1)}.$$

14. We have seen above that
$$\int_{-1}^{1} (H_n^m)^2 \, d\mu = \frac{2}{2n+1} \frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2},$$
and that $(1-\mu^2)^{\frac{1}{2}}\, Y_n^m = mH_n^m$;

hence
$$\int_{-1}^{1} \frac{(H_n^m)^2}{1-\mu^2} \, d\mu = \frac{1}{m^2} \int_{-1}^{1} (Y_n^m)^2 \, d\mu = \frac{1}{m} \frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2}.$$

Also since
$$\frac{\mu^2 (H_n^m)^2}{1-\mu^2} = \frac{(H_n^m)^2}{1-\mu^2} - (H_n^m)^2,$$
it follows that
$$\int_{-1}^{1} \frac{\mu^2 (H_n^m)^2}{1-\mu^2} \, d\mu = \left(\frac{1}{m} - \frac{2}{2n+1}\right) \frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2}.$$

We have also seen that, when n_1 is less than n,
$$\int_{-1}^{1} Y_n^m Y_{n_1}^m \, d\mu = m \frac{(n-m)!\,(n_1+m)!}{1\,.\,3\,.\,5 \ldots (2n-1)\,1\,.\,3\,.\,5 \ldots (2n_1-1)};$$
hence, under the same condition,
$$\int_{-1}^{1} \frac{H_n^m H_{n_1}^m}{1-\mu^2} \, d\mu = \frac{1}{m} \frac{(n-m)!\,(n_1+m)!}{1\,.\,3\,.\,5 \ldots (2n-1)\,1\,.\,3\,.\,5 \ldots (2n_1-1)} \quad \ldots\ldots(14);$$
also
$$\int_{-1}^{1} \frac{\mu^2 H_n^m H_{n_1}^m}{1-\mu^2} \, d\mu = \int_{-1}^{1} \frac{H_n^m H_{n_1}^m}{1-\mu^2} \, d\mu - \int_{-1}^{1} H_n^m H_{n_1}^m \, d\mu$$
$$= \int_{-1}^{1} \frac{H_n^m H_{n_1}^m}{1-\mu^2} \, d\mu,$$
where n and n_1 must evidently be both even or both odd, in order that the integral may have a finite value.

When in these equations we are only concerned with the same value of m, we may conveniently write H_n for H_n^m and H_{n_1} for $H_{n_1}^m$, for the sake of simplification.

15. From the differential equation for H_n we have

$$(1-\mu^2)\frac{d^2 H_n}{d\mu^2} - 2\mu\frac{dH_n}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right] H_n = 0,$$

and
$$(1-\mu^2)\frac{d^2 H_{n_1}}{d\mu^2} - 2\mu\frac{dH_{n_1}}{d\mu} + \left[n_1(n_1+1) - \frac{m^2}{1-\mu^2}\right] H_{n_1} = 0.$$

Multiply by H_{n_1} and H_n respectively and add, then

$$(1-\mu^2)\left[H_{n_1}\frac{d^2 H_n}{d\mu^2} + H_n\frac{d^2 H_{n_1}}{d\mu^2}\right] - 2\mu\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right]$$
$$+\left[n(n+1) + n_1(n_1+1) - \frac{2m^2}{1-\mu^2}\right] H_n H_{n_1} = 0.$$

But $(1-\mu^2)\dfrac{d}{d\mu}\left[H_{n_1}\dfrac{dH_n}{d\mu} + H_n\dfrac{dH_{n_1}}{d\mu}\right] - 2(1-\mu^2)\dfrac{dH_n}{d\mu}\dfrac{dH_{n_1}}{d\mu}$

$$= (1-\mu^2)\left[H_{n_1}\frac{d^2 H_n}{d\mu^2} + H_n\frac{d^2 H_{n_1}}{d\mu^2}\right],$$

therefore

$$(1-\mu^2)\frac{d}{d\mu}\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right] - 2\mu\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right] - 2(1-\mu^2)\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu}$$
$$+\left[n(n+1) + n_1(n_1+1) - \frac{2m^2}{1-\mu^2}\right] H_n H_{n_1} = 0,$$

or
$$\frac{d}{d\mu}(1-\mu^2)\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right] - 2(1-\mu^2)\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu}$$
$$+\left[n(n+1) + n_1(n_1+1) - \frac{2m^2}{1-\mu^2}\right] H_n H_{n_1} = 0.$$

Integrating from $\mu = -1$ to $\mu = 1$, we get

$$\int_{-1}^{1}(1-\mu^2)\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu}\,d\mu + \int_{-1}^{1}\frac{m^2 H_n H_{n_1}}{1-\mu^2}\,d\mu$$
$$= \left[\frac{1}{2}n(n+1) + \frac{1}{2}n_1(n_1+1)\right]\int_{-1}^{1} H_n H_{n_1}\,d\mu.$$

When $n = n_1$, this gives the value of

$$\int_{-1}^{1}(X_n)^2\,d\mu + \int_{-1}^{1}(Y_n)^2\,d\mu = n(n+1)\int_{-1}^{1}(H_n)^2\,d\mu,$$

as we have seen elsewhere.

In all other cases

$$\int_{-1}^{1} X_n X_{n_1}\, d\mu + \int_{-1}^{1} Y_n Y_{n_1}\, d\mu = 0,$$

and $$\int_{-1}^{1} Y_n Y_{n_1}\, d\mu = -\int_{-1}^{1} X_n X_{n_1}\, d\mu = \frac{m\,(n-m)!\,(n_1+m)!}{1\,.\,3\,.\,5\,\ldots\,(2n-1)\,1\,.\,3\,.\,5\,\ldots\,(2n_1-1)} \ldots (15).$$

16. From the differential equation for H_n (taking H_n for H_n^m) we have

$$(1-\mu^2)\frac{d^2 H_n}{d\mu^2} - 2\mu\frac{dH_n}{d\mu} + \left[n\,(n+1) - \frac{m^2}{1-\mu^2}\right] H_n = 0,$$

and $$(1-\mu^2)\frac{d^2 H_{n_1}}{d\mu^2} - 2\mu\frac{dH_{n_1}}{d\mu} + \left[n_1\,(n_1+1) - \frac{m^2}{1-\mu^2}\right] H_{n_1} = 0.$$

Multiply by $(1-\mu^2)\,H_{n_1}$ and $(1-\mu^2)\,H_n$ respectively and add, then

$$(1-\mu^2)^2\left[H_{n_1}\frac{d^2 H_n}{d\mu^2} + H_n\frac{d^2 H_{n_1}}{d\mu^2}\right] - 2\mu\,(1-\mu^2)\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right]$$
$$+\left[n\,(n+1) + n_1\,(n_1+1)\right](1-\mu^2)\,H_n H_{n_1} - 2m^2 H_n H_{n_1} = 0.$$

But $$(1-\mu^2)^2\frac{d}{d\mu}\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right] - 2\,(1-\mu^2)^2\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu}$$
$$= (1-\mu^2)^2\left[H_{n_1}\frac{d^2 H}{d\mu^2} + H_n\frac{d^2 H_{n_1}}{d\mu^2}\right];$$

therefore

$$(1-\mu^2)^2\frac{d}{d\mu}\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right] - 2\mu\,(1-\mu^2)\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right]$$
$$-2\,(1-\mu^2)^2\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu} + \left[n\,(n+1) + n_1\,(n_1+1)\right](1-\mu^2)\,H_n H_{n_1} - 2m^2 H_n H_{n_1} = 0.$$

Or $$\frac{d}{d\mu}\left[(1-\mu^2)^2\left(H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right)\right] + 2\mu\,(1-\mu^2)\left[H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right]$$
$$-2\,(1-\mu^2)^2\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu} + \left[n\,(n+1) + n_1\,(n_1+1)\right](1-\mu^2)\,H_n H_{n_1} - 2m^2 H_n H_{n_1} = 0.$$

Or again

$$\frac{d}{d\mu}\left[(1-\mu^2)^2\left(H_{n_1}\frac{dH_n}{d\mu} + H_n\frac{dH_{n_1}}{d\mu}\right)\right] + \frac{d}{d\mu}\left[2\mu\,(1-\mu^2)\,H_n H_{n_1}\right]$$
$$-2\,(1-\mu^2)\,H_n H_{n_1} + 4\mu^2 H_n H_{n_1} - 2\,(1-\mu^2)^2\frac{dH_n}{d\mu}\frac{dH_{n_1}}{d\mu}$$
$$+\left[n\,(n+1) + n_1\,(n_1+1)\right](1-\mu^2)\,H_n H_{n_1} - 2m^2 H_n H_{n_1} = 0.$$

Now integrate from $\mu = -1$ to $\mu = 1$ and we get

$$\int_{-1}^{1} (1-\mu^2)^2 \frac{dH_n}{d\mu} \frac{dH_{n_1}}{d\mu} d\mu + m^2 \int_{-1}^{1} H_n H_{n_1} d\mu$$

$$= \left[\frac{1}{2} n(n+1) + \frac{1}{2} n_1(n_1+1) - 3\right] \int_{-1}^{1} (1-\mu^2) H_n H_{n_1} d\mu + 2\int_{-1}^{1} H_n H_{n_1} d\mu.$$

Multiplying the equation obtained in Art. 15 by $\frac{2}{3}$ and subtracting from this, we get

$$\int_{-1}^{1} \left(\frac{1}{3}-\mu^2\right)(1-\mu^2) \frac{dH_n}{d\mu} \frac{dH_{n_1}}{d\mu} d\mu + m^2 \int_{-1}^{1} \left(\frac{1}{3}-\mu^2\right) \frac{H_n H_{n_1}}{1-\mu^2} d\mu$$

$$= \int_{-1}^{1} \left(\frac{1}{3}-\mu^2\right) (X_n X_{n_1} + Y_n Y_{n_1}) d\mu$$

$$= \left[\frac{1}{2} n(n+1) + \frac{1}{2} n_1(n_1+1) - 3\right] \int_{-1}^{1} \left(\frac{1}{3}-\mu^2\right) H_n H_{n_1} d\mu \dots\dots\dots(16).$$

17. We have seen above (p. 420) that

$$(X_n^m)^2 + (Y_n^m)^2 = \frac{1}{2}(n+m)^2 (H_n^{m-1})^2 + \frac{1}{2}(n-m)^2 (H_n^{m+1})^2 + m^2 (H_n^m)^2;$$

hence $(1-\mu^2)(X_n^m)^2 + (1-\mu^2)(Y_n^m)^2$

$$= (1-\mu^2)^2 \left(\frac{dH_n^m}{d\mu}\right)^2 + m^2 (H_n^m)^2$$

$$= (1-\mu^2) \left\{\frac{1}{2}(n+m)^2 (H_n^{m-1})^2 + \frac{1}{2}(n-m)^2 (H_n^{m+1})^2 + m^2 (H_n^m)^2\right\}.$$

Multiplying by $d\mu$ and integrating from -1 to $+1$, we have

$$\int_{-1}^{1} (1-\mu^2)^2 \left(\frac{dH_n^m}{d\mu}\right)^2 d\mu + m^2 \int_{-1}^{1} (H_n^m)^2 d\mu = \frac{1}{2}(n+m)^2 \int_{-1}^{1} (1-\mu^2)(H_n^{m-1})^2 d\mu$$

$$+ \frac{1}{2}(n-m)^2 \int_{-1}^{1} (1-\mu^2)(H_n^{m+1})^2 d\mu + m^2 \int_{-1}^{1} (1-\mu^2)(H_n^m)^2 d\mu$$

$$= \int_{-1}^{1} (1-\mu^2)(H_n^m)^2 d\mu \left\{\frac{n^2+n+m^2-2m}{n^2+n+m^2-1} \times \frac{(n-m+1)(n+m)}{2}\right.$$

$$\left. + \frac{n^2+n+m^2+2m}{n^2+n+m^2-1} \times \frac{(n-m)(n+m+1)}{2} + m^2\right\}$$

$$= 2\frac{(n-m)!\,(n+m)!}{(2n-1)(2n+1)(2n+3)\{1\,.\,3\,.\,5 \dots (2n-1)\}^2}$$

$$\times \{[n(n+1) + m(m-2)](n^2+n-m^2+m) + (n^2+n+m^2+2m)(n^2+n-m^2-m)$$

$$+ 2m^2(n^2+n+m^2-1)\}.$$

The factor in large brackets in this expression

$$= 2n^2(n+1)^2 + n(n+1)[m(m-2) - m(m-1) + m(m+2) - m(m+1) + 2m^2]$$
$$+ m^2[2(m^2-1) - (m-2)(m-1) - (m+2)(m+1)]$$
$$= 2n^2(n+1)^2 + 2n(n+1)m^2 - 6m^2$$
$$= 2[n^2(n+1)^2 + (n\overline{n+1} - 3)m^2].$$

Hence

$$\int_{-1}^{1}(1-\mu^2)^2\left(\frac{dH_n^m}{d\mu}\right)^2 d\mu = \int_{-1}^{1}(H_n^m)^2 d\mu\left[\frac{2n^2(n+1)^2 + 2m^2(n\overline{n+1} - 3)}{(2n-1)(2n+3)} - m^2\right]$$
$$= \int_{-1}^{1}(H_n^m)^2 d\mu\left[\frac{2n^2(n+1)^2 - (2n\overline{n+1} + 3)m^2}{(2n-1)(2n+3)}\right].$$

Also $\displaystyle\int_{-1}^{1}(1-\mu^2)^2\left(\frac{dH_n^m}{d\mu}\right)^2 d\mu + m^2\int_{-1}^{1}(H_n^m)^2 d\mu + (n+1)^2\int_{-1}^{1}(1-\mu^2)(H_n^m)^2 d\mu$

$$= \int_{-1}^{1}(H_n^m)^2 d\mu\,\frac{2}{(2n-1)(2n+3)}[n^2(n+1)^2 + m^2(n\overline{n+1} - 3)$$
$$+ (n+1)^2(n\overline{n+1} + m^2 - 1)]$$
$$= \int_{-1}^{1}(H_n^m)^2 d\mu\,\frac{2}{(2n-1)(2n+3)}[(n+1)^2(n\overline{2n+1} - 1) + m^2(\overline{n+1}\,\overline{2n+1} - 3)]$$
$$= \int_{-1}^{1}(H_n^m)^2 d\mu\,\frac{2}{2n+3}[(n+1)^3 + m^2(n+2)] \quad\ldots\ldots\ldots\ldots(17).$$

We also readily find that

$$\int_{-1}^{1}(1-\mu^2)^2\left(\frac{dH_n^m}{d\mu}\right)^2 d\mu + m^2\int_{-1}^{1}(H_n^m)^2 d\mu + n^2\int_{-1}^{1}(1-\mu^2)(H_n^m)^2 d\mu$$
$$= \int_{-1}^{1}(H_n^m)^2 d\mu\,\frac{2}{(2n-1)(2n+3)}[n^2(\overline{n+1}\,\overline{2n+1} - 1) + m^2(n\overline{2n+1} - 3)]$$
$$= \int_{-1}^{1}(H_n^m)^2 d\mu\,\frac{2}{2n-1}[n^3 + m^2(n-1)] \quad\ldots\ldots\ldots\ldots(18).$$

18. We have seen above (p. 413) that, when n_1 is less than n,

$$\int_{-1}^{1}H_n^{m+1}H_{n_1}^{m-1}d\mu = 4m\frac{(n-m-1)!\,(n_1+m-1)!}{1.3.5\ldots(2n-1)\,1.3.5\ldots(2n_1-1)},$$

and
$$\int_{-1}^{1}H_{n_1}^{m+1}H_n^{m-1}d\mu = 0.$$

Also from equations on p. 421 we have

$$\frac{1}{2}\int_{-1}^{1}(\mu Y_n^m + X_n^m)(\mu Y_{n_1}^m - X_{n_1}^m)\,d\mu - \frac{1}{2}\int_{-1}^{1}(\mu Y_n^m - X_n^m)(\mu Y_{n_1}^m + X_{n_1}^m)\,d\mu$$

$$= \int_{-1}^{1}\mu\,(X_n^m Y_{n_1}^m - X_{n_1}^m Y_n^m)\,d\mu = \frac{1}{2}(n-m)(n_1+m)\int_{-1}^{1} H_n^{m+1} H_{n_1}^{m-1}\,d\mu$$

$$-\frac{1}{2}(n+m)(n_1-m)\int_{-1}^{1} H_n^{m-1} H_{n_1}^{m+1}\,d\mu\,;$$

and
$$\int_{-1}^{1}\mu\,(X_n^m Y_{n_1}^m + X_{n_1}^m Y_n^m)\,d\mu = 0\,;$$

hence
$$\int_{-1}^{1}\mu X_n^m Y_{n_1}^m\,d\mu = -\int_{-1}^{1}\mu X_{n_1}^m Y_n^m\,d\mu$$

$$= \frac{1}{4}(n-m)(n_1+m)\int_{-1}^{1} H_n^{m+1} H_{n_1}^{m-1}\,d\mu$$

$$= \frac{m\,(n-m)!\,(n_1+m)!}{1\,.\,3\,.\,5\,\ldots\,(2n-1)\;1\,.\,3\,.\,5\,\ldots\,(2n_1-1)} \quad\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

Hence it appears from equations (15) and (19) that

$$\int_{-1}^{1}\mu^2 Y_n Y_{n_1} d\mu = \int_{-1}^{1} Y_n Y_{n_1} d\mu = -\int_{-1}^{1} X_n X_{n_1} d\mu = \int_{-1}^{1}\mu X_n Y_{n_1} d\mu = -\int_{-1}^{1}\mu X_{n_1} Y_n d\mu.$$

Also from the above equations we have

$$X_n^m = m\frac{\mu}{(1-\mu^2)^{\frac{1}{2}}} H_n^m - (n+m) H_n^{m-1} = \frac{1}{2}(n-m) H_n^{m+1} - \frac{1}{2}(n+m) H_n^{m-1}$$

$$= -nH_n^{m-1} + m\left\{\frac{n-m}{2n-1} H_{n-2}^{m-1} + \frac{(n-m)(n-m-1)(n-m-2)}{(2n-1)(2n-3)(2n-5)} H_{n-4}^{m-1} + \text{\&c.}\right.$$

$$\left. + \frac{(n-m)(n-m-1)\ldots(n-m-2r+2)}{(2n-1)(2n-3)\ldots(2n-4r+3)} H_{n-2r}^{m-1} + \text{\&c.}\right\}.$$

And
$$Y_n^m = m\frac{H_n^m}{(1-\mu^2)^{\frac{1}{2}}} = m\left\{H_{n-1}^{m-1} + \frac{(n-m)(n-m-1)}{(2n-1)(2n-3)} H_{n-3}^{m-1} + \text{\&c.}\right.$$

$$\left. + \frac{(n-m)(n-m-1)\ldots(n-m-2r+3)}{(2n-1)(2n-3)\ldots(2n-4r+5)} H_{n-2r+1}^{m-1} + \text{\&c.}\right\}.$$

Also
$$Z_n^m = (n+1)(1-\mu^2)^{\frac{1}{2}}\left\{H_{n-1}^{m-1} + \frac{(n-m)(n-m-1)}{(2n-1)(2n-3)} H_{n-3}^{m-1} + \text{\&c.}\right.$$

$$\left. + \frac{(n-m)(n-m-1)\ldots(n-m-2r+3)}{(2n-1)(2n-3)\ldots(2n-4r+5)} H_{n-2r+1}^{m-1} + \text{\&c.}\right\}.$$

19. To find the expressions for X_n^m, Y_n^m and Z_n^m in terms of powers of μ.

$$P_n = \frac{1}{2^n\, n!} D^n (\mu^2 - 1)^n = \frac{1}{2^n\, n!} D^n \left\{\mu^{2n} - n\mu^{2n-2} + \&\text{c.} + (-1)^r \frac{n!}{r!\,(n-r)!}\mu^{2n-2r} + \&\text{c.}\right\}$$

$$= \frac{1}{2^n\, n!}\left\{\frac{2n!}{n!}\mu^n - n\frac{(2n-2)!}{(n-2)!}\mu^{n-2} + \&\text{c.} + (-1)^r \frac{n!}{r!\,(n-r)!}\frac{(2n-2r)!}{(n-2r)!}\mu^{n-2r} + \&\text{c.}\right\},$$

so that the general term of P_n is

$$\frac{1}{2^n}(-1)^r \frac{(2n-2r)!}{r!\,(n-r)!\,(n-2r)!}\mu^{n-2r}.$$

If this be divided by the coefficient of the first term in P_n so as to reduce the coefficient of μ^n to unity, we have the general term in

$$H_n^0 \text{ or } G_n^0 = \frac{(n!)^2}{2n!}(-1)^r \frac{(2n-2r)!}{r!\,(n-r)!\,(n-2r)!}\mu^{n-2r}.$$

Similarly the general term in $D^m P_n$ is

$$= \frac{1}{2^n}(-1)^r \frac{(n-2r)!}{(n-m-2r)!}\frac{(2n-2r)!}{r!\,(n-r)!\,(n-2r)!}\mu^{n-m-2r}$$

$$= \frac{1}{2^n}(-1)^r \frac{(2n-2r)!}{(n-m-2r)!\,r!\,(n-r)!}\mu^{n-m-2r},$$

and the first term in the same quantity will be

$$\frac{1}{2^n}\frac{n!}{(n-m)!}\frac{2n!}{n!\,n!}\mu^{n-m} = \frac{1}{2^n}\frac{2n!}{(n-m)!\,n!}\mu^{n-m};$$

therefore dividing by this so as to reduce the coefficient of μ^{n-m} to unity, we have the general term in H_n^m

$$= (1-\mu^2)^{\frac{m}{2}}(-1)^r \frac{n!\,(n-m)!}{2n!}\frac{(2n-2r)!}{(n-m-2r)!\,r!\,(n-r)!}\mu^{n-m-2r}.$$

Hence

$$\mu Y_n^m - X_n^m = (1-\mu^2)^{\frac{m-1}{2}}(n+m)\left\{\mu^{n-m+1} - \&\text{c.}\right.$$
$$\left. + (-1)^r \frac{(n-m+1)!}{(n-m-2r+1)!}\frac{n!\,(2n-2r)!}{2n!\,r!\,(n-r)!}\mu^{n-m-2r+1}\right\},$$

and

$$\mu Y_n^m + X_n^m = (1-\mu^2)^{\frac{m+1}{2}} (n-m) \left\{ \mu^{n-m-1} - \&c. \right.$$

$$\left. + (-1)^r \frac{n!\,(n-m-1)!\,(2n-2r)!}{2n!\,(n-m-2r-1)!\,r!\,(n-r)!} \mu^{n-m-2r-1} \right\};$$

hence

$$\mu Y_n^m + X_n^m = (1-\mu^2)^{\frac{m-1}{2}} (n-m)(1-\mu^2) \left\{ \mu^{n-m-1} - \&c. \right.$$

$$\left. + (-1)^r \frac{n!\,(n-m-1)!\,(2n-2r)!}{2n!\,(n-m-2r-1)!\,r!\,(n-r)!} \mu^{n-m-2r-1} + \&c. \right\}.$$

Multiplying the last two factors together, the coefficient of $\mu^{n-m-2r+1}$ in the general term of the product is

$$-(-1)^r \frac{n!\,(n-m-1)!}{2n!} \left\{ \frac{(2n-2r)!}{(n-m-2r-1)!\,r!\,(n-r)!} \right.$$

$$\left. + \frac{(2n-2r+2)!}{(n-m-2r+1)!\,(r-1)!\,(n-r+1)!} \right\}$$

$$= -(-1)^r \frac{n!\,(n-m-1)!\,(2n-2r)!}{2n!\,r!\,(n-r)!\,(n-m-2r+1)!}$$

$$\times \{(n-m-2r)(n-m-2r+1) + 2r(2n-2r+1)\};$$

$$\therefore\ X_n^m = -(1-\mu^2)^{\frac{m-1}{2}} \left\{ n\mu^{n-m+1} - \&c. + \frac{(-1)^r}{2} \frac{n!\,(n-m)!\,(2n-2r)!}{2n!\,r!\,(n-r)!\,(n-m-2r+1)!} \right.$$

$$\left. \times [(n-m-2r)(n-m-2r+1) + 2r(2n-2r+1) + (n+m)(n-m+1)]\,\mu^{n-m-2r+1} \right\}.$$

The quantity in square brackets becomes

$$(n-m+1)(2n-2r) + 2r(n+m+1) = 2 \times [n(n-m+1) + 2mr].$$

Hence

$$X_n^m = -(1-\mu^2)^{\frac{m-1}{2}} \left\{ n\mu^{n-m+1} - \&c. + (-1)^r \frac{n!\,(n-m)!\,(2n-2r)!}{2n!\,r!\,(n-r)!\,(n-m-2r+1)!} \right.$$

$$\left. \times [n(n-m+1) + 2mr]\,\mu^{n-m-2r+1} + \&c. \right\},$$

$$Y_n^m = m(1-\mu^2)^{\frac{m-1}{2}} \left\{ \mu^{n-m} - \&c. + (-1)^r \frac{n!\,(n-m)!\,(2n-2r)!}{2n!\,r!\,(n-r)!\,(n-m-2r)!} \mu^{n-m-2r} + \&c. \right\},$$

$$Z_n^m = (n+1)(1-\mu^2)^{\frac{m}{2}} \left\{ \mu^{n-m} - \&c. + (-1)^r \frac{n!\,(n-m)!\,(2n-2r)!}{2n!\,r!\,(n-r)!\,(n-m-2r)!} \mu^{n-m-2r} + \&c. \right\}.$$

20. Example of the application of these formulae.

Suppose $n=6$, $n_1=4$ and $m=2$.

Then $$H_4^2=(1-\mu^2)\left\{\mu^2-\frac{2\,.\,1}{2\,.\,7}\right\}=(1-\mu^2)\left\{\mu^2-\frac{1}{7}\right\},$$

and $$H_6^2=(1-\mu^2)\left\{\mu^4-\frac{4\,.\,3}{2\,.\,11}\mu^2+\frac{4\,.\,3\,.\,2\,.\,1}{2\,.\,4\,.\,11\,.\,9}\right\}$$

$$=(1-\mu^2)\left\{\mu^4-\frac{6}{11}\mu^2+\frac{1}{33}\right\}.$$

Hence $$X_4^2=(1-\mu^2)^{\frac{1}{2}}\frac{d}{d\mu}H_4^2=2\mu(1-\mu^2)^{\frac{3}{2}}-2\mu(1-\mu^2)^{\frac{1}{2}}\left(\mu^2-\frac{1}{7}\right)$$

$$=(1-\mu^2)^{\frac{1}{2}}\left\{-2\mu^3+\frac{2}{7}\mu+2\mu-2\mu^3\right\}$$

$$=-(1-\mu^2)^{\frac{1}{2}}\left\{4\mu^3-\frac{16}{7}\mu\right\},$$

also $$Y_4^2=2(1-\mu^2)^{\frac{1}{2}}\left\{\mu^2-\frac{1}{7}\right\},$$

and $$Z_4^2=5(1-\mu^2)\left\{\mu^2-\frac{1}{7}\right\}.$$

But by formula for X_4^2, we have

$$X_4^2=-(1-\mu^2)^{\frac{1}{2}}\left\{4\mu^3-\frac{2}{2}\cdot\frac{1}{7}[4\,.\,3+4]\mu\right\}$$

$$=-(1-\mu^2)^{\frac{1}{2}}\left\{4\mu^3-\frac{16}{7}\mu\right\},\text{ which agrees.}$$

Also $$X_6^2=(1-\mu^2)^{\frac{1}{2}}\frac{d}{d\mu}H_6^2$$

$$=(1-\mu^2)^{\frac{3}{2}}\left\{4\mu^3-\frac{12}{11}\mu\right\}-(1-\mu^2)^{\frac{1}{2}}\,2\mu\left\{\mu^4-\frac{6}{11}\mu^2+\frac{1}{33}\right\}$$

$$=-(1-\mu^2)^{\frac{1}{2}}\left\{4\mu^5-\frac{12}{11}\mu^3-4\mu^3+\frac{12}{11}\mu+2\mu^5-\frac{12}{11}\mu^3+\frac{2}{33}\mu\right\}$$

$$=-(1-\mu^2)^{\frac{1}{2}}\left\{6\mu^5-\frac{68}{11}\mu^3+\frac{38}{33}\mu\right\}.$$

And
$$Y_6^2 = 2(1-\mu^2)^{\frac{1}{2}} \left\{\mu^4 - \frac{6}{11}\mu^2 + \frac{1}{33}\right\},$$

and
$$Z_6^2 = 7(1-\mu^2)\left\{\mu^4 - \frac{6}{11}\mu^2 + \frac{1}{33}\right\}.$$

But by formula for X_6^2, we have

$$X_6^2 = -(1-\mu^2)^{\frac{1}{2}}\left\{6\mu^5 - \frac{4}{2}\cdot\frac{1}{11}[6\,.\,5+4]\mu^3 + \frac{4\,.\,3\,.\,2}{2\,.\,4\,.\,11\,.\,9}[6\,.\,5+8]\mu\right\}$$

$$= -(1-\mu^2)^{\frac{1}{2}}\left\{6\mu^5 - \frac{68}{11}\mu^3 + \frac{38}{33}\mu\right\}, \text{ as before.}$$

Hence by the formula

$$\int_{-1}^{1}(X_4^2)^2 d\mu = 2\,.\,\frac{1\,.\,2}{1\,.\,3\,.\,5\,.\,7}\left\{16\,.\,3\,.\,\frac{1\,.\,2\,.\,3\,.\,4\,.\,5}{1\,.\,3\,.\,5\,.\,7\,.\,9} + 4\,.\,\frac{2}{7}\cdot\frac{1\,.\,2\,.\,3}{1\,.\,3}\right\}$$

$$= \frac{4}{105}\left\{\frac{640}{105} + \frac{16}{7}\right\} = \frac{64}{2205}\{8+3\} = \frac{704}{2205},$$

$$\int_{-1}^{1}(Y_4^2)^2\, d\mu = 2\,.\,4\,.\,\frac{1\,.\,2}{1\,.\,3\,.\,5\,.\,7}\left\{\frac{1\,.\,2\,.\,3\,.\,4}{1\,.\,3\,.\,5} + \frac{2\,.\,1}{7\,.\,5}\cdot\frac{1\,.\,2}{1}\right\}$$

$$= \frac{16}{105}\left\{\frac{8}{5} + \frac{4}{35}\right\} = \frac{64}{3675}\{15\} = \frac{64}{245},$$

and

$$\int_{-1}^{1}(Z_4^2)^2 d\mu = 25\,.\,2\,.\,\frac{1\,.\,2}{1\,.\,3\,.\,5\,.\,7}\left\{\frac{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}{1\,.\,3\,.\,5\,.\,7\,.\,9}\right\} = \frac{20}{21}\left\{\frac{16}{21}\right\} = \frac{320}{441}.$$

Also the formula for $\int_{-1}^{1}(X_4^2)^2\, d\mu + \int_{-1}^{1}(Y_4^2)^2\, d\mu + \int_{-1}^{1}(Z_4^2)^2\, d\mu$ gives

$$2\,.\,\frac{1\,.\,2}{1\,.\,3\,.\,5\,.\,7}\left\{22\,.\,\frac{1\,.\,2\,.\,3\,.\,4\,.\,5}{1\,.\,3\,.\,5\,.\,7} + 4\,.\,\frac{1\,.\,2\,.\,3\,.\,4}{1\,.\,3\,.\,5} + 4\,.\,\frac{2}{7}\cdot\frac{1\,.\,2\,.\,3}{1\,.\,3} + 4\,.\,\frac{2\,.\,1}{7\,.\,5}\cdot\frac{1\,.\,2}{1}\right\}$$

$$= \frac{4}{105}\left\{\frac{176}{7} + \frac{32}{5} + \frac{16}{7} + \frac{16}{35}\right\} = \frac{64}{3675}\{55+14+5+1\} = \frac{64}{49},$$

which agrees with the sum of the separate values just found.

In the same way it is shewn that the separate formulae for

$$\int_{-1}^{1}(X_6^2)^2 d\mu,\ \int_{-1}^{1}(Y_6^2)^2 d\mu \text{ and } \int_{-1}^{1}(Z_6^2)^2 d\mu,$$

when added together give the same result as the formula for

$$\int_{-1}^{1}[(X_6^2)^2+(Y_6^2)^2+(Z_6^2)^2]\,d\mu,$$

and that this result is

$$\frac{2048}{3^3 \,.\, 5 \,.\, 7 \,.\, 11^2 \,.\, 13}[29+13+49]=\frac{2048}{3^3 \,.\, 5 \,.\, 11^2}.$$

Similarly the formulae of Art. 15 (p. 425) give

$$\int_{-1}^{1} Y_6^2 Y_4^2 d\mu = -\int_{-1}^{1} X_6^2 X_4^2 d\mu = \frac{256}{3 \,.\, 5 \,.\, 7^2 \,.\, 11}.$$

21. The important function $\int_{-1}^{1} X_n^m X_{n_1}^m \,d\mu + \int_{-1}^{1} Y_n^m Y_{n_1}^m \,d\mu$ was first proved to vanish in all cases by a different method from that given above.

It was shewn that the function contains the factor

$$\left\{-m+2m^2\left[\frac{1}{n_1+m-1}+\frac{n_1-m-1}{(n_1+m-1)(n_1+m-2)}\right.\right.$$
$$\left.\left.+\frac{(n_1-m-1)(n_1-m-2)}{(n_1+m-1)(n_1+m-2)(n_1+m-3)}+\text{\&c.}\right]\right\}.$$

Also it was proved by induction that

$$\frac{1}{x-1}+\frac{y-1}{(x-1)(x-2)}+\frac{(y-1)(y-2)}{(x-1)(x-2)(x-3)}+\text{\&c.}$$
$$+\frac{(y-1)(y-2)\dots(y-r+1)}{(x-1)(x-2)\dots(x-r)}=\frac{1}{x-y}.$$

For when $y=1$ it becomes $\dfrac{1}{x-1}$,

when $y=2$ it becomes $\dfrac{1}{x-1}+\dfrac{1}{(x-1)(x-2)}=\dfrac{x-1}{(x-1)(x-2)}=\dfrac{1}{x-2}$.

Assume the same law to hold for $y_1=y-1$ and prove it to be true for y thus—

Assume $\frac{1}{x_1-1}+\frac{y_1-1}{(x_1-1)(x_1-2)}+\frac{(y_1-1)(y_1-2)}{(x_1-1)(x_1-2)(x_1-3)}+\&c.$

$$=\frac{1}{x_1-y_1} \text{ for all values of } x_1.$$

Let $x_1=x-1$,

then $$\frac{1}{x-2}+\frac{y_1-1}{(x-2)(x-3)}+\frac{(y_1-1)(y_1-2)}{(x-2)(x-3)(x-4)}+\&c.=\frac{1}{x_1-y_1-1}.$$

Multiply by $\frac{y_1}{x-1}$ and add to $\frac{1}{x-1}$.

$$\therefore \frac{1}{x-1}+\frac{y_1}{(x-1)(x-2)}+\frac{y_1(y_1-1)}{(x-1)(x-2)(x-3)}+\&c.$$

$$=\frac{1}{x-1}+\frac{y_1}{(x-1)(x-y_1-1)}=\frac{1}{x-y_1-1},$$

or $$\frac{1}{x-1}+\frac{y-1}{(x-1)(x-2)}+\frac{(y-1)(y-2)}{(x-1)(x-2)(x-3)}+\&c.=\frac{1}{x-y}.$$

Hence we have

$$\frac{1}{n_1+m-1}+\frac{n_1-m-1}{(n_1+m-1)(n_1+m-2)}+\frac{(n_1-m-1)(n_1-m-2)}{(n_1+m-1)(n_1+m-2)(n_1+m-3)}+\&c.$$
$$=\frac{1}{2m},$$

therefore the above factor reduces to $-m+2m^2\left[\frac{1}{2m}\right]=0$,

hence we get $\int_{-1}^{1} X_n^m X_{n_1}^m \, d\mu+\int_{-1}^{1} Y_n^m Y_{n_1}^m \, d\mu=0$ in all cases.

22. In the series

$$(2n-1)\frac{1}{n+m-1}-(2n-3)\frac{n-m}{(n+m-1)(n+m-2)}$$
$$+(2n-5)\frac{(n-m)(n-m-1)}{(n+m-1)(n+m-2)(n+m-3)}-\&c.,$$

the alternate terms of which occur either in the development of $(X_n^m)^2$ or of $(Y_n^m)^2$, each term may be divided into two parts, so that the series

becomes

$$=1+\frac{n-m}{n+m-1}+\frac{(n-m)(n-m-1)}{(n+m-1)(n+m-2)}+\frac{(n-m)(n-m-1)(n-m-2)}{(n+m-1)(n+m-2)(n+m-3)}+\&c.$$

$$-\frac{n-m}{n+m-1}-\frac{(n-m)(n-m-1)}{(n+m-1)(n+m-2)}-\frac{(n-m)(n-m-1)(n-m-2)}{(n+m-1)(n+m-2)(n+m-3)}+\&c.$$

Hence we see that the sum of this series $=1$.

Also by a similar arrangement of the terms we have

$$(2n-1)\frac{1}{n+m-1}+(2n-3)\frac{n-m}{(n+m-1)(n+m-2)}$$

$$+(2n-5)\frac{(n-m)(n-m-1)}{(n+m-1)(n+m-2)(n+m-3)}+\&c.$$

$$=1+2(n-m)\times\frac{1}{2m}=1+\frac{n-m}{m}=\frac{n}{m}.$$

Hence the odd terms of this series $=\frac{1}{2}\frac{n+m}{m}$,

and the even terms of this series $=\frac{1}{2}\frac{n-m}{m}$.

[By means of these series the simple values for $\int_{-1}^{1}(X_n^m)^2\,d\mu$ and $\int_{-1}^{1}(Y_n^m)^2\,d\mu$, as given above, were first obtained.]

23. Now let us consider the application of the above investigations to the determination of the numerical values of the magnetic constants of terrestrial magnetism. For a given value of μ (i.e. for a given latitude) we have a series of terms forming the coefficients of $\cos m\lambda$ and $\sin m\lambda$, in the values of the magnetic potential and of the magnetic forces X, Y, and Z, which are of the forms

$$\alpha_n H_n^m+\alpha_{n_1}H_{n_1}^m+\&c.$$
$$\alpha_n X_n^m+\alpha_{n_1}X_{n_1}^m+\&c.$$
$$\alpha_n Y_n^m+\alpha_{n_1}Y_{n_1}^m+\&c.$$
$$\alpha_n Z_n^m+\alpha_{n_1}Z_{n_1}^m+\&c.,$$

where α_n, α_{n_1}, &c., are the magnetic constants to be determined.

The numerical values of H_n^m, X_n^m, Y_n^m, and Z_n^m for different values of n and m must be calculated, and in any belt of latitude of breadth corresponding to the numerical value taken for $\delta\mu$, these coefficients must be equated to the values of the forces as derived from the magnetic observations taken in that belt of latitude.

The values of the magnetic forces X, Y, and Z are derived for every 10° of longitude and every 5° of latitude from the declination (δ), the dip (i), and the horizontal force (ω), as given in the charts from which the observations are obtained. These values of the forces X, Y, and Z are analysed for belts of latitude 5° in breadth around the Earth's surface by a formula of the type $a_0 + a_1 \cos\lambda + b_1 \sin\lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda + \&c.$

If we take x_m to represent the coefficient of $\cos m\lambda$ in the expansion of the value of the force X for a given belt of latitude corresponding to the colatitude $\theta = \cos^{-1}\mu$:

then $$\alpha_n X_n^m + \alpha_{n_1} X_{n_1}^m + \alpha_{n_2} X_{n_2}^m + \&c. = x_m,$$

where x_m is derived from the observations. Similar equations, involving on one side the magnetic constants α_n, α_{n_1}, &c., and on the other the values derived from the observations, must be formed for all the successive different belts of latitude from the north pole to the south pole—i.e. for all values of μ between 1 and -1.

The numerical values of X_n^m, $X_{n_1}^m$, &c., as well as the values of H_n^m (as above defined), have been determined for every degree of latitude and recorded for future use, but, in the actual determinations of the magnetic constants which have been made, belts of latitude 5° in breadth have been taken, or $\delta\theta$ has been taken as 5°, and the area of the belt is proportional to $\delta\mu$.

Supposing the observations equally distributed over the surface of the globe, or supposing the weight of any determination proportional to the surface of the corresponding element about the point of observation, then the weight of each of the above equations is proportional to $\delta\mu$, and multiplying the equation in X for each value of μ by X_n^m, and summing up the separate equations for the whole surface of the Earth, we get the final equation—

$$\alpha_n \int_{-1}^{1} (X_n^m)^2 \, d\mu + \alpha_{n_1} \int_{-1}^{1} X_n^m X_{n_1}^m d\mu + \&c. = \int_{-1}^{1} X_n^m \, x_m d\mu.$$

Similarly, the final equation for α_{n_1} is found by multiplying the above equations by $X_{n_1}^m$, $Y_{n_1}^m$, and $Z_{n_1}^m$ respectively, and we get

$$\alpha_n \int_{-1}^{1} X_n^m X_{n_1}^m d\mu + \alpha_{n_1} \int_{-1}^{1} (X_{n_1}^m)^2 d\mu + \&\text{c.} = \int_{-1}^{1} X_{n_1}^m x_m d\mu.$$

Similarly, if y_m denote the coefficient of $\sin m\lambda$ or $-\cos m\lambda$ in the value of the force Y as derived from observations, we have

$$\Sigma (\alpha_n Y_n) = y_m,$$

and the final equations for finding α_n and α_{n_1} respectively will be

$$\alpha_n \int_{-1}^{1} (Y_n^m)^2 d\mu + \alpha_{n_1} \int_{-1}^{1} Y_n^m Y_{n_1}^m d\mu + \&\text{c.} = \int_{-1}^{1} Y_n^m y_m d\mu,$$

and $$\alpha_n \int_{-1}^{1} Y_n^m Y^m d\mu + \alpha_{n_1} \int_{-1}^{1} (Y_{n_1}^m)^2 d\mu + \&\text{c.} = \int_{-1}^{1} Y_{n_1}^m y_m d\mu.$$

Combining the final equations for α_n from X and Y together, we have

$$\alpha_n \int_{-1}^{1} [(X_n^m)^2 + (Y_n^m)^2] d\mu = \int_{-1}^{1} X_n^m x_m d\mu + \int_{-1}^{1} Y_n^m y_m d\mu,$$

since the coefficients of α_{n_1} and all the other terms on the left-hand side of this equation vanish when the integration is taken all over the Earth's surface.

Hence $$\alpha_n \,.\, n(n+1) \int_{-1}^{1} (H_n^m)^2 d\mu = \int_{-1}^{1} X_n^m x_m d\mu + \int_{-1}^{1} Y_n^m y_m d\mu;$$

i.e. $$\alpha_n \times 2n(n+1) \frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2 (2n+1)} = \int_{-1}^{1} X_n^m x_m d\mu + \int_{-1}^{1} Y_n^m y_m d\mu.$$

Similarly, by putting n_1 for n, we may get the value of α_{n_1}.

In the same way the final equation for finding α_n from the equations for Z would give us

$$\alpha_n \int_{-1}^{1} (Z_n^m)^2 d\mu + \alpha_{n_1} \int_{-1}^{1} Z_n^m Z_{n_1}^m d\mu + \&\text{c.} = \int_{-1}^{1} Z_n^m z_m d\mu;$$

or $$\alpha_n (n+1)^2 \int_{-1}^{1} (H_n^m)^2 d\mu = \int_{-1}^{1} Z_n^m z_m d\mu, \text{ since } \int_{-1}^{1} Z_n^m Z_{n_1}^m d\mu = 0;$$

i.e. $$\alpha_n 2(n+1)^2 \frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2 (2n+1)} = \int_{-1}^{1} Z_n^m z_m d\mu.$$

If we combine all the equations of condition involving x, y and z, the final equations for the determination of α_n will be

$$\alpha_n \int_{-1}^{1} [(X_n^m)^2 + (Y_n^m)^2 + (Z_n^m)^2]\, d\mu = \int_{-1}^{1} X_n^m x_m\, d\mu + \int_{-1}^{1} Y_n^m y_m\, d\mu + \int_{-1}^{1} Z_n^m z_m\, d\mu,$$

and similar equations for the other magnetic constants.

Since $$\int_{-1}^{1} [(X_n^m)^2 + (Y_n^m)^2]\, d\mu = n(n+1) \int_{-1}^{1} (H_n^m)^2\, d\mu,$$

and $$\int_{-1}^{1} (Z_n^m)^2\, d\mu = (n+1)^2 \int_{-1}^{1} (H_n^m)\, d\mu,$$

we have

$$\alpha_n 2(n+1) \frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5 \ldots (2n-1)]^2} = \int_{-1}^{1} X_n^m x_m\, d\mu + \int_{-1}^{1} Y_n^m y_m\, d\mu + \int_{-1}^{1} Z_n^m z_m\, d\mu.$$

From the above equations it appears that the weight of a determination of a magnetic constant from the observations of the horizontal force is to the weight of its determination from the observations of the vertical force as n to $(n+1)$.

24. If we take into account separately the parts of the magnetic force at a point due to the internal and external centres of magnetic force, the general terms of the coefficient of $\cos m\lambda$ in the potential function will be of the form

$$\left(\frac{\alpha_n}{r^{n+1}} + \beta_n r^n\right) H_n^m,$$

and the corresponding coefficients in X, Y, and Z will be—

in X, $$\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) \left[\frac{1}{2}(n-m) H_n^{m+1} - \frac{1}{2}(n+m) H_n^{m-1}\right],$$

in Y, $$\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) m (1-\mu^2)^{-\frac{1}{2}} H_n^m,$$

in Z, $$\left[\frac{(n+1)\,\alpha_n}{r^{n+2}} - n\beta_n r^{n-1}\right] H_n^m.$$

If then, as before, we put $r=1$, we shall have the final equation for α_n as follows:

$$\alpha_n \left[\int_{-1}^{1} (X_n^m)^2\, d\mu + \int_{-1}^{1} (Y_n^m)^2\, d\mu + (n+1)^2 \int_{-1}^{1} (H_n^m)^2\, d\mu\right]$$

$$+\beta_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu-n(n+1)\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu+(n+1)\int_{-1}^{1}H_n^m z_m\,d\mu,$$

where the coefficient of $\beta_n = 0$.

And
$$\alpha_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu-n(n+1)\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$+\beta_n\left[\int_{-1}^{1}(X_n^m)^2\,d\mu+\int_{-1}^{1}(Y_n^m)^2\,d\mu+n^2\int_{-1}^{1}(H_n^m)^2\,d\mu\right]$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu-n\int_{-1}^{1}H_n^m z_m\,d\mu,$$

where the coefficient of $\alpha_n = 0$.

Hence α_n and β_n are separately determined from the equations

$$2\alpha_n(n+1)\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\ldots(2n-1)]^2}$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu+(n+1)\int_{-1}^{1}H_n^m z_m\,d\mu,$$

and
$$2\beta_n\,.\,n\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\ldots(2n-1)]^2}$$

$$=\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu-n\int_{-1}^{1}H_n^m z_m\,d\mu.$$

Thus generally from the values of X and Y we derive

$$(\alpha_n+\beta_n)\,2n(n+1)\frac{(n-m)!\,(n+m)!}{[1\,.\,3\,.\,5\ldots(2n-1)]^2}$$

$$=(2n+1)\left[\int_{-1}^{1}X_n^m x_m\,d\mu+\int_{-1}^{1}Y_n^m y_m\,d\mu\right],$$

and from the values of Z we derive

$$[(n+1)\alpha_n-n\beta_n]\int_{-1}^{1}(H_n^m)^2\,d\mu=\int_{-1}^{1}H_n^m z_m\,d\mu.$$

The above theory assumes that the integration is taken over the whole surface of the Earth, and that the observations are uniformly distributed

over the Earth's surface, otherwise the coefficients of the neglected terms on the left-hand side of these equations will not vanish, and each equation may have other terms which are too important to be neglected, and so it will not be so easy to separate the magnetic constants from one another.

Suppose $$\alpha_n + \beta_n = k_n \text{ and } (n+1)\,\alpha_n - n\beta_n = k_n',$$

then $$(2n+1)\,\alpha_n = nk_n + k_n',$$

and $$(2n+1)\,\beta_n = (n+1)\,k_n - k_n',$$

which are expressions analogous to those of Gauss (*Werke*, 1867, Vol. v. p. 173), and α_n, β_n, k_n and k_n' correspond to P', p', Π' and Q' respectively*.

Determination of Special Points on the Earth's Surface.

25. At the Magnetic Poles, we have $X=0$, $Y=0$, two equations which determine the colatitude θ and the longitude λ.

For a line of equal magnetic declination, we have $\frac{Y}{X}$ = a constant, hence for such a line the equation

$$\left(X\frac{dY}{d\theta} - Y\frac{dX}{d\theta}\right)\delta\theta + \left(X\frac{dY}{d\lambda} - Y\frac{dX}{d\lambda}\right)\delta\lambda = 0$$

gives the relation between $\delta\theta$ and $\delta\lambda$ at any point.

On a Mercator's chart the tangent of the angle which the tangent to this line at any point makes with the equator is

$$-\frac{\delta\theta}{\sin\theta\,\delta\lambda} = \frac{X\frac{dY}{d\lambda} - Y\frac{dX}{d\lambda}}{\sin\theta\left(X\frac{dY}{d\theta} - Y\frac{dX}{d\theta}\right)}$$

$$= -\frac{1}{1-\mu^2}\,\frac{X\frac{dY}{d\lambda} - Y\frac{dX}{d\lambda}}{X\frac{dY}{d\mu} - Y\frac{dX}{d\mu}}, \text{ where } \mu = \cos\theta.$$

* In Taylor's *Scientific Memoirs*, vol. II. p. 233, there are some misprints, and the values of $3P'$, &c. there given should have been as follows:

$$3P' = \Pi' + Q', \qquad 3p' = 2\Pi' - Q',$$
$$5P'' = 2\Pi'' + Q'', \qquad 5p'' = 3\Pi'' - Q'',$$
$$7P''' = 3\Pi''' + Q''', \qquad 7p''' = 4\Pi''' - Q''',$$

&c. &c.

At a point where two branches of the line cross each other this expression must have two values, hence at such a point

$$X\frac{dY}{d\lambda} - Y\frac{dX}{d\lambda} = 0 \text{ and } X\frac{dY}{d\theta} - Y\frac{dX}{d\theta} = 0:$$

these are the two equations for finding the values of λ and θ.

At points of maximum or minimum declination, the same two equations must hold good.

The difference between this case and the former is that in the case of maximum declination

$$\frac{d}{d\lambda}\left(X\frac{dY}{d\lambda} - Y\frac{dX}{d\lambda}\right) \text{ and } \frac{d}{d\theta}\left(X\frac{dY}{d\theta} - Y\frac{dX}{d\theta}\right)$$

must both be negative, and in the case of minimum declination they must both be positive, but in the case of two branches crossing each other they must have opposite signs.

Proceeding to a second differentiation we have at such points

$$\left(X\frac{d^2Y}{d\theta^2} - Y\frac{d^2X}{d\theta^2}\right)(\delta\theta)^2 + \left(X\frac{d^2Y}{d\theta d\lambda} - Y\frac{d^2X}{d\theta d\lambda}\right)2\delta\theta\delta\lambda + \left(X\frac{d^2Y}{d\lambda^2} - Y\frac{d^2X}{d\lambda^2}\right)(\delta\lambda)^2 = 0,$$

which will give the two values of $\frac{d\theta}{d\lambda}$ at such points.

At points where the horizontal force is a maximum or a minimum we have

$$X^2 + Y^2 \text{ a maximum or a minimum,}$$

hence the values of θ and λ for such points are given by the equations

$$X\frac{dX}{d\theta} + Y\frac{dY}{d\theta} = 0 \text{ and } X\frac{dX}{d\lambda} + Y\frac{dY}{d\lambda} = 0;$$

similarly the relation between $\delta\theta$ and $\delta\lambda$ for the tangent line to the line of equal horizontal force is given by the equation

$$\left(X\frac{dX}{d\theta} + Y\frac{dY}{d\theta}\right)\delta\theta + \left(X\frac{dX}{d\lambda} + Y\frac{dY}{d\lambda}\right)\delta\lambda = 0.$$

Suppose V to be the magnetic potential and to be a function of μ and λ.

Then
$$X = -\frac{dV}{rd\theta} = \frac{(1-\mu^2)^{\frac{1}{2}}}{r}\frac{dV}{d\mu},$$
$$Y = -\frac{1}{r(1-\mu^2)^{\frac{1}{2}}}\frac{dV}{d\lambda}.$$

Hence
$$\frac{Y}{X} = -\frac{\dfrac{dV}{d\lambda}}{(1-\mu^2)\dfrac{dV}{d\mu}}.$$

On the sphere of unit radius $r=1$, and
$$\frac{dX}{d\theta} = -\sin\theta\frac{dX}{d\mu} = -(1-\mu^2)\frac{d^2V}{d\mu^2} + \mu\frac{dV}{d\mu},$$
$$\frac{dY}{d\theta} = -\sin\theta\frac{dY}{d\mu} = \frac{d^2V}{d\lambda d\mu} + \frac{\mu}{1-\mu^2}\frac{dV}{d\lambda}.$$

Hence
$$X\frac{dY}{d\theta} - Y\frac{dX}{d\theta} = (1-\mu^2)^{\frac{1}{2}}\left[\frac{dV}{d\mu}\frac{d^2V}{d\lambda d\mu} - \frac{dV}{d\lambda}\frac{d^2V}{d\mu^2} + \frac{2\mu}{1-\mu^2}\frac{dV}{d\lambda}\frac{dV}{d\mu}\right],$$

and
$$X\frac{dY}{d\lambda} - Y\frac{dX}{d\lambda} = -\frac{dV}{d\mu}\frac{d^2V}{d\lambda^2} + \frac{dV}{d\lambda}\frac{d^2V}{d\lambda d\mu}.$$

As an example of the application of the above theory, we will find the approximate place of the crossing of two branches of the line of equal declination in the neighbourhood of the point $\theta = 80°$, $\lambda = 260°$. Take this point as the origin, and take x and y as the longitude and latitude respectively of some near point referred to this origin, taking $10°$ of longitude and $5°$ of latitude as the units of x and y respectively. Then if x and y be the coordinates of the point of crossing of the two branches, we have

$$X\frac{dY}{dx} - Y\frac{dX}{dx} = 0 = \left(X\frac{dY}{dx} - Y\frac{dX}{dx}\right)_0 + \left(X\frac{d^2Y}{dx^2} - Y\frac{d^2X}{dx^2}\right)_0 x$$
$$+ \left(X\frac{d^2Y}{dxdy} - Y\frac{d^2X}{dx^2} + \frac{dX}{dy}\frac{dY}{dx} - \frac{dY}{dy}\frac{dX}{dx}\right)_0 y,$$

and
$$X\frac{dY}{dy} - Y\frac{dX}{dy} = 0 = \left(X\frac{dY}{dy} - Y\frac{dX}{dy}\right)_0 + \left(X\frac{d^2Y}{dy^2} - Y\frac{d^2X}{dy^2}\right)_0 y$$
$$+ \left(X\frac{d^2Y}{dxdy} - Y\frac{d^2X}{dxdy} + \frac{dX}{dx}\frac{dY}{dy} - \frac{dY}{dx}\frac{dX}{dy}\right)_0 x.$$

These equations give the values of x and y for the point required.

Also the values of X and Y for the point required are given by the equations

$$X = X_0 + \left(\frac{dX}{dx}\right)_0 x + \left(\frac{dX}{dy}\right)_0 y + \frac{1}{2}\left(\frac{d^2X}{dx^2}\right)_0 x^2 + \frac{1}{2}\left(\frac{d^2X}{dy^2}\right)_0 y^2 + \left(\frac{d^2X}{dxdy}\right)_0 xy,$$

and

$$Y = Y_0 + \left(\frac{dY}{dx}\right)_0 x + \left(\frac{dY}{dy}\right)_0 y + \frac{1}{2}\left(\frac{d^2Y}{dx^2}\right)_0 x^2 + \frac{1}{2}\left(\frac{d^2Y}{dy^2}\right)_0 y^2 + \left(\frac{d^2Y}{dxdy}\right)_0 xy,$$

and $\tan\delta = \frac{Y}{X}$ gives the value of the declination.

Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ be the values of the function X to be determined at the points $\begin{vmatrix} (-1, & 1) & (0, & 1) & (1, & 1) \\ (-1, & 0) & (0, & 0) & (1, & 0) \\ (-1, & -1) & (0, & -1) & (1, & -1) \end{vmatrix}$.

Then taking differences along x, we have

$$e-d,\ f-e \text{ and the mean } \frac{dX}{dx} = \frac{f-d}{2};$$

and taking differences along y,

$$b-e,\ e-h \text{ and the mean } \frac{dX}{dy} = \frac{b-h}{2}.$$

The second differences are

$$\frac{d^2X}{dx^2} = \frac{d+f-2e}{2}, \qquad \frac{d^2X}{dy^2} = \frac{b+h-2e}{2};$$

also taking all the successive first differences with respect to x,

$$\begin{aligned} &b-a, && c-b, \\ &e-d, && f-e, \\ &i-h, && h-g, \end{aligned}$$

and the differences of these with respect to y,

$$\begin{aligned} &b-a-e+d, && c-b-f+e, \\ &e-d-i+h, && f-e-h+g, \end{aligned}$$

and taking the mean of these, we have

$$\frac{d^2X}{dxdy} = \frac{1}{4}(-a-i+c+g).$$

Hence we have

$$X = e + \frac{f-d}{2}x + \frac{b-h}{2}y + \frac{d+f-2e}{2}x^2 + \frac{b+h-2e}{2}y^2 + \frac{c+g-a-i}{4}xy.$$

Similarly the value of Y at the point (x, y) may be determined.

Taking the values of X and Y from the Tables and taking their differences with respect to longitude and latitude, we have for X

	250°	260°	270°	Diff. with respect to latitude			Second diff. with respect to latitude
15°	1000·4	995·3	973·9	$-31·1$	$-35·0$	$-38·5$	
10°	1031·5	1030·3	1012·4	$-15·1$	$-17·7$	$-21·2$	$-16·0$, $-17·3$, $-17·3$
5°	1046·6	1048·0	1033·6				

1st and 2nd diff. with respect to longitude		Second diff. with respect to latitude and longitude
$-5·1$, $-21·4$	$-16·3$	$-3·9$, $-3·5$
$-1·2$, $-17·9$	$-16·7$	$-2·6$, $-3·5$
$+1·4$, $-14·4$	$-15·8$	

Similarly for Y

	250°	260°	270°	Diff. with respect to latitude			Second diff. with respect to latitude
15°	$-152·5$	$-151·4$	$-138·1$	$-5·7$	$+3·0$	$+11·2$	
10°	$-146·8$	$-154·4$	$-149·3$	$-5·9$	$+2·3$	$+10·2$	$+0·2$, $+0·7$, $+1·0$
5°	$-140·9$	$-156·7$	$-159·5$				

1st and 2nd diff. with respect to longitude		Second diff. with respect to latitude and longitude
$+1·1$, $+13·3$	$+12·2$	$+8·7$, $+8·2$
$-7·6$, $+5·1$	$+12·7$	$+8·2$, $+7·9$
$-15·8$, $-2·8$	$+13·0$	

Hence at the origin, where $x=0$, $y=0$, taking the mean differences, we have

$$X_0 = 1030·3,\ \left(\frac{dX}{dx}\right)_0 = -9·55,\ \left(\frac{dX}{dy}\right)_0 = -26·35,\ \left(\frac{d^2X}{dx^2}\right)_0 = -16·7,$$

$$\left(\frac{d^2X}{dy^2}\right)_0 = -17·3,\ \left(\frac{d^2X}{dx\,dy}\right)_0 = -3·4,$$

$$Y_0 = -154·4,\ \left(\frac{dY}{dx}\right)_0 = -1·25,\ \left(\frac{dY}{dy}\right)_0 = 2·65,\ \left(\frac{d^2Y}{dx^2}\right)_0 = 12·7,$$

$$\left(\frac{d^2Y}{dy^2}\right)_0 = 0·7,\ \left(\frac{d^2Y}{dx\,dy}\right)_0 = 8·25.$$

Hence the equations for finding x and y are

$$0 = -2762{\cdot}4 + 10506{\cdot}3x + 8033{\cdot}2y,$$
$$0 = -1338{\cdot}1 + 7916{\cdot}8x - 1949{\cdot}9y,$$

giving $$x = 0{\cdot}19190 \text{ and } y = {\cdot}092895;$$

hence $$\text{Long.} = 261^\circ{\cdot}9 \text{ and Lat.} = 10^\circ{\cdot}45,$$

which agree very well with the chart.

Also we have

$$X = \quad 1030{\cdot}3 - 9{\cdot}55x - 26{\cdot}35y - 8{\cdot}35x^2 - 8{\cdot}65y^2 - 3{\cdot}4xy,$$

and $$Y = -\ 154{\cdot}4 - 1{\cdot}25x + 2{\cdot}65y + 6{\cdot}35x^2 + 0{\cdot}35y^2 + 8{\cdot}25xy.$$

From the equation $\tan\delta = \dfrac{Y}{X}$, we get $\delta = -8^\circ\ 32'{\cdot}4$.

According to Erman's chart, $\delta = -8^\circ\ 33'{\cdot}2$.

The equation which gives the tangent to the two branches of the line of equal declination at their common point is

$$10506{\cdot}3\,(\delta x)^2 + 7975{\cdot}0 \times 2\delta x\delta y - 1949{\cdot}9\,(\delta y)^2 = 0,$$

hence $$\frac{\delta y}{\delta x} = -0{\cdot}6128 \text{ or } +8{\cdot}7928.$$

On a Mercator's chart this must be divided by $\sin\theta$, which gives the values $-31^\circ\ 55'{\cdot}7$ and $83^\circ\ 37'{\cdot}1$ for the directions of the lines.

SECTION VI.

THE THEORY OF TERRESTRIAL MAGNETISM, GIVING THE EXPRESSIONS OF THE MAGNETIC FORCES ON THE EARTH'S SURFACE, TAKING INTO ACCOUNT THE SPHEROIDAL FIGURE OF THE EARTH.

1. LET us now take into account the spheroidal figure of the Earth. Let r, θ', λ be the polar coordinates of a point on the spheroidal surface referred to the Earth's centre as origin and axis of figure as initial line; let θ be the geographical colatitude (the angle which the normal makes with the axis) and let $\mu = \cos\theta$ and $\mu' = \cos\theta'$.

The angle of the vertical $\psi = \theta' - \theta$.

The values of the sines and cosines of these angles for values of θ differing by 1° from 0° to 90° have been computed, the eccentricity e of the elliptic section in the plane of the meridian being derived from Bessel's dimensions of the Earth as given in Encke's tables in the *Berliner Jahrbuch*, 1852.

The expressions for the magnetic potential and for the magnetic forces X, Y, and Z, in terms of the Gaussian magnetic constants g_n^m, h_n^m will be of the same form as those given above for the sphere (see p. 403).

Let X be the total force towards the north perpendicular to the Earth's radius, Y the total force perpendicular to the geographical meridian towards the west, Z the force towards the Earth's centre, then

$$X = -\frac{dV}{rd\theta'}, \quad Y = -\frac{1}{r\sin\theta'}\frac{dV}{d\lambda}, \text{ and } Z = -\frac{dV}{dr}$$

(east longitudes being considered positive).

If X' be the horizontal force in the meridian towards the north,

Y' the horizontal force perpendicular to the meridian towards the west,

Z' the vertical downward force on the spheroidal surface of the Earth,

then
$$X' = X\cos\psi + Z\sin\psi,$$
$$Y' = Y,$$
$$Z' = -X\sin\psi + Z\cos\psi.$$

We may conveniently denote the values of the coefficients of $g_n^m \cos m\lambda$ and $h_n^m \sin m\lambda$ in the potential function and in the forces X, Y and Z by the same symbols V_n^m, X_n^m, Y_n^m and Z_n^m respectively as in the case of the sphere, regarding them as functions of H'^m_n, where H'^m_n is the same function of μ' that H^n is of μ.

The coefficients of $g_n^m \cos m\lambda$ and $h_n^m \sin m\lambda$ in X' and Z' may be denoted by X_n' and Z_n', where
$$X_n' = X_n^m \cos\psi + Z_n^m \sin\psi,$$
and
$$Z_n' = -X_n^m \sin\psi + Z_n^m \cos\psi.$$

2. If a be the equatorial radius of the spheroid, then, taking into account only the terms to the order e^2,
$$\frac{a^2}{r^2} = \frac{1 - e^2\sin^2\theta'}{1 - e^2} = 1 + e^2\cos^2\theta' = 1 + e^2\mu^2.$$

We have also
$$\sin\psi = \sin(\theta' - \theta) = \frac{e^2 \sin\theta\cos\theta}{[1 - e^2(2 - e^2)\cos^2\theta]^{\frac{1}{2}}} = e^2\cos\theta\sin\theta' = e^2\cos\theta\sin\theta,$$
also
$$\mu' = \cos\theta' = \cos\theta - e^2\mu(1-\mu^2)^{\frac{1}{2}}\sin\theta,$$
or
$$\mu' = \mu - e^2\mu(1 - \mu^2) = \mu - \sin\theta\sin\psi;$$
hence
$$\frac{d\mu'}{d\mu} = 1 - e^2(1 - 3\mu^2),$$
and
$$(1 - \mu'^2)^{\frac{1}{2}} = \{1 - [\mu - e^2\mu(1-\mu^2)]^2\}^{\frac{1}{2}} = (1-\mu^2)^{\frac{1}{2}}(1 + e^2\mu^2).$$

Let b be the polar axis, and let x and y be rectangular coordinates of the point on the spheroid. Then
$$x = r(1 - \mu'^2)^{\frac{1}{2}} = a(1-\mu^2)^{\frac{1}{2}}(1 + e^2\mu^2)^{\frac{1}{2}},$$
$$y = r\mu' = b\mu'(1 - e^2 + e^2\mu'^2)^{-\frac{1}{2}} = b\mu[1 - e^2(1 - \mu^2)]^{\frac{1}{2}}.$$

If N be the normal terminated by the minor axis,

$$N^2 = x^2 + \frac{y^2}{(1-e^2)^2} = \frac{b^2}{1-e^2+e^2\mu'^2}\left[1-\mu'^2+\frac{\mu'^2}{(1-e^2)^2}\right]$$

$$= \frac{a^2}{1-e^2}\left[\frac{(1-e^2)^2+(2e^2-e^4)\,\mu'^2}{1-e^2+e^2\mu'^2}\right]$$

$$= \frac{a^2}{1-e^2}\left[\frac{1-2e^2\,(1-\mu^2)}{1-e^2\,(1-\mu^2)}\right].$$

The radius of curvature of the meridian is

$$\rho = \frac{1-e^2}{a^2}N^3.$$

If δS be the elementary area of a belt of the Earth's surface between two parallels of latitude and δs be the length of the arc of the meridian, we have

$$\delta S = 2\pi x\,.\,\delta s = 2\pi N \sin\theta\,.\,\rho\,.\,\delta\theta = 2\pi\frac{1-e^2}{a^2}N^4\sin\theta\,.\,\delta\theta$$

$$= 2\pi b^2\frac{N^4}{a^4}\sin\theta\,.\,\delta\theta,$$

or
$$\frac{dS}{d\mu} = -2\pi b^2\frac{N^4}{a^4} = -2\pi a^2\,(1-e^2)\,\frac{1}{(1-e^2)^2}\left[1-e^2\,(1-\mu^2)\right]^2,$$

and
$$\frac{dS}{d\mu'} = -2\pi a^2\,\frac{1-2e^2(1-\mu^2)}{(1-e^2)\left[1-e^2(1-3\mu^2)\right]} = -2\pi a^2\,(1-e^2\mu^2) = -2\pi r^2.$$

Taking the equatorial radius $=1$, we have

$$\frac{dS}{d\mu'} = -2\pi\,(1-e^2\mu^2),$$

$$\frac{1}{r^2} = 1+e^2\mu^2,$$

$$\frac{1}{r^{n+2}} = 1+\frac{n+2}{2}e^2\mu^2,$$

and
$$r^{n-1} = 1-\frac{n-1}{2}e^2\mu^2.$$

3. In the following investigation of the coefficients of $\cos m\lambda$, &c., in which m remains the same, while n may have different values, it will be convenient to denote H_n^m by H_n, H'^m_n by H_n', X_n^m by X_n, &c., regarding H_n', X_n, &c. as functions of μ' or $\cos\theta'$, where θ' is the geocentric colatitude.

Regarding H_n' and $\dfrac{dH_n'}{d\mu'}$, &c. as functions of μ', we have by Taylor's theorem

$$H_n' = H_n - e^2\mu\,(1-\mu^2)\frac{dH_n}{d\mu}$$

to the order e^2, and

$$\frac{dH_n'}{d\mu'} = \frac{dH_n}{d\mu} - e^2\mu\,(1-\mu^2)\frac{d^2H_n}{d\mu^2}.$$

Hence

$$(1-\mu'^2)^{\frac{1}{2}}\frac{dH_n'}{d\mu'} = (1-\mu^2)^{\frac{1}{2}}\frac{dH_n}{d\mu}(1+e^2\mu^2) - e^2\mu\,(1-\mu^2)^{\frac{3}{2}}\frac{d^2H_n}{d\mu^2}$$

$$= (1-\mu^2)^{\frac{1}{2}}\frac{dH_n}{d\mu}(1+e^2\mu^2) - 2e^2\mu^2\,(1-\mu^2)^{\frac{1}{2}}\frac{dH_n}{d\mu}$$

$$+ e^2\mu\,(1-\mu^2)^{\frac{1}{2}}\left[n\,(n+1) - \frac{m^2}{1-\mu^2}\right]H_n$$

$$= (1-\mu^2)^{\frac{1}{2}}\frac{dH_n}{d\mu}(1-e^2\mu^2) + e^2\mu\,(1-\mu^2)^{\frac{1}{2}}\left[n\,(n+1) - \frac{m^2}{1-\mu^2}\right]H_n.$$

4. *Expressions for the Magnetic Forces on the Earth's Surface.*

If α_n and β_n be taken to represent magnetic constants depending on the internal and external sources of magnetic force respectively, the coefficient of $\cos m\lambda$ in the general term of the potential function V is

$$\Sigma\left[\left(\frac{\alpha_n}{r^{n+1}} + \beta_n r^n\right)H_n'\right].$$

The coefficients of $\cos m\lambda$ in the general terms of the values of the forces X, Y and Z are:—

for X, $$\Sigma\left[\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right)(1-\mu'^2)^{\frac{1}{2}}\frac{dH_n'}{d\mu'}\right],$$

for Y, $$\Sigma\left[\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right)m\,(1-\mu'^2)^{-\frac{1}{2}}H_n'\right],$$

for Z, $$\Sigma\left[\left(\frac{\alpha_n(n+1)}{r^{n+2}} - \beta_n n r^{n-1}\right)H_n'\right].$$

If we resolve the forces X and Z in the horizontal and vertical directions instead of along and perpendicular to the Earth's radius, the change in the value of X is $Z\sin\psi$, and the change in the value of Z is $-X\sin\psi$, where

$$\sin\psi = e^2\sin\theta\cos\theta = e^2\mu\,(1-\mu^2)^{\frac{1}{2}}.$$

Hence the term $\quad (1-\mu^2)^{-\frac{1}{2}} H_n\left[(n+1)\,\alpha_n - n\beta_n\right] e^2\mu\,(1-\mu^2)$

must be added to the coefficient of $\cos m\lambda$ in the value of X, and the term

$$-\frac{dH_n}{d\mu}(\alpha_n+\beta_n)\,e^2\mu\,(1-\mu^2)$$

must be added to the coefficient of $\cos m\lambda$ in the value of Z. Hence taking, as in the case of a sphere, x_m, y_m and z_m for the values of the coefficients of $\cos m\lambda$ as derived from observation, and substituting the values just obtained for H_n', $\dfrac{dH_n'}{d\mu'}$, &c. and collecting terms, we get for the equations of condition

$$\Sigma\left[(1-\mu^2)^{\frac{1}{2}}\frac{dH_n}{d\mu}\left\{(\alpha_n+\beta_n)-(\alpha_n+\beta_n)\,e^2\mu^2+\frac{1}{2}\left[\alpha_n(n+2)-\beta_n(n-1)\right]e^2\mu^2\right\}\right.$$
$$\left.+(1-\mu^2)^{-\frac{1}{2}}H_n\left\{(n+1)^2\alpha_n+n^2.\,\beta_n-(\alpha_n+\beta_n)\frac{m^2}{1-\mu^2}\right\}e^2\mu\,(1-\mu^2)\right]=x_m,$$

$$\Sigma\left[m\,(1-\mu^2)^{-\frac{1}{2}}H_n\left\{(\alpha_n+\beta_n)-(\alpha_n+\beta_n)\,e^2\mu^2+\frac{1}{2}\left[\alpha_n(n+2)-\beta_n(n-1)\right]e^2\mu^2\right\}\right.$$
$$\left.-m\,(1-\mu^2)^{-\frac{1}{2}}\frac{dH_n}{d\mu}(\alpha_n+\beta_n)\,e^2\mu\,(1-\mu^2)\right]=y_m,$$

$$\Sigma\left[H_n\left\{(n+1)\,\alpha_n-n\beta_n+\frac{1}{2}\left[\alpha_n(n+1)(n+2)+\beta_n.\,n\,(n-1)\right]e^2\mu^2\right\}\right.$$
$$\left.-\frac{dH_n}{d\mu}\left[\alpha_n(n+2)-\beta_n(n-1)\right]e^2\mu\,(1-\mu^2)\right]=z_m.$$

5. Multiplying the equation for x_m by $(1-\mu^2)^{\frac{1}{2}}\dfrac{dH_n}{d\mu}$, and the equation for y_m by $m\,(1-\mu^2)^{-\frac{1}{2}}H_n$ and adding, we get

$$\left[(1-\mu^2)\left(\frac{dH_n}{d\mu}\right)^2+\frac{m^2}{1-\mu^2}(H_n)^2\right]\left\{(\alpha_n+\beta_n)+\frac{1}{2}\left[n\alpha_n-(n+1)\,\beta_n\right]e^2\mu^2\right\}$$
$$+H_n\frac{dH_n}{d\mu}\left\{\mu\,(1-\mu^2)\,e^2\left[(n+1)^2\alpha_n+n^2.\,\beta_n\right]-2\mu m^2e^2\,(\alpha_n+\beta_n)\right\}$$
$$+\text{terms involving other magnetic constants}=X_nx_m+Y_ny_m.$$

Then taking the weight of the observations in a belt of latitude as

proportional to its breadth ($\delta\mu$), and multiplying by $\delta\mu$ and integrating from -1 to $+1$, we get

$$n(n+1)\int_{-1}^{1}(H_n)^2\,d\mu\left\{(\alpha_n+\beta_n)+\frac{1}{2}[n\alpha_n-(n+1)\beta_n]e^2\right\}$$

$$-\frac{1}{2}[n\alpha_n-(n+1)\beta_n]e^2\left\{\int_{-1}^{1}(1-\mu^2)^2\left(\frac{dH_n}{d\mu}\right)^2d\mu+m^2\int_{-1}^{1}(H_n)^2\,d\mu\right\}$$

$$+[(n+1)^2\alpha_n+n^2\beta_n]e^2\int_{-1}^{1}\mu(1-\mu^2)H_n\frac{dH_n}{d\mu}d\mu$$

$$-2m^2e^2(\alpha_n+\beta_n)\int_{-1}^{1}\mu H_n\frac{dH_n}{d\mu}d\mu+\text{terms involving }\alpha_{n_1},\ \beta_{n_1},\ \&\text{c.}$$

$$=\int_{-1}^{1}X_nx_m d\mu+\int_{-1}^{1}Y_ny_m d\mu.$$

Hence referring for the values of the above definite integrals to Section V. Art. 9 (p. 417), we get

$$\int_{-1}^{1}(H_n)^2\,d\mu\left\{\left[(\alpha_n+\beta_n)+\frac{1}{2}e^2[n\alpha_n-(n+1)\beta_n]\right]n(n+1)\right.$$

$$-e^2[n\alpha_n-(n+1)\beta_n]\frac{n^2(n+1)^2+(n^2+n-3)m^2}{(2n-1)(2n+3)}$$

$$\left.+e^2[(n+1)^2\alpha_n+n^2\beta_n]\frac{n^2+n-3m^2}{(2n-1)(2n+3)}+m^2e^2(\alpha_n+\beta_n)\right\}+\&\text{c.}$$

$$=\int_{-1}^{1}X_nx_m d\mu+\int_{-1}^{1}Y_ny_m d\mu.$$

In the same way from the equation for z_m we get

$$\int_{-1}^{1}(H_n)^2\,d\mu\left\{(n+1)\alpha_n-n\beta_n+\frac{1}{2}e^2[(n+1)(n+2)\alpha_n+n(n-1)\beta_n]\right.$$

$$\times\frac{2n^2+2n-2m^2-1}{(2n-1)(2n+3)}$$

$$\left.-e^2[(n+2)\alpha_n-(n-1)\beta_n]\frac{n^2+n-3m^2}{(2n-1)(2n+3)}\right\}+\&\text{c.}=\int_{-1}^{1}H_nz_m d\mu.$$

Since $\int_{-1}^{1}(H_n)^2d\mu=2\dfrac{(n-m)!\,(n+m)!}{\{1.3.5\ldots(2n-1)\}^2(2n+1)}$ (as proved above, p. 411), we see that the coefficients of α_n and β_n in the above final equations are determined.

6. The above equations may be combined so as to give the final equations for α_n and β_n respectively.

Multiplying the last equation by $(n+1)$ and adding to the former, we get the final equation for α_n:—

$$\int_{-1}^{1}(H_n)^2\,d\mu\left\{(2n+1)(n+1)\alpha_n-\beta_n e^2\frac{(n-1)[n(n+1)-3m^2]}{(2n-1)(2n+3)}\right.$$

$$\left.+\alpha_n e^2\frac{n+2}{2(2n+3)}[(n+1)(2n^2+2n+1)-2m^2(n-1)]\right\}$$

$$+\text{terms in } e^2 \text{ involving other magnetic constants}$$

$$=\int_{-1}^{1}X_n x_m\,d\mu+\int_{-1}^{1}Y_n y_m\,d\mu+(n+1)\int_{-1}^{1}H_n z_m\,d\mu.$$

The principal term in α_n has the coefficient

$$2(n+1)\frac{(n-m)!\,(n+m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2},$$

as before found. We see that when $n=1$, and also when $n(n+1)=3m^2$, i.e. when $n=3$ and $m=2$, the term containing β_n disappears from this equation.

In the same way by multiplying the last equation of Art. 5 by n and subtracting it from the former, we get the final equation for β_n:—

$$\int_{-1}^{1}(H_n)^2\,d\mu\left\{(2n+1)\,n\beta_n+\alpha_n e^2\frac{(n+2)[n(n+1)-3m^2]}{(2n-1)(2n+3)}\right.$$

$$\left.-\frac{1}{2}\beta_n e^2\frac{n-1}{2n-1}[n(2n^2+2n+1)-2m^2(n+2)]\right\}$$

$$+\text{terms in } e^2 \text{ involving other magnetic constants}$$

$$=\int_{-1}^{1}X_n x_m\,d\mu+\int_{-1}^{1}Y_n y_m\,d\mu-n\int_{-1}^{1}H_n z_m\,d\mu.$$

7. The formulae for finding the numerical values of the coefficients V_n^m, X_n^m, Y_n^m, Z_n^m, &c. are:—

$$V_n^m=\frac{1}{r^{n+1}}H'^m_n,\quad\text{and}\quad V_{-n}^m=r^n H'^m_n,$$

$$Z_n^m=\frac{n+1}{r^{n+2}}H'^m_n,\ \text{and}\quad Z_{-n}^m=-nr^{n-1}H'^m_n,$$

$$Y^m_n = \frac{m}{r^{n+2}}(1-\mu'^2)^{-\frac{1}{2}} H'^m_n, \text{ and } Y^m_{-n} = mr^{n-1}(1-\mu'^2)^{-\frac{1}{2}} H'^m_n,$$

$$\mu' Y^m_n - X^m_n = \frac{n+m}{r^{n+2}} H'^{m-1}_n \text{ or } \mu' Y^m_n + X^m_n = \frac{n-m}{r^{n+2}} H'^{m+1}_n.$$

These formulae may be simplified in the cases when $m=n$, and when $m=n-1$.

When $m=n$, we have

$$V^m_n = \frac{1}{r^{n+1}}(1-\mu'^2)^{\frac{n}{2}} = \frac{(\sin\theta')^n}{r^{n+1}}, \text{ and } V^m_{-n} = r^n(\sin\theta')^n,$$

$$Z^m_n = \frac{(n+1)(\sin\theta')^n}{r^{n+2}}, \qquad \text{and } Z^m_{-n} = -nr^{n-1}(\sin\theta')^n,$$

$$Y^m_n = \frac{n(\sin\theta')^{n-1}}{r^{n+2}}, \qquad \text{and } Y^m_{-n} = nr^{n-1}(\sin\theta')^{n-1},$$

$$X^m_n + \mu' Y^m_n = 0, \qquad \text{and } X^m_{-n} + \mu' Y^m_{-n} = 0.$$

When $m=n-1$, we have

$$G'^{n-1}_n = \mu', \qquad \text{and } H'^{n-1}_n = \mu'(1-\mu'^2)^{\frac{n-1}{2}} = \mu'(\sin\theta')^{n-1},$$

$$Z^m_n = \frac{(n+1)\mu'(\sin\theta')^{n-1}}{r^{n+2}}, \quad \text{and} \quad Z^m_{-n} = -nr^{n-1}\mu'(\sin\theta')^{n-1},$$

$$Y^m_n = \frac{(n-1)\mu'(\sin\theta')^{n-2}}{r^{n+2}}, \quad \text{and} \quad Y^m_{-n} = (n-1)r^{n-1}\mu'(\sin\theta')^{n-2},$$

$$X^m_n = \frac{n(\sin\theta')^n - (n-1)(\sin\theta')^{n-2}}{r^{n+2}},$$

and
$$X^m_{-n} = r^{n-1}\left[n(\sin\theta')^n - (n-1)(\sin\theta')^{n-2}\right].$$

The above formulae have been employed to determine the numerical values of X^m_n, Y^m_n and Z^m_n and of X^m_{-n}, Y^m_{-n} and Z^m_{-n}, and also the values of X_n', Z_n' and of X'_{-n}, Z'_{-n}, for every degree of the geographical colatitude over the surface of the Earth.

8. We will now give a more complete investigation of the case of the spheroid.

For a given value of μ', i.e. for a given narrow belt of latitude, we

may express the horizontal and vertical magnetic forces in terms of the magnetic constants as in the case of the sphere.

We may also analyse the observations of horizontal and vertical forces in the same belt of latitude in a series of the form

$$a_0 + a_1 \cos\lambda + b_1 \sin\lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda + \&c.,$$

and equate the coefficients of $\cos m\lambda$ and of $\sin m\lambda$ in the two series.

Thus we shall have

$$\alpha_n X_n' + \beta_n X_{-n}' + \alpha_{n_1} X_{n_1}' + \beta_{n_1} X_{-n_1}' + \&c. = x_m',$$

$$\alpha_n Y_n' + \beta_n Y_{-n}' + \alpha_{n_1} Y_{n_1}' + \beta_{n_1} Y_{-n_1}' + \&c. = y_m',$$

$$\alpha_n Z_n' + \beta_n Z_{-n}' + \alpha_{n_1} Z_{n_1}' + \beta_{n_1} Z_{-n_1}' + \&c. = z_m',$$

where x_m', y_m' and z_m' are the coefficients derived from the observations of horizontal and vertical forces.

Substituting the values of X_n', Y_n', Z_n', &c., in terms of H_n', $\frac{dH_n'}{d\mu'}$, &c., in the above equations, we get

$$\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) \frac{dH_n'}{d\mu'} (1-\mu'^2)^{\frac{1}{2}} \cos\psi + \left[\frac{(n+1)\,\alpha_n}{r^{n+2}} - n\beta_n r^{n-1}\right] H_n' \sin\psi$$

$$+ \text{similar terms involving other magnetic constants} = x_m',$$

$$\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) m H_n' (1-\mu'^2)^{-\frac{1}{2}} + \text{similar terms} = y_m',$$

$$-\left(\frac{\alpha_n}{r^{n+2}} + \beta_n r^{n-1}\right) \frac{dH_n'}{d\mu'} (1-\mu'^2)^{\frac{1}{2}} \sin\psi$$

$$+ \left[\frac{(n+1)\,\alpha_n}{r^{n+2}} - n\beta_n r^{n-1}\right] H_n' \cos\psi + \text{similar terms} = z_m'.$$

On multiplying these equations by X_n', Y_n' and Z_n' respectively, i.e. each equation by the coefficient of α_n in that equation, and adding them all together, we shall get the partial equation of condition for α_n: the coefficient of α_n will be

$$\frac{1}{r^{2n+4}}\left[(1-\mu'^2)\left(\frac{dH_n'}{d\mu'}\right)^2 + (n+1)^2 (H_n')^2 + \frac{m^2 (H_n')^2}{1-\mu'^2}\right];$$

the coefficient of α_{n_1} will be

$$\left[\frac{1}{r^{n+2}}\frac{dH_n'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\cos\psi+\frac{n+1}{r^{n+2}}H_n'\sin\psi\right]$$

$$\times\left[\frac{1}{r^{n_1+2}}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\cos\psi+\frac{n_1+1}{r^{n_1+2}}H_{n_1}'\sin\psi\right]$$

$$+\frac{m}{r^{n+2}}\frac{m}{r^{n_1+2}}\frac{H_n'H_{n_1}'}{1-\mu'^2}$$

$$+\left[-\frac{1}{r^{n+2}}\frac{dH_n'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\sin\psi+\frac{n+1}{r^{n+2}}H_n'\cos\psi\right]$$

$$\times\left[-\frac{1}{r^{n_1+2}}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\sin\psi+\frac{n_1+1}{r^{n_1+2}}H_{n_1}'\cos\psi\right]$$

$$=\frac{1}{r^{n+n_1+4}}\left\{\frac{dH_n'}{d\mu'}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)+(n+1)(n_1+1)H_n'H_{n_1}'+\frac{m^2H_n'H_{n_1}'}{1-\mu'^2}\right\}.$$

Also the coefficient of β_{n_1} in the same partial equation of condition for α_n will be

$$\left(\frac{1}{r^{n+2}}\frac{dH_n'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\cos\psi+\frac{n+1}{r^{n+2}}H_n'\sin\psi\right)$$

$$\times\left(r^{n_1-1}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\cos\psi-n_1r^{n_1-1}H_{n_1}'\sin\psi\right)$$

$$+\frac{1}{r^{n+2}}\frac{r^{n_1-1}m^2}{1-\mu'^2}H_n'H_{n_1}'$$

$$+\left(-\frac{1}{r^{n+2}}\frac{dH_n'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\sin\psi+\frac{n+1}{r^{n+2}}H_n'\cos\psi\right)$$

$$\times\left(-r^{n_1-1}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)^{\frac{1}{2}}\sin\psi-n_1r^{n_1-1}H_{n_1}'\cos\psi\right)$$

$$=\frac{1}{r^{n-n_1+3}}\left\{\frac{dH_n'}{d\mu'}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)-(n+1)n_1H_n'H_{n_1}'+\frac{m^2H_n'H_{n_1}'}{1-\mu'^2}\right\}.$$

Similarly the coefficient of α_{n_1} in the partial equation of condition for β_n is

$$r^{n-n_1-3}\left\{\frac{dH_n'}{d\mu'}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)-n(n_1+1)H_n'H_{n_1}'+\frac{m^2H_n'H_{n_1}'}{1-\mu'^2}\right\}.$$

Also the coefficient of β_{n_1} in the same partial equation for β_n is

$$r^{n+n_1-2}\left\{\frac{dH_n'}{d\mu'}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)+n\,n_1 H_n' H_{n_1}'+\frac{m^2 H_n' H_{n_1}'}{1-\mu'^2}\right\}.$$

The coefficient of α_n in the partial equation of condition for α_n will be found from the coefficient of α_{n_1} in the same equation by putting n for n_1.

Similarly the coefficient of β_n in any partial equation of condition for β_n will be found from the coefficient of β_{n_1} in the same equation by putting n for n_1.

Hence the coefficient of α_n in the partial equation of condition for α_n will be

$$\frac{1}{r^{2n+4}}\left\{(1-\mu'^2)\left(\frac{dH_n'}{d\mu'}\right)^2+(n+1)^2(H_n')^2+\frac{m^2(H_n')^2}{1-\mu'^2}\right\};$$

and the coefficient of β_n in the partial equation of condition for β_n will be

$$r^{2n-2}\left\{(1-\mu'^2)\left(\frac{dH_n'}{d\mu'}\right)^2+n^2(H_n')^2+\frac{m^2(H_n')^2}{1-\mu'^2}\right\}.$$

9. Since $\frac{1}{r^2}=1+e^2\mu^2$, we have

$$\frac{1}{r^{n+n_1+4}}=(1+e^2\mu^2)^{\frac{n+n_1}{2}+2}=\left(1+\frac{n+n_1+4}{2}e^2\mu^2\right),$$

$$\frac{1}{r^{n-n_1+3}}=1+\frac{n-n_1+3}{2}e^2\mu^2,$$

$$r^{n-n_1-3}=1-\frac{n-n_1-3}{2}e^2\mu^2,$$

and

$$r^{n+n_1-2}=1-\frac{n+n_1-2}{2}e^2\mu^2.$$

Also the area of a small belt of the surface of the spheroid is

$$\begin{aligned}\delta S &= -\,2\pi r^2\delta\mu' \\ &= -2\pi(1-e^2\mu^2)\,\delta\mu' = -2\pi(1-e^2+2e^2\mu^2)\,\delta\mu.\end{aligned}$$

If we assume the weight of the observations in any belt of latitude to be proportional to the area of the belt, then to form the final

equations for α_n and β_n, &c. we must multiply each of the terms of the respective partial equations of condition similar to the above equations for α_n and β_n by $(1-e^2\mu^2)\,\delta\mu'$, or $(1-e^2+2e^2\mu^2)\,\delta\mu$, and integrate between the limits $+1$ and -1, i.e. over the whole surface of the Earth.

The coefficient of α_{n_1} in the final equation for α_n will be the same quantity as the coefficient of α_n in the final equation for α_{n_1}; and similarly the coefficient of β_{n_1} in the final equation for β_n will be the same quantity as the coefficient of β_n in the final equation for β_{n_1}.

10. Since H_n' is the same function of μ' that H_n is of μ, it follows from the results given above (p. 421) that

$$\int_{-1}^{1}\left[(1-\mu'^2)\left(\frac{dH_n'}{d\mu'}\right)^2+\frac{m^2(H_n')^2}{1-\mu'^2}+(n+1)^2(H_n')^2\right]d\mu'=\frac{2\,(n-m)\,!\,(n+m)\,!}{[1\,.\,3\,.\,5\ldots(2n-1)]^2}(n+1),$$

also
$$\int_{-1}^{1}\frac{dH_n'}{d\mu'}\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)\,d\mu'+\int_{-1}^{1}\frac{m^2H_n'H_{n_1}'}{1-\mu'^2}\,d\mu'=0,$$

and
$$\int_{-1}^{1}H_n'H_{n_1}'d\mu'=0.$$

Hence we need only consider the terms involving $e^2\mu^2$ in the above expressions for the coefficients of the magnetic constants.

The coefficient of α_{n_1} in the equation for α_n will be

$$\left(1+\frac{n+n_1+2}{2}\,e^2\mu^2\right)$$
$$\times\left\{\frac{dH_n'}{d\mu'}\,\frac{dH_{n_1}'}{d\mu'}(1-\mu'^2)+(n+1)\,(n_1+1)\,H_n'H_{n_1}'+\frac{m^2H_n'H_{n_1}'}{1-\mu'^2}\right\}.$$

Putting this under the form

$$\left[1+\frac{1}{6}(n+n_1+2)\,e^2-\frac{n+n_1+2}{2}\,e^2\left(\frac{1}{3}-\mu^2\right)\right]$$
$$\times\left\{(1-\mu'^2)\,\frac{dH_n'}{d\mu}\,\frac{dH_{n_1}'}{d\mu}+\frac{m^2H_n'H_{n_1}'}{1-\mu'^2}+(n+1)\,(n_1+1)H_n'H_{n_1}'\right\},$$

and putting μ for μ' in terms of the second order, we see that all the terms are readily integrable by means of the above definite integrals.

On integrating this expression from $\mu = -1$ to $\mu = 1$, we get

$$-\frac{n+n_1+2}{2} e^2 \left\{\frac{1}{2} n(n+1) + \frac{1}{2} n_1 (n_1+1) - 3 + (n+1)(n_1+1)\right\} \int_{-1}^{1} \left(\frac{1}{3} - \mu^2\right) H_n H_{n_1} d\mu$$

$$= \frac{n+n_1+2}{2} e^2 \left\{\frac{1}{2}(n+n_1+1)(n+n_1+2) - 3\right\} \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu$$

$$= \frac{1}{4} e^2 (n+n_1+2)(n+n_1+4)(n+n_1-1) \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu.$$

This vanishes, except when $n_1 = n-2$,

or when $n_1 = n+2$,

or when $n = n_1$.

When $n_1 = n-2$, the value of the coefficient of α_{n-2} is

$$e^2 \frac{2n(n+1)(2n-3)}{2n+1} \frac{(n+m)!\,(n-m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2};$$

and when $n_1 = n+2$, the value of the coefficient of α_{n+2} is

$$e^2 \frac{2(n+2)(n+3)(2n+1)}{2n+5} \frac{(n+2+m)!\,(n+2-m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n+3)]^2}.$$

Similarly the coefficient of β_{n_1} in the final equation for α_n is

$$e^2 \frac{n-n_1+1}{2} \left[\frac{1}{2} n(n+1) + \frac{1}{2} n_1 (n_1+1) - 3 - (n+1) n_1\right] \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu$$

$$= e^2 \frac{n-n_1+1}{4} [(n-n_1)(n-n_1+1) - 6] \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu$$

$$= e^2 \frac{(n-n_1+1)(n-n_1+3)(n-n_1-2)}{4} \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu;$$

hence, when $n_1 = n-2$, the coefficient of β_{n-2} vanishes.

When $n_1 = n+2$, the coefficient of β_{n+2} is

$$e^2 \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu = e^2 \int_{-1}^{1} (H_{n_1})^2 d\mu.$$

Similarly the coefficient of α_{n_1} in the final equation for β_n is

$$e^2 \frac{n_1-n+1}{2} \left[\frac{1}{2} n(n+1) + \frac{1}{2} n_1 (n_1+1) - 3 - n(n_1+1)\right] \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu$$

$$= e^2 \frac{(n_1-n+1)(n_1-n+3)(n_1-n-2)}{4} \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu.$$

When $n_1 = n-2$, the coefficient of α_{n-2} in the final equation for β_n is

$$e^2 \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu = e^2 \frac{2}{2n+1} \frac{(n+m)!\,(n-m)!}{[1\,.\,3\,.\,5\,\ldots\,(2n-1)]^2};$$

when $n_1 = n+2$, the coefficient of α_{n+2} vanishes.

The coefficient of β_{n_1} in the same final equation for β_n is

$$-e^2 \frac{n+n_1}{2} \left[\frac{1}{2} n(n+1) + \frac{1}{2} n_1(n_1+1) - 3 + nn_1\right] \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu$$

$$= -e^2 \frac{(n+n_1)(n+n_1+3)(n+n_1-2)}{4} \int_{-1}^{1} \mu^2 H_n H_{n_1} d\mu.$$

Hence the coefficient of β_{n-2} is

$$-e^2 (n-1)(2n+1)(n-2) \frac{2}{2n+1} \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2},$$

and the coefficient of β_{n+2} is

$$-e^2 (n+1)(2n+5)\, n \frac{2}{2n+5} \frac{(n+m+2)!\,(n-m+2)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n+3)\}^2}.$$

The coefficient of α_n in the final equation for α_n is

$$\left\{2(n+1)\left(1 + \frac{n+1}{3} e^2\right) + \frac{4(n+1)(n+2)}{(2n+1)(2n+3)} \left[\frac{1}{3} n(n+1) - m^2\right] e^2\right\}$$
$$\times \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}.$$

11. Hence the final equation for α_n becomes

$$\alpha_n \left\{2(n+1)\left(1 + \frac{n+1}{3} e^2\right) + \frac{4(n+1)(n+2)}{(2n+1)(2n+3)} \left[\frac{1}{3} n(n+1) - m^2\right] e^2\right\}$$
$$\times \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}$$

$$-\beta_n \,.\, e^2 \frac{2[n(n+1) - 3m^2]}{(2n-1)(2n+1)(2n+3)} \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\ldots(2n-1)\}^2}$$

$$+\alpha_{n+2} \,.\, e^2 \frac{2(n+2)(n+3)(2n+1)}{2n+5} \frac{(n+m+2)!\,(n-m+2)!}{\{1\,.\,3\,.\,5\ldots(2n+3)\}^2}$$

$$+\beta_{n+2} \,.\, e^2 \frac{2}{2n+5} \frac{(n+m+2)!\,(n-m+2)!}{\{1\,.\,3\,.\,5\,\ldots(2n+3)\}^2}.$$

$$+\alpha_{n-2} \,.\, e^2 \frac{2n(n+1)(2n-3)}{2n+1} \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,\ldots\,(2n-1)\}^2}$$

= a known quantity of the form

$$\int_{-1}^{1} x_m' X_n' (w)\, d\mu + \int_{-1}^{1} y_m' Y_n' (w)\, d\mu + \int_{-1}^{1} z_m' Z_n' (w)\, d\mu,$$

where $(w)\, d\mu$ represents the weight, and where x_m', y_m' and z_m' are derived from the observations of the horizontal and vertical magnetic forces.

Similarly the final equation for β_n becomes

$$- \alpha_n \,.\, e^2 \frac{2\,[n(n+1) - 3m^2]}{(2n-1)(2n+1)(2n+3)} \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,...\,(2n-1)\}^2}$$

$$+ \beta_n \left\{ 2n \left(1 - \frac{n}{3} e^2\right) - e^2 \frac{4n(n-1)}{(2n-1)(2n+1)} \left[\frac{1}{3} n(n+1) - m^2\right] \right\} \times \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,...\,(2n-1)\}^2}$$

$$+ \alpha_{n-2} \,.\, e^2 \frac{2}{2n+1} \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,...\,(2n-1)\}^2}$$

$$- \beta_{n+2} \,.\, e^2 \,.\, 2n(n+1) \frac{(n+m+2)!\,(n-m+2)!}{\{1\,.\,3\,.\,5\,...\,(2n+3)\}^2}$$

$$- \beta_{n-2} \,.\, e^2 \,.\, 2(n-1)(n-2) \frac{(n+m)!\,(n-m)!}{\{1\,.\,3\,.\,5\,...\,(2n-1)\}^2}$$

= a known quantity of the form

$$\int_{-1}^{1} x_m' X'_{-n} (w)\, d\mu + \int_{-1}^{1} y_m' Y'_{-n} (w)\, d\mu + \int_{-1}^{1} z_m' Z'_{-n} (w)\, d\mu.$$

In the same way the other final equations for α_{n-2}, β_{n+2}, &c. may be formed from the equations of condition, and the coefficients of the other magnetic constants in all the final equations determined.

Also it appears that in the final equation for each magnetic constant, for a given value of m, there will only be five unknown magnetic constants when the equations for X, Y and Z are combined, since the coefficients of the other magnetic constants will severally vanish.

12. The coefficient of α_n in the final equation for α_n (as found from the

coefficient of α_{n_1}; see Art. 8) will be

$$\int_{-1}^{1}[1+(n+1)e^2\mu^2]\left\{(1-\mu'^2)\left(\frac{dH_n'}{d\mu}\right)^2+(n+1)^2(H_n')^2+\frac{m^2(H_n')^2}{1-\mu'^2}\right\}d\mu'$$

$$=[1+(n+1)e^2]\int_{-1}^{1}\left[(1-\mu^2)\left(\frac{dH_n}{d\mu}\right)^2+(n+1)^2(H_n)^2+\frac{m^2(H_n)^2}{1-\mu^2}\right]d\mu$$

$$-(n+1)e^2\int_{-1}^{1}\left[(1-\mu^2)^2\left(\frac{dH_n}{d\mu}\right)^2+m^2(H_n)^2+(n+1)^2(1-\mu^2)(H_n)^2\right]d\mu.$$

Hence we see from the investigations of these integrals on pp. 421 and 427, that the coefficient of α_n in the final equation for α_n, arising from combining X, Y and Z, will be

$$\frac{(n-m)!\,(n+m)!}{\{1.3.5\ldots(2n-1)\}^2}\,2(n+1)$$
$$\times\left\{1+(n+1)e^2-e^2\frac{2}{(2n+1)(2n+3)}[(n+1)^3+m^2(n+2)]\right\}.$$

Similarly the coefficient of β_n in the final equation for β_n is

$$\int_{-1}^{1}(1-ne^2\mu^2)\left\{(1-\mu'^2)\left(\frac{dH_n'}{d\mu}\right)^2+n^2(H_n')^2+\frac{m^2(H_n')^2}{1-\mu'^2}\right\}d\mu'$$

$$=(1-ne^2)\int_{-1}^{1}\left[(1-\mu^2)\left(\frac{dH_n}{d\mu}\right)^2+n^2(H_n)^2+\frac{m^2(H_n)^2}{1-\mu^2}\right]d\mu$$

$$+ne^2\int_{-1}^{1}\left[(1-\mu^2)^2\left(\frac{dH_n}{d\mu}\right)^2+m^2(H_n)^2+n^2(1-\mu^2)(H_n)^2\right]d\mu.$$

Hence the coefficient of β_n in the combined final equation for β_n is

$$2n\frac{(n-m)!\,(n+m)!}{\{1.3.5\ldots(2n-1)\}^2}\left[1-ne^2+e^2\frac{2}{(2n-1)(2n+1)}\{n^3+m^2(n-1)\}\right].$$

If the polar radius instead of the equatorial radius be taken as the unit of length, then we must multiply the coefficient of α_n in the final equation for α_n by $(1-e^2)^{n+1}$ or $1-(n+1)e^2$, and we multiply the coefficient of β_n in the final equation for β_n by $(1-e^2)^{-n}$ or $1+ne^2$, and the equations are somewhat simplified.

13. Hence when the polar radius of the Earth is taken as the unit of length, the final equation for α_n for a given value of m becomes

$$\frac{2(n-m)!\,(n+m)!}{[1.3.5\ldots(2n-1)]^2}\left\{\alpha_n(n+1)\left[1-e^2\frac{2}{(2n+1)(2n+3)}\{(n+1)^2+(n+2)m^2\}\right]\right.$$
$$-\beta_n e^2\frac{n(n+1)-3m^2}{(2n-1)(2n+1)(2n+3)}$$
$$+\alpha_{n-2}e^2\frac{n(n+1)(2n-3)}{2n+1}$$
$$+\alpha_{n+2}e^2\frac{(n+2)(n+3)(n+m+1)(n+m+2)(n-m+1)(n-m+2)}{(2n+1)(2n+3)^2(2n+5)}$$
$$\left.+\beta_{n+2}e^2\frac{(n+m+1)(n+m+2)(n-m+1)(n-m+2)}{(2n+1)^2(2n+3)^2(2n+5)}\right\}$$

$=$a known quantity of the form

$$\int_{-1}^{1}x_m'X_n'(w)\,d\mu+\int_{-1}^{1}y_m'Y_n'(w)\,d\mu+\int_{-1}^{1}z_m'Z_n'(w)\,d\mu.$$

Similarly the final equation for β_n becomes

$$\frac{2(n-m)!\,(n+m)!}{[1.3.5\ldots(2n-1)]^2}\left\{\beta_n n\left[1+e^2\frac{2}{(2n-1)(2n+1)}\{n^2+(n-1)m^2\}\right]\right.$$
$$-\alpha_n e^2\frac{n(n+1)-3m^2}{(2n-1)(2n+1)(2n+3)}$$
$$+\alpha_{n-2}e^2\frac{1}{2n+1}-\beta_{n-2}e^2(n-1)(n-2)$$
$$\left.-\beta_{n+2}e^2\frac{n(n+1)(n+m+1)(n+m+2)(n-m+1)(n-m+2)}{(2n+1)^2(2n+3)^2}\right\}$$

$=$a known quantity of the form

$$\int_{-1}^{1}x_m'X'_{-n}(w)\,d\mu+\int_{-1}^{1}y_m'Y'_{-n}(w)\,d\mu+\int_{-1}^{1}z_m'Z'_{-n}(w)\,d\mu.$$

14. If we had expressed the magnetic potential V and the magnetic forces X, Y and Z in terms of the functions Q_n, Q_{n_1}, &c., instead of in terms of H_n, H_{n_1}, &c., we should have obtained another series of magnetic constants; but the two series are related to one another, and the one series may be derived from the other by multiplying each constant in one series by a factor depending on the values of n and m to get the corresponding magnetic constant for the other series.

Thus let a_n and b_n be two magnetic constants derived from the function Q_n (as defined above), and let α_n and β_n be the corresponding magnetic Gaussian constants as derived from the function H_n. Then these magnetic constants a_n and b_n are connected with α_n and β_n by the relations

$$\frac{\alpha_n}{a_n} = \frac{\beta_n}{b_n} = \frac{Q_n}{H_n} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{(n-m)\,!}$$

for a given value of m, and similarly

$$\frac{\alpha_{n_1}}{a_{n_1}} = \frac{\beta_{n_1}}{b_{n_1}} = \frac{Q_{n_1}}{H_{n_1}} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n_1-1)}{(n_1-m)\,!},$$

and in particular

$$\frac{\alpha_{n-2}}{a_{n-2}} = \frac{\beta_{n-2}}{b_{n-2}} = \frac{Q_{n-2}}{H_{n-2}} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-5)}{(n-m-2)\,!},$$

and

$$\frac{\alpha_{n+2}}{a_{n+2}} = \frac{\beta_{n+2}}{b_{n+2}} = \frac{Q_{n+2}}{H_{n+2}} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n+3)}{(n-m+2)\,!}.$$

We may find the final equations for a_n and b_n from the final equations for α_n and β_n respectively by multiplying them by $\dfrac{Q_n}{H_n}$, and then substituting the values of α_n and β_n in terms of a_n and b_n respectively.

Hence in the final equations for α_n and β_n the coefficient of α_n or of β_n will be multiplied by

$$\left(\frac{Q_n}{H_n}\right)^2 \text{ or } \left(\frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{(n-m)\,!}\right)^2$$

in order to find the coefficients of a_n and b_n respectively. Also the coefficient of α_{n-2} or of β_{n-2} in the same equations will be multiplied by

$$\frac{Q_n\,Q_{n-2}}{H_n H_{n-2}} \text{ or } \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)\; 1 \,.\, 3 \,.\, 5 \ldots (2n-5)}{(n-m)\,!\;(n-m-2)\,!}$$

to find the coefficients of a_{n-2} and b_{n-2} respectively. And the coefficients of α_{n+2} and β_{n+2} will be multiplied by

$$\frac{Q_n\,Q_{n+2}}{H_n H_{n+2}} \text{ or } \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)\, 1 \,.\, 3 \,.\, 5 \ldots (2n+3)}{(n-m)\,!\;(n-m+2)\,!}$$

to find the coefficients of a_{n+2} and b_{n+2}. Or generally the coefficients of α_{n_1} and β_{n_1} in the final equations for α_n and β_n will be multiplied by $\dfrac{Q_n\,Q_{n_1}}{H_n H_{n_1}}$ to find the coefficients of a_{n_1} and b_{n_1} in the corresponding final equations.

Hence the constants a_n and b_n will have to be multiplied by $\frac{Q_n}{H_n}$, i.e. by the factor

$$\frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{(n-m)!},$$

in order to obtain the corresponding Gaussian constants α_n and β_n.

Again, let A_n, B_n be two magnetic constants connected with α_n and β_n by the relations

$$\frac{\alpha_n}{A_n} = \frac{\beta_n}{B_n} = \frac{\Pi_n}{H_n} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{[(n-m)!\,(n+m)!]^{\frac{1}{2}}}.$$

Then the values of the magnetic constants A_n, B_n, &c., as determined from the function Π_n, can be converted into the corresponding Gaussian magnetic constants derived by means of the function H_n by multiplying each magnetic constant A_n or B_n for each value of m by the factor

$$\frac{\Pi_n}{H_n} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{[(n-m)!\,(n+m)!]^{\frac{1}{2}}}.$$

Also in the final equations for α_n and β_n the coefficients of α_n or of β_n will be multiplied by

$$\left(\frac{\Pi_n}{H_n}\right)^2 \quad \text{or} \quad \frac{[1 \,.\, 3 \,.\, 5 \ldots (2n-1)]^2}{(n-m)!\,(n+m)!}$$

in order to find the coefficients of A_n and B_n respectively.

Also the coefficients of α_{n-2} or of β_{n-2} in the same equations will be multiplied by $\frac{\Pi_n \Pi_{n-2}}{H_n H_{n-2}}$ to find the coefficients of A_{n-2} or of B_{n-2} respectively.

Also the coefficients of α_{n+2} or of β_{n+2} will be multiplied by $\frac{\Pi_n \Pi_{n+2}}{H_n H_{n+2}}$ to find the coefficients of A_{n+2} or of B_{n+2} respectively.

From the final equations for the determination of the Gaussian constants α_n, β_n, &c., and taking the equatorial radius of the Earth as unity, we may write down the final equations for the determination of a_n, b_n, &c., where

$$\frac{\alpha_n}{a_n} = \frac{\beta_n}{b_n} = \frac{Q_n}{H_n}, \quad \text{\&c.}$$

Hence the final equation for a_n becomes

$$2\left\{a_n(n+1)\left[1+(n+1)e^2-e^2\frac{2}{(2n+1)(2n+3)}\{(n+1)^3+(n+2)m^2\}\right]\right.$$

$$\left.-b_ne^2\frac{n(n+1)-3m^2}{(2n-1)(2n+1)(2n+3)}\right\}\frac{(n+m)!}{(n-m)!}$$

$$+2a_{n-2}e^2\frac{n(n+1)}{(2n-1)(2n+1)}\frac{(n+m)!}{(n-m-2)!}$$

$$+2\left[a_{n+2}e^2\frac{(n+2)(n+3)}{(2n+3)(2n+5)}+b_{n+2}e^2\frac{1}{(2n+1)(2n+3)(2n+5)}\right]\frac{(n+m+2)!}{(n-m)!}$$

$$=\frac{1.3.5\ldots(2n-1)}{(n-m)!}\left[\int_{-1}^{1}x_m'X_n'(w)\,d\mu+\int_{-1}^{1}y_m'Y_n'(w)\,d\mu+\int_{-1}^{1}z_m'Z_n'(w)\,d\mu\right].$$

Also the final equation for b_n will be

$$2\left\{b_nn\left[1-ne^2+e^2\frac{2}{(2n-1)(2n+1)}\{n^3+(n-1)m^2\}\right]\right.$$

$$\left.-a_ne^2\frac{n(n+1)-3m^2}{(2n-1)(2n+1)(2n+3)}\right\}\frac{(n+m)!}{(n-m)!}$$

$$+2\left[a_{n-2}e^2\frac{1}{(2n-3)(2n-1)(2n+1)}-b_{n-2}e^2\frac{(n-2)(n-1)}{(2n-3)(2n-1)}\right]\frac{(n+m)!}{(n-m-2)!}$$

$$-2b_{n+2}e^2\frac{n(n+1)}{(2n+1)(2n+3)}\frac{(n+m+2)!}{(n-m)!}$$

$$=\frac{1.3.5\ldots(2n-1)}{(n-m)!}\times\left[\int_{-1}^{1}x_m'X'_{-n}(w)\,d\mu+\int_{-1}^{1}y_m'Y'_{-n}(w)d\mu+\int_{-1}^{1}z_m'Z'_{-n}(w)d\mu\right].$$

In these equations x_m', y_m' and z_m' are quantities derived from the magnetic observations.

Also since $$\frac{a_n}{A_n}=\frac{b_n}{B_n}=\frac{\Pi_n}{Q_n}=\left[\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}},$$

we have $$\frac{\Pi_n\Pi_{n-2}}{Q_nQ_{n-2}}=\left[\frac{(n-m)!\,(n-m-2)!}{(n+m)!\,(n+m-2)!}\right]^{\frac{1}{2}},$$

and $$\frac{\Pi_n\Pi_{n+2}}{Q_nQ_{n+2}}=\left[\frac{(n-m)!\,(n-m+2)!}{(n+m)!\,(n+m+2)!}\right]^{\frac{1}{2}}.$$

Hence the final equation for A_n becomes

$$A_n(n+1)\left[1+(n+1)e^2-e^2\frac{2}{(2n+1)(2n+3)}\{(n+1)^3+(n+2)m^2\}\right]$$

$$-B_n e^2\frac{n(n+1)-3m^2}{(2n-1)(2n+1)(2n+3)}$$

$$+A_{n-2}e^2\frac{n(n+1)}{(2n-1)(2n+1)}$$

$$\times[(n-m-1)(n-m)(n+m-1)(n+m)]^{\frac{1}{2}}$$

$$+\left[A_{n+2}e^2\frac{(n+2)(n+3)}{(2n+3)(2n+5)}+B_{n+2}e^2\frac{1}{(2n+1)(2n+3)(2n+5)}\right]$$

$$\times[(n-m+1)(n-m+2)(n+m+1)(n+m+2)]^{\frac{1}{2}}$$

$$=\frac{1.3.5\ldots(2n-1)}{2[(n-m)!\,(n+m)!]^{\frac{1}{2}}}\left[\int_{-1}^{1}x_m'X_n'(w)\,d\mu+\int_{-1}^{1}y_m'Y_n'(w)\,d\mu+\int_{-1}^{1}z_m'Z_n'(w)\,d\mu\right].$$

And the final equation for B_n becomes

$$B_n n\left[1-ne^2+e^2\frac{2}{(2n-1)(2n+1)}\{n^3+(n-1)m^2\}\right]$$

$$-A_n e^2\frac{n(n+1)-3m^2}{(2n-1)(2n+1)(2n+3)}$$

$$+\left[A_{n-2}e^2\frac{1}{(2n-3)(2n-1)(2n+1)}-B_{n-2}e^2\frac{(n-1)(n-2)}{(2n-3)(2n-1)}\right]$$

$$\times[(n-m-1)(n-m)(n+m-1)(n+m)]^{\frac{1}{2}}$$

$$-B_{n+2}e^2\frac{n(n+1)}{(2n+1)(2n+3)}$$

$$\times[(n-m+1)(n-m+2)(n+m+1)(n+m+2)]^{\frac{1}{2}}$$

$$=\frac{1.3.5\ldots(2n-1)}{2[(n-m)!\,(n+m)!]^{\frac{1}{2}}}\left[\int_{-1}^{1}x_m'X'_{-n}(w)\,d\mu+\int_{-1}^{1}y_m'Y'_{-n}(w)\,d\mu+\int_{-1}^{1}z_m'Z'_{-n}(w)\,d\mu\right].$$

SECTION VII.

NUMERICAL CALCULATION OF THE MAGNETIC FORCES ON THE EARTH'S SURFACE—REGARDED AS A SPHEROID.

1. *Expressions for the Magnetic Forces at the Equator* ($\mu = 0$).

SINCE

$$G_n^{m-1} = (-1)^{\frac{n-m+1}{2}} \frac{1 \,.\, 3 \,.\, 5 \ldots (n-m)\, 1 \,.\, 3 \,.\, 5 \ldots (n+m-2)}{1 \,.\, 3 \,.\, 5 \ldots (2n-1)},$$

and

$$G_n^{m+1} = (-1)^{\frac{n-m-1}{2}} \frac{1 \,.\, 3 \,.\, 5 \ldots (n-m-2)\, 1 \,.\, 3 \,.\, 5 \ldots (n+m)}{1 \,.\, 3 \,.\, 5 \ldots (2n-1)},$$

we have

$$(n-m)\, G_n^{m+1} = -(n+m)\, G_n^{m-1} = X_n^m = X_{-n}^m,$$

$$mG_n^m = Y_n^m = Y_{-n}^m,$$

$$(n+1)\, G_n^m = Z_n^m \text{ and } -nG_n^m = Z_{-n}^m.$$

If $n-m$ is odd, the value of $G_n^m = 0$, and the forces Y_n, Y_{-n}^m, Z_n and Z_{-n}^m vanish.

If $n-m$ is even and $=2r$,

$$G_n^m = (-1)^r \frac{1 \,.\, 3 \,.\, 5 \ldots (2r-1)\, 1 \,.\, 3 \,.\, 5 \ldots (2n-2r-1)}{1 \,.\, 3 \,.\, 5 \ldots (2n-1)},$$

and the forces X_n^m and X_{-n}^m vanish.

For $m=0$ and $n=2r$,

$$G_n^0 = (-1)^r \frac{\{1 \,.\, 3 \,.\, 5 \ldots (2r-1)\}^2}{1 \,.\, 3 \,.\, 5 \ldots (4r-1)} = -\frac{(2r-1)^2}{(4r-3)(4r-1)}\, G_{n-2}^0.$$

For $m=1$ and $n=2r+1$,

$$G_n^1 = (-1)^r \frac{1 \,.\, 3 \,.\, 5 \ldots (2r-1)\, 1 \,.\, 3 \,.\, 5 \ldots (2r+1)}{1 \,.\, 3 \,.\, 5 \ldots (4r+1)} = -\frac{(2r-1)(2r+1)}{(4r-1)(4r+1)}\, G_{n-}^1.$$

For $m=2$ and $n=2r+2$,

$$G_n^2 = (-1)^r \frac{1 \,.\, 3 \,.\, 5 \ldots (2r-1)\, 1 \,.\, 3 \,.\, 5 \ldots (2r+3)}{1 \,.\, 3 \,.\, 5 \ldots (4r+3)} = -\frac{(2r-1)(2r+3)}{(4r+1)(4r+3)}\, G_{n-2}^2.$$

For $m=3$ and $n=2r+3$,

$$G_n^3=(-1)^r \frac{1.3.5\ldots(2r-1)\,1.3.5\ldots(2r+5)}{1.3.5\ldots(4r+5)}=-\frac{(2r-1)(2r+5)}{(4r+3)(4r+5)}G_{n-2}^3.$$

For $m=4$ and $n=2r+4$,

$$G_n^4=(-1)^r \frac{1.3.5\ldots(2r-1)\,1.3.5\ldots(2r+7)}{1.3.5\ldots(4r+7)}=-\frac{(2r-1)(2r+7)}{(4r+5)(4r+7)}G_{n-2}^4.$$

The law of formation of these quantities is evident, and their numerical values for all values of m and n from 0 to 10 have been determined from these formulae.

The numerical calculation of these functions from one another in succession is greatly simplified by putting $n-1=x$, when we get the following relations :—

$$G_n^0=-\frac{x^2}{4x^2-1}G_{n-2}^0,$$

$$G_n^1=-\frac{x^2-1}{4x^2-1}G_{n-2}^1,$$

$$G_n^2=-\frac{x^2-4}{4x^2-1}G_{n-2}^2,$$

$$G_n^3=-\frac{x^2-9}{4x^2-1}G_{n-2}^3,$$

$$G_n^4=-\frac{x^2-16}{4x^2-1}G_{n-2}^4,$$

and generally

$$G_n^m=-\frac{x^2-m^2}{4x^2-1}G_{n-2}^m.$$

2. *Values of the Expressions for the Magnetic Forces at the Pole* ($\mu=1$).

Here we have

$$G_n^0=\frac{n!}{1.3.5\ldots(2n-1)}=\frac{n}{2n-1}G_{n-1}^0,$$

$$G_n^1=\frac{1}{2}\frac{(n+1)!}{1.3.5\ldots(2n-1)}=\frac{n+1}{2n-1}G_{n-1}^1,$$

$$G_n^2=\frac{1}{2^2}\frac{1}{1.2}\frac{(n+2)!}{1.3.5\ldots(2n-1)}=\frac{n+2}{2n-1}G_{n-1}^2,$$

and generally

$$G_n^m = \frac{1}{2^m}\frac{1}{m!}\frac{(n+m)!}{1\,.\,3\,.\,5\,\ldots\,(2n-1)} = \frac{n+m}{2n-1}G_{n-1}^m.$$

The value of X_n^m is always 0, except when $m=1$.

The value of Y_n^m is always 0, except when $m=1$,

and the value of Z_n^m is always 0, except when $m=0$.

$$X_n^1 = -\frac{\mu}{r^{n+2}}G_n^1, \text{ and } X_{-n}^1 = -\mu r^{n-1}G_n^1,$$

$$Y_n^1 = \frac{1}{r^{n+2}}G_n^1, \qquad Y_{-n}^1 = r^{n-1}G_n^1,$$

$$Z_n^0 = \frac{n+1}{r^{n+2}}G_n^0, \qquad Z_{-n}^0 = -nr^{n-1}G_n^0.$$

$$r^2 = 1-e^2 \text{ and } \log\left(\frac{1}{r}\right) = 0{\cdot}0014542.$$

The logarithms of the values of the coefficients X_n^m, Y_n^m, Z_n^m at the Pole for $m=0$, $m=1$ and for the several values of n are here given.

	Log Y_n^1		Log Y^1_{-n}		Log Z_n^0		Log Z^0_{-n}
$Y_1^1=-X_1^1$	0·0043626	Y^1_{-1}	0·	Z_1^0	0·3053926	Z^0_{-1}	n 0·
$Y_2^1=-X_2^1$	0·0058168	Y^1_{-2}	9·9985458	Z_2^0	0·3068468	Z^0_{-2}	n 0·1234845
$Y_3^1=-X_3^1$	9·9103610	Y^1_{-3}	9·9001816	Z_3^0	0·2113910	Z^0_{-3}	n 0·0762729
$Y_4^1=-X_4^1$	9·7656872	Y^1_{-4}	9·7525993	Z_4^0	0·0667172	Z^0_{-4}	n 9·9567193
$Y_5^1=-X_5^1$	9·5910501	Y^1_{-5}	9·5750539	Z_5^0	9·8920801	Z^0_{-5}	n 9·7969026
$Y_6^1=-X_6^1$	9·3962097	Y^1_{-6}	9·3773050	Z_6^0	9·6972397	Z^0_{-6}	n 9·6113883
$Y_7^1=-X_7^1$	9·1868105	Y^1_{-7}	9·1649975	Z_7^0	9·4878405	Z^0_{-7}	n 9·4080355
$Y_8^1=-X_8^1$	8·9664160	Y^1_{-8}	8·9416945	Z_8^0	9·2674460	Z^0_{-8}	n 9·1915720
$Y_9^1=-X_9^1$	8·7374212	Y^1_{-9}	8·7097914	Z_9^0	9·0384512	Z^0_{-9}	n 8·9650639
$Y_{10}^1=-X_{10}^1$	8·5015145	Y^1_{-10}	8·4709763	Z_{10}^0	8·8025445	Z^0_{-10}	n 8·7306136

For X these are to be multiplied by $-(g'\cos\lambda+h'\sin\lambda)$, and for Y they are to be multiplied by $(g'\sin\lambda-h'\cos\lambda)$.

NOTE. In the tables n before a logarithm indicates that the corresponding number is negative.

3. *Calculation of the weights of the observations for every* 5*th degree of geographical colatitude, on the assumption that the weight of any determination of magnetic force is proportional to the area of the surface over which the observations extend.*

If ρ be the radius of curvature of the terrestrial meridian for the geographical colatitude θ, and if N be the normal terminated by the axis of revolution, we have seen that

$$\rho = \frac{1-e^2}{a^2} N^3,$$

and that an element of the surface

$$= 2\pi \frac{1-e^2}{a^2} N^4 \sin\theta d\theta.$$

Take a as unity, then the weight of the observation for any belt

$$= (1-e^2) N^4 \sin\theta d\theta.$$

When the observations extend over a zone of 5° in breadth, the area of the zone will contain the factor $2\sin\frac{\omega}{2}$, where ω is the circular measure of $5° = \frac{\pi}{36}$.

The weight found by the above formula must be divided by $2\sin\frac{\omega}{2}$ or ω in order to reduce it to the scale employed when observations are given to every 5° of latitude. This summation will only extend to latitude 87°·5.

The area of the small circle of radius 2°·5 round the pole is very approximately

$$2\pi N^4 (1-e^2)\left(1-\cos\frac{\omega}{2}\right),$$

hence the weight for this small area

$$= N^4 (1-e^2) \frac{1-\cos\frac{\omega}{2}}{2\sin\frac{\omega}{2}}$$

$$= \frac{1}{2} N^4 (1-e^2) \tan\frac{\omega}{4} = \frac{1}{2(1-e^2)} \tan\frac{\omega}{4}.$$

4. *Table of the values of the common logarithms of*

$$\mu', \quad \frac{a}{r}, \quad \cos\psi, \quad \sin\psi, \quad \frac{N}{r}e^2,$$

and the weight (w) *of the observations for every* 5° *of geographical colatitude.*

Colatitude θ	$\mu' = \cos\theta'$	$L\cos\psi$	$L\sin\psi$	$\log\frac{a}{r}$	$\log\frac{N}{r}e^2$
0°				0·00145,41798	7·82731,87745
5° (a)	9·99832,19863	9·99999,99261	6·76593,67691	0·00144,30230	7·82729,65348
10° (b)	9·99326,31892	9·99999,97134	7·06025,21937	0·00140,99016	7·82723,05047
15° (c)	9·98474,77330	9·99999,93878	7·22506,27380	0·00135,58511	7·82712,27293
20° (d)	9·97264,35843	9·99999,89889	7·33401,40434	0·00128,25582	7·82697,65423
25° (e)	9·95675,33949	9·99999,85652	7·41002,04288	0·00119,23041	7·82679,64579
30° (f)	9·93679,98818	9·99999,81680	7·46308,86606	0·00108,78888	7·82658,80246
35° (g)	9·91240,33923	9·99999,78453	7·49831,34682	0·00097,25389	7·82635,76474
40° (h)	9·88304,75765	9·99999,76361	7·51843,38477	0·00084,98028	7·82611,23844
45° (i)	9·84802,59533	9·99999,75655	7·52482,97299	0·00072,34381	7·82585,97256
50° (k)	9·80635,60827	9·99999,76416	7·51792,88237	0·00059,72936	7·82560,73604
55° (l)	9·75663,54698	9·99999,78552	7·49731,87637	0·00047,51916	7·82536,29430
60° (m)	9·69678,50910	9·99999,81802	7·46163,44972	0·00036,08132	7·82513,38611
65° (n)	9·62355,65018	9·99999,85775	7·40815,09877	0·00025,75897	7·82492,70168
70° (o)	9·53148,15336	9·99999,89992	7·33178,61245	0·00016,86039	7·82474,86234
75° (p)	9·41028,14816	9·99999,93949	7·22254,40342	0·00009,65027	7·82460,40255
80° (q)	9·23684,90010	9·99999,97170	7·05751,92345	0·00004,34238	7·82449,75455
85° (r)	8·93740,95949	9·99999,99271	6·76307,25954	0·00001,09366	7·82443,23610

θ	log (w)	θ	log (w)
0°	8·0407347,	50°	9·8837442,1
5°	8·9431600,5	55°	9·9123655,2
10°	9·2424026,3	60°	9·9360727,9
15°	9·4155137,1	65°	9·9554033,9
20°	9·5362776,0	70°	9·9707558,6
25°	9·6278148,6	75°	9·9824238,6
30°	9·7004205,2	80°	9·9906179,4
35°	9·7595817,0	85°	9·9954799,1
40°	9·8085678,2	90°	9·9970916,4
45°	9·8494801,2		

Calculation of the Values of the Quantities H'^m_n for every 5° of Latitude.

5. *Scheme of Calculation of Quantities H'^m_n from G'^m_n.*

$m=$	-1	0	1	2	3	4	5	6	7	8	9	10
	$\log(1-\mu'^2)^{-\frac{1}{2}}$		$\log(1-\mu'^2)^{\frac{1}{2}}$	$\log(1-\mu'^2)$	$\log(1-\mu'^2)^{\frac{3}{2}}$	$\log(1-\mu'^2)^2$	$\log(1-\mu'^2)^{\frac{5}{2}}$	$\log(1-\mu'^2)^3$	$\log(1-\mu'^2)^{\frac{7}{2}}$	$\log(1-\mu'^2)^4$	$\log(1-\mu'^2)^{\frac{9}{2}}$	$\log(1-\mu'^2)^5$
n												
$n=1$	$\log G_1'^{-1}$	$\log G_1'^{0}$	$\log G_1'^{1}$									
2	$\log G_2'^{-1}$	$\log G_2'^{0}$	$\log G_2'^{1}$	$\log G_2'^{2}$								
3	$\log G_3'^{-1}$	$\log G_3'^{0}$	$\log G_3'^{1}$	$\log G_3'^{2}$	$\log G_3'^{3}$							
4	$\log G_4'^{-1}$	$\log G_4'^{0}$	$\log G_4'^{1}$	$\log G_4'^{2}$	$\log G_4'^{3}$	$\log G_4'^{4}$						
5	$\log G_5'^{-1}$	$\log G_5'^{0}$	$\log G_5'^{1}$	$\log G_5'^{2}$	$\log G_5'^{3}$	$\log G_5'^{4}$	$\log G_5'^{5}$					
6	$\log G_6'^{-1}$	$\log G_6'^{0}$	$\log G_6'^{1}$	$\log G_6'^{2}$	$\log G_6'^{3}$	$\log G_6'^{4}$	$\log G_6'^{5}$	$\log G_6'^{6}$				
7	$\log G_7'^{-1}$	$\log G_7'^{0}$	$\log G_7'^{1}$	$\log G_7'^{2}$	$\log G_7'^{3}$	$\log G_7'^{4}$	$\log G_7'^{5}$	$\log G_7'^{6}$	$\log G_7'^{7}$			
8	$\log G_8'^{-1}$	$\log G_8'^{0}$	$\log G_8'^{1}$	$\log G_8'^{2}$	$\log G_8'^{3}$	$\log G_8'^{4}$	$\log G_8'^{5}$	$\log G_8'^{6}$	$\log G_8'^{7}$	$\log G_8'^{8}$		
9	$\log G_9'^{-1}$	$\log G_9'^{0}$	$\log G_9'^{1}$	$\log G_9'^{2}$	$\log G_9'^{3}$	$\log G_9'^{4}$	$\log G_9'^{5}$	$\log G_9'^{6}$	$\log G_9'^{7}$	$\log G_9'^{8}$	$\log G_9'^{9}$	
10	$\log G_{10}'^{-1}$	$\log G_{10}'^{0}$	$\log G_{10}'^{1}$	$\log G_{10}'^{2}$	$\log G_{10}'^{3}$	$\log G_{10}'^{4}$	$\log G_{10}'^{5}$	$\log G_{10}'^{6}$	$\log G_{10}'^{7}$	$\log G_{10}'^{8}$	$\log G_{10}'^{9}$	$\log G_{10}'^{10}$
1	$\log H_1'^{-1}$	$\log H_1'^{0}$	$\log H_1'^{1}$									
2	$\log H_2'^{-1}$	$\log H_2'^{0}$	$\log H_2'^{1}$	$\log H_2'^{2}$								
3	$\log H_3'^{-1}$	$\log H_3'^{0}$	$\log H_3'^{1}$	$\log H_3'^{2}$	$\log H_3'^{3}$							
4	$\log H_4'^{-1}$	$\log H_4'^{0}$	$\log H_4'^{1}$	$\log H_4'^{2}$	$\log H_4'^{3}$	$\log H_4'^{4}$						
5	$\log H_5'^{-1}$	$\log H_5'^{0}$	$\log H_5'^{1}$	$\log H_5'^{2}$	$\log H_5'^{3}$	$\log H_5'^{4}$	$\log H_5'^{5}$					
6	$\log H_6'^{-1}$	$\log H_6'^{0}$	$\log H_6'^{1}$	$\log H_6'^{2}$	$\log H_6'^{3}$	$\log H_6'^{4}$	$\log H_6'^{5}$	$\log H_6'^{6}$				
7	$\log H_7'^{-1}$	$\log H_7'^{0}$	$\log H_7'^{1}$	$\log H_7'^{2}$	$\log H_7'^{3}$	$\log H_7'^{4}$	$\log H_7'^{5}$	$\log H_7'^{6}$	$\log H_7'^{7}$			
8	$\log H_8'^{-1}$	$\log H_8'^{0}$	$\log H_8'^{1}$	$\log H_8'^{2}$	$\log H_8'^{3}$	$\log H_8'^{4}$	$\log H_8'^{5}$	$\log H_8'^{6}$	$\log H_8'^{7}$	$\log H_8'^{8}$		
9	$\log H_9'^{-1}$	$\log H_9'^{0}$	$\log H_9'^{1}$	$\log H_9'^{2}$	$\log H_9'^{3}$	$\log H_9'^{4}$	$\log H_9'^{5}$	$\log H_9'^{6}$	$\log H_9'^{7}$	$\log H_9'^{8}$	$\log H_9'^{9}$	
10	$\log H_{10}'^{-1}$	$\log H_{10}'^{0}$	$\log H_{10}'^{1}$	$\log H_{10}'^{2}$	$\log H_{10}'^{3}$	$\log H_{10}'^{4}$	$\log H_{10}'$[illegible]	$\log H_{10}'^{6}$	$\log H_{10}'^{7}$	$\log H_{10}'^{8}$	$\log H_{10}'^{9}$	$\log H_{10}'^{10}$

CALCULATION OF THE QUANTITIES H'^m_n.

	−1	0	1	2	3	4	5	6	7	8	9	10
(a)	1·0568179	0·	−10 8·9431821	7·8863643	6·8295464	5·7727285	4·7159106	3·6590928	2·6022749	1·5454570	0·4886392	−20 9·4318213
1	n 7·8863643		0·									
2	n 7·8846863		9·9983220	0·								
3	n 7·7852552		9·8988909	9·9983220	0·							
4	n 7·6357581		9·7493938	9·9291353	9·9983220	0·						
5	n 7·4554430		9·5690787	9·8171869	9·9450701	9·9983220	0·					
6	n 7·2540616		9·3676973	9·6755134	9·8553980	9·9549142	9·9983220	0·				
7	n 7·0372481		9·1508838	9·5118717	9·7385086	9·8800106	9·9616010	9·9983220	0·			
8	n 6·8085534		8·9221891	9·3312655	9·6004383	9·7804090	9·8972128	9·9664400	9·9983220	0·		
9	n 6·5703582		8·6839939	9·1371157	9·4453711	9·6608721	9·8103697	9·9099225	9·9701044	9·9983220	0·	
10	n 6·3243367		8·4379724	8·9318655	9·2763313	9·5248740	9·7049024	9·8328874	9·9197001	9·9729757	9·9983220	0·
1	n 8·9431822	9·9983220	8·9431821									
2	n 8·9415042	9·8188649	8·9415041	7·8863643								
3	n 8·8420731	9·5919427	8·8420730	7·8846863	6·8295464							
4	n 9·6925760	9·3420933	8·6925759	7·8154996	6·8278684	5·7727285						
5	n 8 5122609	9·0782298	8·5122608	7·7035512	6·7746165	7·7710505	4·7159106					
6	n 8·3108795	8·8045633	8·3108794	7·5618777	6·6849444	5·7276427	4·7142326	3·6590928				
7	n 8·0940660	8·5233900	8·0940659	7·3982360	6·5680550	5·6527391	4·6775116	3·6574148	2·6022749			
8	n 7·8653713	8·2360735	7·8653712	7·2176298	6·4299847	5·5531375	4·6131234	3·6255328	2·6005969	1·5454570		
9	n 7 6271761	7·9434648	7·6271760	7·0234800	6·2749175	5·4336006	4·5262803	3·5690153	2·5723793	1·5437790	0·4886392	
10	n 7·3811546	7·6460899	7·3811545	6·8182298	6·1058777	5·2976025	4·4208130	3·4919802	2·5219750	1·5184327	0·4869612	9·4318213

	−1	0	1	2	3	4	5	6	7	8	9	10
(b)	0·7575097	0·	9·2424903	8·4849806	7·7274710	6·9699613	6·2124516	5·4549419	4·6974322	3·9399226	3·1824129	2·4249032
1	n 8·4849806		0·									
2	n 8·4782437		9·9932631 +	0·								
3	n 8·3711622		9·8861816	9·9932631 +	0·							
4	n 8·2113453		9·7263647	9·9172928	9·9932631 +	0·						
5	n 8·0179325		9·5329519	9·7968015	9·9336600	9·9932631 +	0·					
6	n 7·8005303		9·3155497	9·6447815	9·8363245	9·9437631	9·9932631 +	0·				
7	n 7·5645847		9·0796041	9·4689347	9·7104364	9·8617213	9·9506224	9·9932631 +	0·			
8	n 7·3134059		8·8284253	9·2741964	9·5620035	9·7539208	9·8794448	9·9555845	9·9932631 +	0·		
9	n 7·0490629		8·5640823	9·0639062	9·3951761	9·6251076	9·7849328	9·8925263	9·9593413	9·9932631 +	0·	
10	n 6·7728293		8·2878487	8·8404072	9·2129394	9·4787360	9·6709058	9·8081992	9·9025824	9 9622844	9·9932631 +	0·
1	n 9·2424903	9·9932631 +	9·2424903									
2	n 9·2357534	9·8035383	9·2357534 +	8·4849806								
3	n 9·1286719	9·5608214	9·1286719	8·4782437 +	7·7274710							
4	n 8·9688550	9·2891268	8·9688550	8·4022734	7·7207341 +	6·9699613						
5	n 8·7754422	8·9965673	8·7754422	8·2817821	7·6611310	6·9632244 +	6·2124516					
6	n 8·5580400	8·6861526	8·5580400	8·1297621	7·5637955	6·9137244	6·2057147 +	5·4549419				
7	n 8·3220944	8·3583556	8·3220944	7·9539153	7·4379074	6·8316826	6·1630740	5·4482050 +	4·6974322			
8	n 8·0709156	8·0116592	8·0709156	7·7591770	7·2894745	6·7238821	6·0918964	5·4105264	4·6906953 +	3·9399226		
9	n 7·8065726	7·6420785	7·8065726	7·5488868 −	7·1226471	6·5950689	5·9973844	5·3474682	4·6567735	3·9331857 +	3·1824129	
10	n 7·5303390	7·2411977	7·5303390	7·3253878	6·9404104	6·4486973	5·8833574	5·2631411	4·6000146	3·9022070	3·1756760 +	2 4249032

In the tables, where necessary, 10 has been added to the common logarithm to avoid the use of negative characteristics.

CALCULATION OF THE QUANTITIES H'^{m}_{n} (*continued*).

	− 1	0	1	2	3	4	5	6	7	8	9	10
(*c*)	0·5842915	0·	9·4157085	8·8314171	8 2471256	7·6628342	7·0785427	6·4942513	5·9099598	5·3256684	4·7413769	4·1570855
1	*n* 8·8314171		0·									
2	*n* 8·8161649		9·9847478 −	0·								
3	*n* 8·6960294		9·8646123	9·9847478 −	0·							
4	*n* 8·5182500		9·6868329	9·8972496	9·9847478 −	0·						
5	*n* 8·3014072		9·4699901	9·7620567	9·9143747	9·9847478 −	0·					
6	*n* 8·0542849		9·2228678	9·5919418	9·8039253	9·9249310	9·9847478 −	0·				
7	*n* 7·7811110		8·9496939	9·3943139	9·6624639	9·8307144	9·9320919	9·9847478 −	0·			
8	*n* 7·4833422		8·6519251	9·1737210 +	9·4958528	9·7088105	9·8493601	9·9372691	9·9847478 −	0·		
9	*n* 7·1600983		8·3286812	8·9329854	9·3080617	9·5638793	9·7417073	9·8630972	9·9411869	9·9847478 −	0·	
10	*n* 6·8076157		7·9761986	8·6737524	9·1018484	9·3992730	9·6128971	9·7663090	9·8736436	9·9442552	9·9847478 −	0·
1	*n* 9·4157086	9·9847478 −	9·4157085									
2	*n* 9·4004564	9·7773089	9·4004563 −	8·8314171								
3	*n* 9·2803209	9·5061091	9·2803208	8·8161649 −	8·2471256							
4	*n* 9·1025415	9·1921578	9·1025414	8·7286667	8·2318734 −	7·6628342						
5	*n* 8·8856987	8·8377898	8·8856986	8·5934738	8·1615003	7·6475820 −	7·0785427					
6	*n* 8·6385764	8·4337730	8·6385763	8·4233589	8·0510509	7·5877652	7·0632905 −	6·4942513				
7	*n* 8·3654025	7·9486574	8·3654024	8·2257310	7·9095895	7·4935486	7·0106346	6·4789991 −	5·9099598			
8	*n* 8·0676337	7·2445245	8·0676336	8·0051381 +	7·7429784	7·3716447	6·9279028	6·4315204	5·8947076 −	5·3256684		
9	*n* 7·7443898	*n* 6·7280290	7·7443897	7·7644025	7·5551873	7·2267135	6·8202500	6·3573485	5·8511467	5 3104162 −	4·7413769	
10	*n* 7·3919072	*n* 6·9806850	7·3919071	7·5051695	7·3489740	7·0621072	6·6914398	6·2605603	5·7836034	5·2699236	4·7261247 −	4·1570855

	− 1	0	1	2	3	4	5	6	7	8	9	10
(*d*)	0·4633822	0·	9·5366178	9·0732356	8·6098534	8·1464712	7·6830891	7·2197069	6·7563247	6·2929425	5·8295603	5·3661781
1	*n* 9·0732356		0·									
2	*n* 9·0458792		9·9726436	0·								
3	*n* 8·9067853		9·8335497	9·9726436	0·							
4	*n* 8·7020351		9·6287995	9·8685119	9·9726436	0·						
5	*n* 8·4482386		9·3750030	9·7116604	9·8867842	9·9726436	0·					
6	*n* 8·1507289		9·0774933	9·5141323	9·7571927	9·8980241	9·9726436	0·				
7	*n* 7·8072036		8·7339680	9·2822551 +	9·5925614	9·7861327	9·9056386	9·9726436	0·			
8	*n* 7·4049422		8·3317066	9·0189321	9·3982479	9·6434589	9·8061932	9·9111389	9·9726436	0·		
9	*n* 6·9034264		7·8301908	8·7243685	9·1775264	9·4743751	9·6793143	9·8209318	9·9152985	9·9726436	0·	
10	*n* 6·0880920		7·0148564	8·3957421	8·9321799	9·2818602 +	9·5285808	9·7059938	9·8322242	9·9185545	9·9726436	0·
1	*n* 9·5366178	9·9726436	9·5366178									
2	*n* 9·5092614	9·7390169	9·5092614	9·0732356								
3	*n* 9·3701675	9·4223250	9·3701675	9·0458792	8·6098534							
4	*n* 9·1654173	9·0306175	9·1654173	8·9417475	8·5824970	8·1464712						
5	*n* 8·9116208	8·5262707	8·9116208	8·7848960	8·4966376	8·1191148	7·6830891					
6	*n* 8·6141111	7·6480378	8·6141111	8·5873679	8·3670461	8·0444953	7·6557327	7·2197069				
7	*n* 8·2705858	*n* 7·6316568	8·2705858	8·3554907 +	8·2024148	7·9326039	7·5887277	7·1923505	6·7563247			
8	*n* 7·8683244	*n* 7·7108026	7·8683244	8·0921677	8·0081013	7·7899301	7·4892823	7·1308458	6·7289683	6·2929425		
9	*n* 7·3668086	*n* 7·5739965	7·3668086	7·7976041	7·7873798	7·6208463	7·3624034	7·0406387	6·6716232	6·2655861	5·8295603	
10	*n* 6·5514742	*n* 7·3487526	6·5514742	7·4689777	7·5420333	7·4283314 +	7·2116699	6·9257007	6·5885489	6·2114970	5·8022039	5·3661781

	−1	0	1	2	3	4	5	6	7	8	9	10
(*e*)	0·3716657	0·	9·6283343	9·2566686	8·8850029	8·5133372	8·1416715	7·7700058	7·3983401	7·0266744	6·6550087	6·2833430
1	*n* 9·2566686		0·									
2	*n* 9·2134219		9·9567533+	0·								
3	*n* 9·0486541		9·7919855	9·9567533+	0·							
4	*n* 8·8054311		9·5487625	9·8303084	9·9567533+	0·						
5	*n* 8·4941834		9·2375148	9·6434675	9·8502230	9·9567533+	0·					
6	*n* 8·1057248		8·8490562	9·4061929	9·6944971	9·8624363	9·9567533+	0·				
7	*n* 7·5922147		8·3355461	9·1211707−	9·4972541	9·7266115	9·8706949	9·9567533+	0·			
8	*n* 6·6017677		7·3450991	8·7842877	9·2623041+	9·5551745	9·7487390	9·8766529	9·9567533+	0·		
9	6·7812205		*n* 7·5245519	8·3799343	8·9904392	9·3516628	9·5955041	9·7649306	9·8811545	9·9567533+	0·	
10	6·8102240		*n* 7·5535554	7·8575918	8·6791761	9·1178425	9·4140534	9·6252850	9·7773002	9·8846757	9·9567533+	0·
1	*n* 9·6283343	9·9567533+	9·6283343									
2	*n* 9·5850876	9·6867141	9·5850876+	9·2566686								
3	*n* 9·4203198	9·2980306	9·4203198	9·2134219+	8·8850029							
4	*n* 9·1770968	8·7388099	9·1770968	9·0869770	8·8417562+	8·5133372						
5	*n* 8·8658491	*n* 6·9214784	8·8658491	8·9001361	8·7352259	8·4700905+	8·1416715					
6	*n* 8·4773905	*n* 8·1642011	8·4773905	8·6628615	8·5795000	8·3757735	8·0984248+	7·7700058				
7	*n* 7·9638804	*n* 8·1139901	7·9638804	8·3778393−	8·3822570	8·2399487	8·0123664	7·7267591+	7·3983401			
8	*n* 6·9734334	*n* 7·9085762	6·9734334	8·0409563	8·1473070+	8·0685117	7·8904105	7·6466587	7·3550934+	7·0266744		
9	7·1528862	*n* 7·6096691	*n* 7·1528862	7·6366029	7·8754421	7·8650000	7·7371756	7·5349364	7·2794946	6·9834277+	6·6550087	
10	7·1818897	*n* 7·2182991	*n* 7·1818897	7·1142604	7·5641790	7·6311797	7·5557249	7·3952908	7·1756403	6·9113501	6·6117620+	6·2833430

	−1	0	1	2	3	4	5	6	7	8	9	10
(*f*)	0·2988524	0·	9·7011476	9·4022952	9·1034428	8·8045905	8·5057381	8·2068857	7·9080333	7·6091809	7·3103285	7·0114761
1	*n* 9·4022952		0·									
2	*n* 9·3390951		9·9367999	0·								
3	*n* 9·1406637		9·7383685	9·9367999	0·							
4	*n* 8·8427618		9·4404666	9·7814848	9·9367999	0·						
5	*n* 8·4358227		9·0335275	9·5539543	9·8037092	9·9367999	0·					
6	*n* 7·8264366		8·4241414	9·2584301+	9·6132675	9·8172818	9·9367999	0·				
7	6·9440367		*n* 7·5417415	8·8844152	9·3705410	9·6500475	9·8264358	9·9367999	0·			
8	7·3821394		*n* 7·9798442	8·3876230	9·0745641	9·4395214	9·6751682	9·8330282	9·9367999	0·		
9	7·2711394		*n* 7·8688442	7·4879016	8·7162254	9·1868682	9·4866785	9·6934432	9·8380028	9·9367999	0·	
10	7·0073523		*n* 7·6050571	*n* 7·4997970	8·2675026	8·8897470	9·2626348	9·5210969	9·7073466	9·8418903	9·9367999	0·
1	*n* 9·7011476	9·9367999	9·7011476									
2	*n* 9·6379475	9·6171546	9·6379475	9·4022952								
3	*n* 9·4395161	9·1055340	9·4395161	9·3390951	9·1034428							
4	*n* 9·1416142	7·5733126	9·1416142	9·1837800	9·0402427	8·8045905						
5	*n* 8·7346751	*n* 8·4645805	8·7346751	8·9562495	8·9071520	8·7413904	8·5057381					
6	*n* 8·1252890	*n* 8·4173737	8·1252890	8·6607253+	8·7167103	8·6218723	8·4425380	8·2068857				
7	7·2428891	*n* 8·1837253	*n* 7·2428891	8·2867104	8·4739838	8·4546380	8·3321739	8·1436856	7·9080333			
8	7·6809918	*n* 7·8214415	*n* 7·6809918	7·7899182	8·1780069	8·2441119	8·1809063	8·0399139	7·8448332	7·6091809		
9	7·5699918	*n* 7·2786850	*n* 7·5699918	6·8901968	7·8196682	7·9914587	7·9924166	7·9003289	7·7460361	7·5459808	7·3103285	
10	7·3062047	5·2988531	*n* 7·3062047	*n* 6·9020922	7·3709454	7·6943375	7·7683729	7·7279826	7·6153799	7·4510712	7·2471284	7·0114761

CALCULATION OF THE QUANTITIES H'^{m}_{n} (*continued*).

	−1	0	1	2	3	4	5	6	7	8	9	10
(*g*)	0·2394615	0·	9·7605385	9·5210771	9·2816156	9·0421541	8·8026927	8·5632312	8·3237697	8·0843083	7·8448468	7·6053853
1	*n* 9·5210771		0·									
2	*n* 9·4334805		9·9124034	0·								
3	*n* 9·1913662		9·6702891	9·9124034	0·							
4	*n* 8·8127409		9·2916638	9·7203160	9·9124034	0·						
5	*n* 8·2071849		8·6861078	9·4370762	9·7458049	9·9124034	0·					
6	7·4118278		*n* 7·8907507	9·0499937	9·5093513	9·7612793	9·9124034	0·				
7	7·7820115		*n* 8·2609344	8·4841047	9·2012223	9·5531602	9·7716783	9·9124034	0·			
8	7·6347264		*n* 8·1136493	*n* 6·9888264	8·8036339	9·2888128	9·5826924	9·7791489	9·9124034	0·		
9	7·3073102		*n* 7·7862331	*n* 7·9014529	8·2485568	8·9621685	9·3468804	9·6039940	9·7847763	9·9124034	0·	
10	6·7711554		*n* 7·2500783	*n* 7·7980772	6·4820156	8·5536937	9·0619992 −	9·3884779	9·6201028	9·7891681	9·9124034	0·
1	*n* 9·7605386	9·9124034	9·7605385									
2	*n* 9·6729420	9·5246730	9·6729419	9·5210771								
3	*n* 9·4308277	8·7452099	9·4308276	9·4334805	9·2816156							
4	*n* 9·0522024	*n* 8·6086416	9·0522023	9·2413931	9·1940190	9·0421541						
5	*n* 8·4466464	*n* 8·6750255	8·4466463	8·9581533	9·0274205	8·9545575	8·8026927					
6	7·6512893	*n* 8·4536180	*n* 7·6512892	8·5710708	8·7909669	8·8034334	8·7150961	8·5632312				
7	8·0214730	*n* 8·0537083	*n* 8·0214729	8·0051818	8·4828379	8·5953143	8·5743710	8·4756346	8·3237697			
8	7·8741879	*n* 7·3238500	*n* 7·8741878	*n* 6·5099035	8·0852495	8·3309669	8·3853851	8·3423801	8·2361731	8·0843083		
9	7·5467717	7·0481698	*n* 7·5467716	*n* 7·4225300	7·5301724	8·0043226	8·1495731	8·1672252	8·1085460	7·9967117	7·8448468	
10	7·0106169	7·1589050	*n* 7·0106168	*n* 7·3191543	5·7636312	7·5958478	7·8646919 −	7·9517091	7·9438725	7·8734764	7·7572502	7·6053853

	−1	0	1	2	3	4	5	6	7	8	9	10
(*h*)	0·1902305	0·	9·8097695	9·6195389	9·4293084	9·2390779	9·0488473	8·8586168	8·6683863	8·4781557	8·2879252	8·0976947
1	*n* 9·6195389		0·									
2	*n* 9·5025865		9·8830476	0·								
3	*n* 9·2033868		9·5838479	9·8830476	0·							
4	*n* 8·6929225		9·0733836	9·6441586	9·8830476	0·						
5	6·5601550		*n* 6·9406161	9·2814036	9·6743667	9·8830476	0·					
6	8·0874650		*n* 8·4679261	8·7205554	9·3755925	9·6925506	9·8830476	0·				
7	7·9663611		*n* 8·3468222	*n* 7·4208465	8·9647074	9·4305809	9·7047078	9·8830476	0·			
8	7·6084277		*n* 7·9888888	*n* 8·1504402	8·3368518	9·0881815	9·4668957	9·7134117	9·8830476	0·		
9	6·9108189		*n* 7·2912800	*n* 8·0078118	*n* 7·5176181	8·6320309	9·1654079	9·4927513	9·7199518	9·8830476	0·	
10	*n* 6·5834683		6·9639294	*n* 7·6441529	*n* 7·8667185	7·9077176	8·7853406	9·2189528	9·5121292	9·7250465	9·8830476	0·
1	*n* 9·8097694	9·8830476	9·8097695									
2	*n* 9·6928170	9·3983560	9·6928171	9·6195389								
3	*n* 9·3936173	*n* 8·0986057	9·3936174	9·5025865	9·4293084							
4	*n* 8·8831530	*n* 8·8688424	8·8831531	9·2636975	9·3123560	9·2390779						
5	6·7503855	*n* 8·7266653	*n* 6·7503856	8·9009425	9·1036751	9·1221255	9·0488473					
6	8·2776955	*n* 8·3432313	*n* 8·2776956	8·3400943	8·8049009	8·9316285	8·9318949	8·8586168				
7	8·1565916	*n* 7·5341785	*n* 8·1565917	*n* 7·0403854	8·3940158	8·6696588	8·7535551	8·7416644	8·6683863			
8	7·7986582	7·4661259	*n* 7·7986583	*n* 7·7699791	7·7661602	8·3272594	8·5157430	8·5720285	8·5514339	8·4781557		
9	7·1010494	7·4903940	*n* 7·1010495	*n* 7·6273507	*n* 6·9469265	7·8711088	8·2142552	8·3513681	8·3883381	8·3612033	8·2879252	
10	*n* 6·7736988	7·2120277	6·7736989	*n* 7·2636918	*n* 7·2960269	7·1467955	7·8341879	8·0775696	8·1805155	8·2032022	8·1709728	8·0976947

	−1	0	1	2	3	4	5	6	7	8	9	10
(*i*)	0·1490657	0·	9·8509343	9·7018686	9·5528029	9·4037373	9·2546716	9·1056059	8·9565402	8·8074745	8·6584088	8·5093431
1	*n* 9·7018686		0·									
2	*n* 9·5498946		9·8480260	0·								
3	*n* 9·1741155		9·4722469	9·8480260	0·							
4	*n* 8·3829153		8·6810467	9·5487512	9·8480260	0·						
5	8·2679430		*n* 8·5660744	9·0610606	9·5860701	9·8480260	0·					
6	8·2768138		*n* 8·5749452	7·7828308	9·1981271	9·6082506	9·8480260	0·				
7	7·9441988		*n* 8·2423302	*n* 8·3320666	8·5845359	9·2727154	9·6229686	9·8480260	0·			
8	7·1879589		*n* 7·4860903	*n* 8·2184935	*n* 7·7239644	8·8019995	9·3202726	9·6334534	9·8480260	0·		
9	*n* 7·0359603		7·3340917	*n* 7·8195900	*n* 8·0798539	7·9731279	8·9198038	9·3534215	9·6413036	9·8480260	0·	
10	*n* 7·0597082		7·3578396	*n* 6·8507075	*n* 7·8626918	*n* 7·7875526	8·3491785	8·9961328	9·3779141	9·6474026	9·8480260	0·
1	*n* 9·8509343	9·8480260	9·8509343									
2	*n* 9·6989603	9·2130351	9·6989603	9·7018686								
3	*n* 9·3231812	*n* 8·8623294	9·3231812	9·5498946	9·5528029							
4	*n* 8·5319810	*n* 8·9699947	8·5319810	9·2506198	9·4008289	9·4037373						
5	8·4170087	*n* 8·6746003	*n* 8·4170087	8·7629292	9·1388730	9·2517633	9·2546716					
6	8·4258795	*n* 7·9888799	*n* 8·4258795	7·4846994	8·7509300	9·0119879	9·1026976	9·1056059				
7	8·0932645	7·7016802	*n* 8·0932645	*n* 8·0339352	8·1373388	8·6764527	8·8776402	8·9536319	8·9565402			
8	7·3370246	7·7777964	*n* 7·3370246	*n* 7·9203621	*n* 7·2767673	8·2057368	8·5749442	8·7390593	8·8045662	8·8074745		
9	*n* 7·1850260	7·4716144	7·1850260	*n* 7·5214586	*n* 7·6326568	7·3768652	8·1744754	8·4590274	8·5978438	8·6555005	8·6584088	
10	*n* 7·2087739	6·7665616	7·2087739	*n* 6·5525761	*n* 7·4154947	*n* 7·1912899	7·6038501	8·1017387	8·3344543	8·4548771	8·5064348	8·5093431

	−1	0	1	2	3	4	5	6	7	8	9	10
(*k*)	0·1145491	0·	9·8854509	9·7709018	9·6563527	9·5418036	9·4272546	9·3127055	9·1981564	9·0836073	8·9690582	8·8545091
1	*n* 9·7709018		0·									
2	*n* 9·5772579		9·8063561	0·								
3	*n* 9·0929811		9·3220793	9·8063561	0·							
4	7·8476842		*n* 8·0767824	9·4266336	9·8063561	0·						
5	8·5315105		*n* 8·7606087	8·6905788	9·4754114	9·8063561	0·					
6	8·3024025		*n* 8·5315007	*n* 8·4022941	8·9437259	9·5038223	9·8063561	0·				
7	7·6554649		*n* 7·8845631	*n* 8·4336498	*n* 6·2455127	9·0596043	9·5224563	9·8063561	0·			
8	*n* 7·3101661		7·5392643	*n* 8·0626121	*n* 8·2584601	8·2890580	9·1284351	9·5356303	9·8063561	0·		
9	*n* 7·3847648		7·6138630	*n* 7·0038912	*n* 8·0633897	*n* 7·9607418	8·5456436	9·1745715	9·5454416	9·8063561	0·	
10	*n* 7·0199044		7·2490026	7·3237882	*n* 7·5272173	*n* 7·9896456	*n* 6·9100905	8·6801550	9·2078212	9·5530339	9·8063561	0·
1	*n* 9·8854509	9·8063561	9·8854509									
2	*n* 9·6918070	8·8842231	9·6918070	9·7709018								
3	*n* 9·2075302	*n* 9·0852645	9·2075302	9·5772579	9·6563527							
4	7·9622333	*n* 8·9895023	*n* 7·9622333	9·1975354	9·4627088	9·5418036						
5	8·6460596	*n* 8·4995606	*n* 8·6460596	8·4614806	9·1317641	9·3481597	9·4272546					
6	8·4169516	7·6457366	*n* 8·4169516	*n* 8·1731959	8·6000786	9·0456259	9·2336107	9·3127055				
7	7·7700140	8·0328161	*n* 7·7700140	*n* 8·2045516	*n* 5·9018654	8·6014079	8·9497109	9·1190616	9·1981564			
8	*n* 7·4247152	7·7629560	7·4247152	*n* 7·8335139	*n* 7·9148128	7·8308616	8·5556897	8·8483358	9·0045125	9·0836073		
9	*n* 7·4993139	7·0011277	7·4993139	*n* 6·7747930	*n* 7·7197424	*n* 7·5025454	7·9728982	8·4872770	8·7435980	8·8899634	8·9690582	
10	*n* 7·1344535	*n* 6·9090209	7·1344535	7·0946900	*n* 7·1835700	*n* 7·5314492	*n* 6·3373451	7·9928605	8·4059776	8·6366412	8·7754143	8·8545091

CALCULATION OF THE QUANTITIES H'^{m}_{n} (*continued*).

	− 1	0	1	2	3	4	5	6	7	8	9	10
(*l*)	0·0856830	0·	9·9143170	9·8286341	9·7429511	9·6572682	9·5715852	9·4859023	9·4002193	9·3145364	9·2288534	9·1431705
1	*n* 9·8286341		0·									
2	*n* 9·5852696		9·7566355	0·								
3	*n* 8·9291425		9·1005084	9·7566355	0·							
4	8·5961261		*n* 8·7674920	9·2628850	9·7566355	0·						
5	8·6309897		*n* 8·8023556	*n* 7·6195524	9·3322948	9·7566355	0·					
6	8·1716624		*n* 8·3430283	*n* 8·6152650	8·4834675	9·3713097	9·7566355	0·				
7	*n* 7·2980879		7·4694538	*n* 8·3543601	*n* 8·3654674	8·7355957	9·3964032	9·7566355	0·			
8	*n* 7·6802532		7·8516191	*n* 7·5310060	*n* 8·2898319	*n* 7·9409794	8·8571443	9·4139252	9·7566355	0·		
9	*n* 7·3509870		7·5223529	7·5290965	*n* 7·7871486	*n* 8·1825373	7·4762228	8·9314787	9·4268631	9·7566355	0·	
10	*n* 5·9781611		6·1495270	7·4378298	6·9278834	*n* 7·8411341	*n* 8·0321350	8·1014034	8·9823206	9·4368119	9·7566355	0·
1	*n* 9·9143171	9·7566355	9·9143170									
2	*n* 9·6709526	*n* 7·8629241	9·6709525	9·8286341								
3	*n* 9·0148255	*n* 9·1943225	9·0148254	9·5852696	9·7429511							
4	8·6818091	*n* 8·9417424	*n* 8·6818090	9·0915191	9·4995866	9·6572682						
5	8·7166727	*n* 8·0087449	*n* 8·7166726	*n* 7·4481865	9·0752459	9·4139037	9·5715852					
6	8·2573454	8·2110244	*n* 8·2573453	*n* 8·4438991	8·2264186	9·0285779	9·3282207	9·4859023				
7	*n* 7·3837709	8·0737550	7·3837708	*n* 8·1829942	*n* 8·1084185	8·3928639	8·9679884	9·2425378	9·4002193			
8	*n* 7·7659362	7·4284588	7·7659361	*n* 7·3596401	*n* 8·0327830	*n* 7·5982476	8·4287295	8·8998275	9·1568548	9·3145364		
9	*n* 7·4366700	*n* 7·1592362	7·4366699	7·3577306	*n* 7·5300997	*n* 7·8398055	7·0478080	8·4173810	8·8270824	9·0711719	9·2288534	
10	*n* 6·0638441	*n* 7·1750767	6·0638440	7·2664639	6·6708345	*n* 7·4984023	*n* 7·6037202	7·5873057	8·3825399	8·7513483	8·9854889	9·1431705

	− 1	0	1	2	3	4	5	6	7	8	9	10
(*m*)	0·0617459	0·	9·9382541	9·8765082	9·8147622	9·7530163	9·6912704	9·6295245	9·5677785	9·5060326	9·4442867	9·3825408
1	*n* 9·8765082		0·									
2	*n* 9·5732933		9·6967851	0·								
3	*n* 8·5531753		8·6766671	9·6967851	0·							
4	8·8311505		*n* 8·9546423	9·0196977	9·6967851	0·						
5	8·6256584		*n* 8·7491502	*n* 8·6304564	9·1347698	9·6967851	0·					
6	7·6606332		*n* 7·7841250	*n* 8·6378958	*n* 8·0987060	9·1947585	9·6967851	0·				
7	*n* 7·9063139		8·0298057	*n* 8·0811852	*n* 8·5050903	7·9202225	9·2319128	9·6967851	0·			
8	*n* 7·7106618		7·8341536	7·6050032	*n* 8·1252502	*n* 8·3494602	8·3734508	9·2572715	9·6967851	0·		
9	*n* 6·7513378		6·8748296	7·6848633	6·4199557	*n* 8·1034685	*n* 8·1563189	8·5482051	9·2757110 −	9·6967851	0·	
10	6·9963247		*n* 7·1198165	7·1607086	7·4920896	*n* 7·2585655	*n* 8·0502676	*n* 7·8816243	8·6491040	9·2897350	9·6967851	0·
1	*n* 9·9382541	9·6967851	9·9382541									
2	*n* 9·6350392	*n* 8·9336705	9·6350392	9·8765082								
3	*n* 8·6149212	*n* 9·2439479	8·6149212	9·5732933	9·8147622							
4	8·8928964	*n* 8·8140564	*n* 8·8928964	8·8962059	9·5115473	9·7530163						
5	8·6874043	8·0833413	*n* 8·6874043	*n* 8·5069646	8·9495320	9·4498014	9·6912704					
6	7·7223791	8·3518871	*n* 7·7223791	*n* 8·5144040	*n* 7·9134682	8·9477748	9·3880555	9·6295245				
7	*n* 7·9680598	7·9104109	7·9680598	*n* 7·9576934	*n* 8·3198525	7·6732388	8·9231832	9·3263096	9·5677785			
8	*n* 7·7724077	*n* 7·2047980	7·7724077	7·4815114	*n* 7·9400124	*n* 8·1024765	8·0647212	8·8867960	9·2645636	9·5060326		
9	*n* 6·8130837	*n* 7·4531960	6·8130837	7·5613715	6·2347179	*n* 7·8564848	*n* 7·8475893	8·1777296	8·8434895 −	9·2028177	9·4442867	
10	7·0580706	*n* 7·0045793	*n* 7·0580706	7·0372168	7·3068518	*n* 7·0115818	*n* 7·7415380	*n* 7·5111488	8·2168825	8·7957676	9·1410718	9·3825408

	−1	0	1	2	3	4	5	6	7	8	9	10
(*n*)	0·0422077	0·	9·9577923	9·9155846	9·8733769	9·8311693	9·7889616	9·7467539	9·7045462	9·6623385	9·6201308	9·5779231
1	*n* 9·9155846		0·									
2	*n* 9·5391411		9·6235565	0·								
3	8·2838771		*n* 8·3682925	9·6235565	0·							
4	8·9404067		*n* 9·0248221	8·5288216	9·6235565	0·						
5	8·5060062		*n* 8·5904216	*n* 8·8185802	8·8164972	9·6235565	0·					
6	*n* 7·8841189		7·9685343	*n* 8·5421554	*n* 8·6061782	8·9331865	9·6235565	0·				
7	*n* 8·0440001		8·1284155	5·9580858	*n* 8·4675563	*n* 8·3569107	8·9988110	9·6235565	0·			
8	*n* 7·4418592		7·5262746	7·9073845	*n* 7·6075515	*n* 8·3814797	*n* 7·9918505	9·0413256	9·6235565	0·		
9	7·1962995		*n* 7·2807149	7·5282094	7·6652995	*n* 7·7660334	*n* 8·2872910	5·8762178	9·0712415	9·6235565	0·	
10	7·1288898		*n* 7·2133052	*n* 6·7056927	7·4545095	7·3786522	*n* 7·8090612	*n* 8·1844387	7·8966695	9·0934851	9·6235565	0·
1	*n* 9·9577923	9·6235565	9·9577923									
2	*n* 9·5813488	*n* 9·1950236	9·5813488	9·9155846								
3	8·3260848	*n* 9·2502563	*n* 8·3260848	9·5391411	9·8733769							
4	8·9826144	*n* 8·5377511	*n* 8·9826144	8·4444062	9·4969334	9·8311693						
5	8·5482139	8·4870167	*n* 8·5482139	*n* 8·7341648	8·6898741	9·4547258	9·7889616					
6	*n* 7·9263266	8·3346588	7·9263266	*n* 8·4577400	*n* 8·4795551	8·7643558	9·4125181	9·7467539				
7	*n* 8·0862078	7·1323237	8·0862078	5·8736704	*n* 8·3409332	*n* 8·1880800	8·7877726	9·3703104	9·7045462			
8	*n* 7·4840669	*n* 7·6866541	7·4840669	7·8229691	*n* 7·4809284	*n* 8·2126490	*n* 7·7808121	8·7880795	9·3281027	9·6623385		
9	7·2385072	*n* 7·3771423	*n* 7·2385072	7·4437940	7·5386764	*n* 7·5972027	*n* 8·0762526	5·6229717	8·7757877	9·2858950	9·6201308	
10	7·1710975	6·3368598	*n* 7·1710975	*n* 6·6212773	7·3278864	7·2098215	*n* 7·5980228	*n* 7·9311926	7·6012157	8·7558236	9·2436873	9·5779231

	−1	0	1	2	3	4	5	6	7	8	9	10
(*o*)	0·0266760	0·	9·9733240	9·9466480	9·9199721	9·8932961	9·8666201	9·8399441	9·8132682	9·7865922	9·7599162	9·7332402
1	*n* 9·9466480		0·									
2	*n* 9·4781295		9·5314815	0·								
3	8·8729833		*n* 8·9263353	9·5314815	0·							
4	8·9736320		*n* 9·0269840	*n* 8·4354573	9·5314815	0·						
5	8·1530637		*n* 8·2064157	*n* 8·8694036	7·6522754	9·5314815	0·					
6	*n* 8·2547320		8·3080840	*n* 8·2875442	*n* 8·7277291	8·3925616	9·5314815	0·				
7	*n* 7·9820097		8·0353617	7·9988607	*n* 8·2790210	*n* 8·5928127	8·5874674	9·5314815	0·			
8	7·0656728		*n* 7·1190248	7·8957263	7·6526814	*n* 8·2428444	*n* 8·4578168	8·6896169	9·5314815	0·		
9	7·4418175		*n* 7·4951695	6·5150787	7·7504158	7·1532964	*n* 8·1951244	*n* 8·3158801	8·7541786	9·5314815	0·	
10	6·8144689		*n* 6·8678209	*n* 7·2464001	6·9598996	7·6025049	*n* 6·5483894	*n* 8·1415752	*n* 8·1577559	8·7991323	9·5314815	0·
1	*n* 9·9733240	9·5314815	9·9733240									
2	*n* 9·5048055	*n* 9·3379221	9·5048055	9·9466480								
3	8·8996593	*n* 9·2166844	*n* 8·8996593	9·4781295	9·9199721							
4	9·0003080	*n* 4·9493900	*n* 9·0003080	*n* 8·3821053	9·4514536	9·8932961						
5	8·1797397	8·6214318	*n* 8·1797397	*n* 8·8160516	7·5722475	9·4247776	9·8666201					
6	*n* 8·2814080	8·1529820	8·2814080	*n* 8·2341922	*n* 8·6477012	8·2858577	9·3981016	9·8399441				
7	*n* 8·0086857	*n* 7·7553793	8·0086857	7·9455087	*n* 8·1989931	*n* 8·4861088	8·4540875	9·3714256	9·8132682			
8	7·0923488	*n* 7·7411280	*n* 7·0923488	7·8423743	7·5726535	*n* 8·1361405	*n* 8·3244369	8·5295610	9·3447497	9·7865922		
9	7·4684935	*n* 6·6477741	*n* 7·4684935	6·4617267	7·6703879	7·0465925	*n* 8·0617445	*n* 8·1558242	8·5674468	9·3180737	9·7599162	
10	6·8411449	7·0901169	*n* 6·8411449	*n* 7·1930481	6·8798717	7·4958010	*n* 6·4150095	*n* 7·9815193	*n* 7·9710241	8·5857245	9·2913977	9·7332402

CALCULATION OF THE QUANTITIES H'^{m}_{n} (continued).

	−1	0	1	2	3	4	5	6	7	8	9	10
(p)	0·0148626	0·	9·9851374	9·9702748	9·9554122	9·9405496	9·9256869	9·9108243	9·8959617	9·8810991	9·8662365	9·8513739
1	n 9·9702748		0·									
2	n 9·3805562		9·4102814 +	0·								
3	9·0968770		n 9·1266022	9·4102814 +	0·							
4	8·9397639		n 8·9694891	n 8·8848073	9·4102814 +	0·						
5	n 7·8674729		7·8971981	n 8·8370826	n 8·6527887	9·4102814 +	0·					
6	n 8·3616969		8·3914221	n 7·1476763	n 8·7253529	n 8·3936472	9·4102814 +	0·				
7	n 7·6139939		7·6437191	8·1765658	n 7·7141201	n 8·6267478	n 8·0321391	9·4102814 +	0·			
8	7·6631041		n 7·6928293	7·6218511	7·9807758	n 7·8261267	n 8·5368828	n 6·7090154	9·4102814 +	0·		
9	7·3423763		n 7·3721015	n 7·3903168	7·5535675	7·7956159	n 7·8577847	n 8·4529175	7·8651869	9·4102814 +	0·	
10	n 6·7592205		6·7889457	n 7·2121343	n 7·0836817	7·4705575	7·6143276	n 7·8589280	n 8·3728384	8·1310859	9·4102814 +	0·
1	n 9·9851374	9·4102814 +	9·9851374									
2	n 9·3954188	n 9·4268011	9·3954188 +	9·9702748								
3	9·1117396	n 9·1376964 +	n 9·1117396	9·3805562 +	9·9554122							
4	8·9546265	8·5235709	n 8·9546265	n 8·8550821	9·3656936 +	9·9405496						
5	n 7·8823355	8·6380807	7·8823355	n 8·8073574	n 8·6082009	9·3508310 +	9·9256869					
6	n 8·3765595	7·4388587	8·3765595	n 7·1179511	n 8·6807651	n 8·3341968	9·3359683 +	9·9108243				
7	n 7·6288565	n 8·0100539	7·6288565	8·1468406	n 7·6695323	n 8·5672974	n 7·9578260	9·3211057 +	9·8959617			
8	7·6779667	n 7·5214650	n 7·6779667	7·5921259	7·9361880	n 7·7666763	n 8·4625697	n 6·6198397	9·3062431 +	9·8810991		
9	7·3572389	7·2340108	n 7·3572389	n 7·3605916	7·5089797	7·7361655	n 7·7834716	n 8·3637418	7·7611486	9·2913805 +	9·8662365	
10	n 6·7740831	7·1051694	6·7740831	n 7·1824091	n 7·0390939	7·4111071	7·5400145	n 7·7697523	n 8·2688001	8·0121850	9·2765179 +	9·8513739

	−1	0	1	2	3	4	5	6	7	8	9	10
(q)	0·0065614	0·	9·9934386	9·9868772	9·9803158	9·9737544	9·9671929	9·9606315	9·9540701	9·9475087	9·9409473	9·9343859
1	n 9·9868772		0·									
2	n 9·2237262		9·2368490	0·								
3	9·2179273		n 9·2310501	9·2368490	0·							
4	8·8244890		n 8·8376118	n 9·0534342	9·2368490	0·						
5	n 8·4441852		8·4573080	n 8·7191067	n 8·9103394	9·2368490	0·					
6	n 8·3218253		8·3349481	8·1747545	n 8·6223892	n 8·7863581	9·2368490	0·				
7	7·5033465		n 7·5164693	8·1553269	7·9099623	n 8·5400553	n 8·6735609	9·2368490	0·			
8	7·7569704		n 7·7700932	n 6·9929510	8·0000217	7·6399743	n 8·4678995	n 8·5670523	9·2368490	0·		
9	6·2979950		n 6·3111178	n 7·5483280	n 5·4424798	7·8621672	7·3315690	n 8·4032973	n 8·4632822	9·2368490	0·	
10	n 7·1402042		7·1533270	n 6·5743786	n 7·3491026	6·5771470	7·7373517	6·8891336	n 8·3445004	n 8·3592111	9·2368490	0·
1	n 9·9934386	9·2368490	9·9934386									
2	n 9·2302876	n 9·4822572	9·2302876	9·9868772								
3	9·2244887	n 8·9929033	n 9·2244887	9·2237262	9·9803158							
4	8·8310504	8·7859552	n 8·8310504	n 9·0403114	9·2171648	9·9737544						
5	n 8·4507466	8·5505256	8·4507466	n 8·7059839	n 8·8906552	9·2106034	9·9671929					
6	n 8·3283867	n 7·9683662	8·3283867	8·1616317	n 8·6027050	n 8·7601125	9·2040419	9·9606315				
7	7·5099079	n 8·0231413	n 7·5099079	8·1422041	7·8902781	n 8·5138097	n 8·6407538	9·1974805	9·9540701			
8	7·7635318	6·7131544	n 7·7635318	n 6·9798282	7·9803375	7·6137287	n 8·4350924	n 8·5276838	9·1909191	9·9475087		
9	6·3045564	7·4371637	n 6·3045564	n 7·5352052	n 5·4227956	7·8359216	7·2987619	n 8·3639288	n 8·4173523	9·1843577	9·9409473	
10	n 7·1467656	6·5346606	7·1467656	n 6·5612558	n 7·3294184	6·5509014	7·7045446	6·8497651	n 8·2985705	n 8·3067198	9·1777963	9·9343859

	−1	0	1	2	3	4	5	6	7	8	9	10
(r)	0·0016338	0·	9·9983662	9·9967323	9·9950985	9·9934647	9·9918309	9·9901970	9·9885632	9·9869294	9·9852955	9·9836617
1	n 9·9967323		0·									
2	n 8·9341419		8·9374096	0·								
3	9·2811725		n 9·2844402	8·9374096	0·							
4	8·5585013		n 8·5617690	n 9·1314946 –	8·9374096	0·						
5	n 8·6269364		8·6302041	n 8·4504093	n 9·0154239	8·9374096	0·					
6	n 8·0947804		8·0980481	8·4194700	n 8·3610324	n 8·9212353	8·9374096	0·				
7	7·9680360		n 7·9713037	7·9338566	8·2449102 –	n 8·2862431	n 8·8415303	8·9374096	0·			
8	7·5872844		n 7·5905521	n 7·7258300	7·7946344	8·0949076	n 8·2218444	n 8·7721082	8·9374096	0·		
9	n 7·2923654		7·2956331	n 7·3946268	n 7·5120570	7·6735738	7·9626203	n 8·1652265	n 8·7103526	8·9374096	0·	
10	n 7·0520342		7·0553019	7·0224696	n 7·2228725	n 7·3213084	7·5663196	7·8436687	n 8·1146477	n 8·6545212	8·9374096	0·
1	n 9·9983661	8·9374096	9·9983662									
2	n 8·9357757	n 9·5130011	8·9357758	9·9967323								
3	9·2828063	n 8·7101004	n 9·2828064	8·9341419	9·9950985							
4	8·5601351	8·8995223	n 8·5601352	n 9·1282269 –	8·9325081	9·9934647						
5	n 8·6285702	8·2988007	8·6285703	n 8·4471416	n 9·0105224	8·9308743	9·9918309					
6	n 8·0964142	n 8·2627832	8·0964143	8·4162023	n 8·3561309	n 8·9147000	8·9292405	9·9901970				
7	7·9696698	n 7·8192016	n 7·9696699	7·9305889	8·2400087 –	n 8·2797078	n 8·8333612	8·9276066	9·9885632			
8	7·5889182	7·6054128	n 7·5889183	n 7·7225623	7·7897329	8·0883723	n 8·2136753	n 8·7623052	8·9259728	9·9869294		
9	n 7·2939992	7·3019194	7·2939993	n 7·3913591	n 7·5071555	7·6670385	7·9544512	n 8·1554235	n 8·6989158	8·9243390	9·9852955	
10	n 7·0536680	n 6·9228811	7·0536681	7·0192019	n 7·2179710	n 7·3147731	7·5581505	7·8338657	n 8·1032109	n 8·6414506	8·9227051	9·9836617

Values of Log G_n^m at the Equator.

(s)	−1	0	1	2	3	4	5	6	7	8	9	10
1			0·									
2		n 9·5228787		0·								
3			n 9·3010300		0·							
4		8·9330532		n 9·1549020		0·						
5			8·6777807		n 9·0457575		0·					
6		n 8·3353580		8·4814861		n 8·9586073		0·				
7			n 8·0665127		8·3217852		n 8·8860566		0·			
8		7·7355195		n 7·8446640		8·1870866		n 8·8239087		0·		
9			7·4593131		n 7·6556077		8·0705811		n 8·7695511		0·	
10		n 7·1348020		7·2219522		n 7·4907975		7·9679187		n 8·7212464		0·

Values of Log G_n^m at the Pole.

For the Pole $\mu=1$.

	−1	0	1	2	3	4	5	6	7	8	9	10
1		0·										
2		9·8239087	0·									
3		9·6020600	9·9030900	0·								
4		9·3590219	9·7569619	9·9330532	0·							
5		9·1037494	9·5808707	9·8239087	9·9488475	0·						
6		8·8405080	9·3845760	9·6856060	9·8616973	9·9586073	0·					
7		8·5716627	9·1737227	9·5259052	9·7477539	9·8860566	9·9652379	0·				
8		8·2986614	8·9518739	9·3498139	9·6130554	9·7891466	9·9030900	9·9700368	0·			
9		8·0224550	8·7214250	9·1607577	9·4617877	9·6726411	9·8187691	9·9156791	9·9736711	0·		
10		7·7437014	8·4840641	8·9611853	9·2969774	9·5400155	9·7161068	9·8410455	9·9253664	9·9765189	0·	

Tables of the Values of X^m_n, Y^m_n, Z^m_n *and* X^m_{-n}, Y^m_{-n}, Z^m_{-n} *for every* 5° *of geographical colatitude and for the equator and the pole, beginning at* 5° *geographical colatitude or* 85° *latitude for series (a), regarding* X^m_n, Y^m_n, *&c. as functions of* $\mu' = \cos\theta'$, *where* θ' *is the geocentric colatitude corresponding to the geographical colatitude* θ.

From these values of X^m_n, Y^m_n, &c. are derived the values of $X^m_n \cos\psi + Z^m_n \sin\psi$ or X'^m_n and $-X^m_n \sin\psi + Z^m_n \cos\psi$ or Z'^m_n, the coefficients of g^m_n or of h^m_n in the equations of condition for finding these magnetic functions.

VALUES OF LOG Y^m_n FROM THE FORMULA $Y^m_n = mG'^m_n = Y^m_{-n}$ FOR $\mu' = 0$.

(s)	m = 1	2	3	4	5	6	7	8	9	10
1	0·0									
2		·3010300								
3	n 9·3010300		·4771213							
4		n 9·4559320		·6020600						
5	8·6777807		n 9·5228788		·6989700					
6		8·7825161		n 9·5606673		·7781513				
7	n 8·0665127		8·7989065		n 9·5850266		·8450980			
8		n 8·1456940		8·7891466		n 9·6020600		·9030900		
9	7·4593131		n 8·1327290		8·7695511		n 9·6146491		·9542425	
10		7·5229822		n 8·0928575		8·7460700		n 9·6243364		1·

VALUES OF LOG X_n^m FROM THE FORMULA $X_n^m = (n-m)\, G'^{m+1}_n = X_{-n}^m$ FOR $\mu' = 0$.

(s)	m = 0	1	2	3	4	5	6	7
1	0·							
2		0·						
3	n 9·7781513							
4		n 9·6320233						
5	9·3767507		n 9·5228788					
6		9·1804561		n 9·4357286				
7	n 8·9116107		9·0207552		n 9·3631779			
8		n 8·6897620		8·8860566		n 9·3010300		
9	8·4135556		n 8·5007057		8·7695511		n 9·2466724	
10		8·1761947		n 8·3358955		8·6668887		n 9·1983677

VALUES OF LOG Z_n^m FROM THE FORMULA $Z_n^m = (n+1)\, G_n^m$ WHEN $\mu' = 0$.

(s)	m = 0	1	2	3	4	5	6	7	8	9	10
1		·3010300									
2	n 0·		·4771213								
3		n 9·9030900		·6020600							
4	9·6320232		n 9·8538720		·6989700						
5		9·4559320		n 9·8239088		·7781513					
6	n 9·1804560		9·3265841		n 9·8037053		·8450980				
7		n 8·9696027		9·2248752		n 9·7891466		·9030900			
8	8·6897620		n 8·7989065		9·1413291		n 9·7781512		·9542425		
9		8·4593131		n 8·6556077		9·0705811		n 9·7695511		1·	
10	n 8·1761947		8·2633449		n 8·5321902		9·0093114		n 9·7626391		1·0413927

VALUES OF LOG Z_{-n}^m FROM THE FORMULA $Z_{-n}^m = -nG_n^m$ WHEN $\mu' = 0$.

(s)	m = 0	1	2	3	4	5	6	7	8	9	10
1		n 0·									
2	9·8239087		n ·3010300								
3		9·7781513		n ·4771213							
4	n 9·5351132		9·7569620		n ·6020600						
5		n 9·3767507		9·7447275		n ·6989700					
6	9·1135093		n 9·2596374		9·7367586		n ·7781513				
7		8·9116107		n 9·1668832		9·7311546		n ·8450980			
8	n 8·6386095		8·7477540		n 9·0901766		9·7269987		n ·9030900		
9		n 8·4135556		8·6098502		n 9·0248236		9·7237936		n ·9542425	
10	8·1348020		n 8·2219522		8·4907975		n 8·9679187		9·7212464		n 1·

VALUES OF LOG Y_n^m FROM THE FORMULA $Y_n^m = \dfrac{1}{\gamma^{n+2}} m (1-\mu'^2)^{-\frac{1}{2}} H'^m_n$.

(a)	$m = 1$	2	3	4	5	6	7	8	9	10
$n=1$	·0043291									
2	·0040941	9·2499843								
3	9·9061060	9·2497493	8·3707007							
4	9·7580519	9·1820056	8·3704657	7·4402645						
5	9·5791799	9·0715003	8·3186569	7·4400296	6·4817997					
6	9·3792415	8·9312698	8·2304278	7·3980648	6·4815647	5·5056062				
7	9·1638710	8·7690711	8·1149814	7·3246042	6·4462867	5·5053712	4·5171780			
8	8·9366193	8·5899079	7·9783541	7·2264456	6·3833415	5·4749322	4·5169430	3·5197951		
9	8·6998672	8·3972012	7·8247300	7·1083518	6·2979415	5·4198578	4·4901685	3·5195602	2·5155729	
10	8·4552887	8·1933940	7·6571332	6·9737967	6·1939172	5·3442657	4·4412072	3·4956569	2·5153379	1·5059555

(b)	$m = 1$	2	3	4	5	6	7	8	9	10
$n=1$	·0042297									
2	9·9989027	9·5491599								
3	9·8932311	9·5438329	8·9691515							
4	9·7348241	9·4692725	8·9638245	8·3379904						
5	9·5428212	9·3501911	8·9056313	8·3326634	7·6788006					
6	9·3268289	9·1995810	8·8097057	8·2845733	7·6734736	7·0018821				
7	9·0922932	9·0251441	8·6852275	8·2039414	7·6322428	6·9965551	6·3127290			
8	8·8425243	8·8318157	8·5382045	8·0975508	7·5624751	6·9602864	6·3074020	5·6146213		
9	8·5795912	8·6229354	8·3727870	7·9701475	7·4693730	6·8986381	6·2748901	5·6092943	4·9096740	
10	8·3047675	8·4008463	8·1919602	7·8251858	7·3567559	6·8157209	6·2195411	5·5797255	4·9043470	4·1993317

(c)	$m = 1$	2	3	4	5	6	7	8	9	10
$n=1$	·0040676									
2	9·9901712	9·7221620								
3	9·8713916	9·7082657	9·3153177							
4	9·6949680	9·6221233	9·3014213	8·8573208						
5	9·4794811	9·4882863	9·2324041	8·8434245	8·3712952					
6	9·2337146	9·3195272	9·1233105	8·7849635	8·3573988	7·8675409				
7	8·9618966	9·1232552	8·9832050	8·6921028	8·3060988	7·8536446	7·3515520			
8	8·6654836	8·9040184	8·8179497	8·5715547	8·2247228	7·8075217	7·3376556	6·8266084		
9	8·3435956	8·6646384	8·6315145	8·4279794	8·1184259	7·7347057	7·2954506	6·8127121	6·2948253	
10	7·9924688	8·4067612	8·4266570	8·2647289	7·9909715	7·6392733	7·2292631	6·7735753	6·2809289	5·7576472

(d)	m = 1	2	3	4	5	6	7	8	9	10
n = 1	·0038477									
2	9·9777738	9·8427780								
3	9·8399625	9·8167042	9·5567697							
4	9·6364948	9·7138550	9·5306958	9·2196087						
5	9·3839809	9·5582861	9·4461190	9·1935349	8·8544192					
6	9·0877538	9·3620406	9·3178101	9·1201980	8·8283454	8·4715009				
7	8·7455110	9·1314459	9·1544613	9·0095891	8·7626229	8·4454270	8·0763479			
8	8·3445322	8·8694055	8·9614304	8·8681979	8·6644601	8·3852049	8·0502741	7·6722403		
9	7·8442989	8·5761244	8·7419914	8·7003966	8·5388637	8·2962803	7·9942115	7·6461664	7·2612931	
10	7·0302471	8·2487806	8·4979275	8·5091643	8·3894128	8·1826249	7·9124198	7·5933599	7·2352193	6·8449510

(e)	m = 1	2	3	4	5	6	7	8	9	10
n = 1	·0035769									
2	9·9615225	9·9341335								
3	9·7979470	9·8920791	9·7397514							
4	9·5559163	9·7668265	9·6976970	9·4942167						
5	9·2458609	9·5811779	9·5923590	9·4521623	9·2206533					
6	8·8585946	9·3450956	9·4378254	9·3590376	9·1785989	8·9293612				
7	8·3462768	9·0612657	9·2417747	9·2244051	9·0937328	8·8873068	8·6258345			
8	7·3570221	8·7255750	9·0080170	9·0541604	8·9729692	8·8083987	8·5837801	8·3133531		
9	n 7·5376672	8·3224139	8·7373444	8·8518410	8·8209266	8·6978687	8·5093736	8·2712987	7·9940322	
10	n 7·5678631	7·8012638	8·4272737	8·6192131	8·6406683	8·5594155	8·4067117	8·2004135	7·9519779	7·6693164

(f)	m = 1	2	3	4	5	6	7	8	9	10
n = 1	·0032637									
2	9·9411515	·0065292								
3	9·7438079	9·9444169	9·8848559							
4	9·4469939	9·7901897	9·8227437	9·7120302						
5	9·0411427	9·5637471	9·6907409	9·6499180	9·5111757					
6	8·4328445	9·2693108	9·5013871	9·5314878	9·4490635	9·2925925				
7	n 7·5515325	8·8963838	9·2597485	9·3653414	9·3397873	9·2304803	9·0617747			
8	n 7·9907231	8·4006795	8·9648595	9·1559032	9·1896076	9·1277965	8·9996625	8·8220022		
9	n 7·8808110	7·5020460	8·6076087	8·9043379	9·0022058	8·9892994	8·9019533	8·7598900	8·5753902	
10	n 7·6181118	n 7·5150293	8·1599738	8·6083046	8·7792500	8·8180410	8·7723850	8·6660683	8·5132780	8·3233832

VALUES OF LOG Y_n^m FROM THE FORMULA $Y_n^m = \frac{1}{r^{n+2}} m (1-\mu'^2)^{-\frac{1}{2}} H'^m_n$ (*continued*).

(*g*)	*m* = 1	2	3	4	5	6	7	8	9	10
n = 1	·0029176									
2	9·9162936	·0654588								
3	9·6751518	9·9788347	·0030611							
4	9·2974990	9·7877198	9·9164370	9·8895108						
5	8·6929156	9·5054526	9·7508111	9·8028868	9·7479320					
6	*n* 7·8985310	9·1193426	9·5153300	9·6527352	9·6613079	9·5886243				
7	*n* 8·2696872	8·5544261	9·2081735	9·4455886	9·5215553	9·5020002	9·4170820			
8	*n* 8·1233747	*n* 7·0601204	8·8115577	9·1822138	9·3335420	9·3697183	9·3304580	9·2365852		
9	*n* 7·7969310	*n* 7·9737194	8·2574531	8·8565420	9·0987025	9·1955359	9·2038034	9·1499611	9·0492487	
10	*n* 7·2617488	*n* 7·8713163	6·4918845	8·4490398	8·8147939	8·9809924	9·0401025	9·0276984	8·9626247	8·[illegible]565173

(*h*)	*m* = 1	2	3	4	5	6	7	8	9	10
n = 1	·0025494									
2	9·8864468	0·1141986								
3	9·5880969	9·9980960	·1009092							
4	9·0784824	9·7600568	9·9848066	·0364672						
5	*n* 6·9465647	9·3981516	9·7769755	9·9203646	9·9439964					
6	*n* 8·4747245	8·8381532	9·4790511	9·7307174	9·8278938	9·8337970				
7	*n* 8·3544704	*n* 7·5392941	9·0690158	9·4695975	9·6504038	9·7176944	9·7113630			
8	*n* 7·9973868	*n* 8·2697376	8·4420100	9·1280479	9·4134415	9·5489083	9·5952604	9·5799742		
9	*n* 7·3006278	*n* 8·1279590	*n* 7·6236261	8·6727471	9·1128035	9·3290977	9·4330144	9·4638716	9·4417460	
10	6·9741270	*n* 7·7651499	*n* 7·9735763	7·9492836	8·7335860	9·0561490	9·2260416	9·3067203	9·3256434	9·2981228

(*i*)	*m* = 1	2	3	4	5	6	7	8	9	10
n = 1	·0021703									
2	9·8509198	0·1548581								
3	9·4758641	·0036075	·1826071							
4	8·6853873	9·7050561	·0313565	·1592036						
5	*n* 8·5711385	9·2180890	9·7701241	·0079531	·1077714					
6	*n* 8·5807327	7·9405826	9·3829045	9·7689011	9·9565208	·0386104				
7	*n* 8·2488411	*n* 8·4905418	8·7700367	9·4340893	9·7321868	9·8873598	9·9572148			
8	*n* 7·4933247	*n* 8·3776922	*n* 7·9101887	8·9640969	9·4302143	9·6735107	9·8059643	9·8668646		
9	7·3420495	*n* 7·9795121	*n* 8·2668016	8·1359487	9·0304689	9·3942022	9·5999653	9·7156140	9·7696748	
10	7·3665209	*n* 7·0113531	*n* 8·0503630	*n* 7·9510969	8·4605671	9·0376370	9·3372993	9·5157141	9·6184243	9·6670901

(*k*)	$m = 1$	2	3	4	5	6	7	8	9	10
$n = 1$	·0017919									
2	9·8087453	·1888701								
3	9·3250658	9·9958235	·2510096							
4	*n* 8·0803662	9·6166983	·0579630	·2619965						
5	*n* 8·7647898	8·8812408	9·7276156	·0689499	·2449548					
6	*n* 8·5362790	*n* 8·5935533	9·1965273	9·7670133	·0519081	·2101842				
7	*n* 7·8899387	*n* 8·6255063	*n* 6·4989114	9·3233926	9·7686056	·0171376	·1631791			
8	7·5452372	*n* 8·2550659	*n* 8·5124561	8·5534436	9·3751817	9·7470091	9·9701325	·1072193		
9	7·6204332	*n* 7·1969423	*n* 8·3179830	*n* 8·2257247	8·7929875	9·3865476	9·7098153	9·9141727	·0444200	
10	7·2561701	7·5174366	*n* 7·7824079	*n* 8·2552258	*n* 7·1580317	8·8927284	9·3727922	9·6614478	9·8513734	9·9762257

(*l*)	$m = 1$	2	3	4	5	6	7	8	9	10
$n = 1$	·0014256									
2	9·7585363	·2172479								
3	9·1028844	9·9743586	·3081314							
4	*n* 8·7703431	9·4810832	·0652420	·3478623						
5	*n* 8·8056819	*n* 7·8382258	9·6413765	·1049730	·3595645					
6	*n* 8·3468298	*n* 8·8344136	8·7930244	9·7201224	·1166752	·3535381				
7	7·4737305	*n* 8·5739839	*n* 8·6754995	9·0848836	9·7569181	·1106488	·3352770			
8	7·8563710	*n* 7·7511050	*n* 8·6003392	*n* 8·2907425	9·2181344	9·7684137	·0923877	·3080613		
9	7·5275800	7·7496707	*n* 8·0981311	*n* 8·5327756	7·8376881	9·2864424	9·7630905	·0651720	·2740060	
10	6·1552293	7·6588792	7·2393411	*n* 8·1918476	*n* 8·3940755	8·4568423	9·3190232	9·7458236	·0311167	·2345558

(*m*)	$m = 1$	2	3	4	5	6	7	8	9	10
$n = 1$	·0010824									
2	9·6982284	·2407274								
3	8·6784712	9·9378733	·3554335							
4	*n* 8·9568072	9·2611467	·0525794	·4189871						
5	*n* 8·7516759	*n* 8·8722662	9·4909249	·1161330	·4545120					
6	*n* 7·7870115	*n* 8·8800664	*n* 8·4552219	9·6144672	·1516579	·4723082				
7	8·0330530	*n* 8·3237166	*n* 8·8619670	8·3402920	9·6871464	·1694541	·4778697			
8	7·8377617	7·8478954	*n* 8·4824877	*n* 8·7698905	8·8290452	9·7303013	·1750156	·4744766		
9	6·8787985	7·9281163	6·7775540	*n* 8·5242596	*n* 8·6122741	9·0215957	9·7543023	·1716225	·4642440	
10	*n* 7·1241463	7·4043225	7·8500488	*n* 7·6797175	*n* 8·5065837	*n* 8·3553758	9·1280562	9·7649333	·1613900	·4486165

VALUES OF LOG Y_n^m FROM THE FORMULA $Y_n^m = \frac{1}{\gamma^{n+2}} m (1-\mu'^2)^{-\frac{1}{2}} H'^m_n$ (*continued*).

(*n*)	*m* = 1	2	3	4	5	6	7	8	9	10
n = 1	·0007728									
2	9·6245869	·2598527								
3	*n* 8·3695804	9·8836667	·3939938							
4	*n* 9·0263676	8·7891894	·0178079	·4769825						
5	*n* 8·5922247	*n* 9·0792056	9·2110062	·1007966	·5319424					
6	7·9705950	*n* 8·8030384	*n* 9·0009448	9·4106842	·1557565	·5691736				
7	8·1307338	6·2192264	*n* 8·8625805	*n* 8·8346660	9·5312686	·1929877	·5941702			
8	7·5288505	8·1687827	*n* 8·0028333	*n* 8·8594926	*n* 8·5245657	9·6110144	·2179843	·6102121		
9	*n* 7·2835484	7·7898652	8·0608389	*n* 8·2443039	*n* 8·8202638	6·4461642	9·6659269	·2340262	·6194145	
10	*n* 7·2163963	*n* 6·9676061	7·8503065	7·8571803	*n* 8·3422916	*n* 8·7546427	8·4916125	9·7042124	·2432286	·6232219

(*o*)	*m* = 1	2	3	4	5	6	7	8	9	10
n = 1	·0005058									
2	9·5321559	·2750284								
3	*n* 8·9271783	9·8066785	·4246124							
4	*n* 9·0279956	*n* 8·7108229	9·9562625	·5230437						
5	*n* 8·2075959	*n* 9·1449378	8·0772250	·0546938	·5934463					
6	8·3094328	*n* 8·5632470	*n* 9·1528473	8·9159425	·1250964	·6461202				
7	8·0368791	8·2747321	*n* 8·7043078	*n* 9·1163622	9·1812509	·1777703	·6865596			
8	*n* 7·1207108	8·1717663	8·0781368	*n* 8·7665625	*n* 9·0517689	9·3360743	·2182097	·7180442		
9	*n* 7·4970241	6·7912873	8·1760398	7·6771831	*n* 8·7892451	*n* 8·9625061	9·4410754	·2496943	·7426893	
10	*n* 6·8698441	*n* 7·5227773	7·3856922	8·1265602	*n* 7·1426787	*n* 8·7883698	*n* 8·8448213	9·5175137	·2743394	·7619394

(*p*)	*m* = 1	2	3	4	5	6	7	8	9	10
n = 1	·0002895									
2	9·4106674	·2865534								
3	*n* 9·1270847	9·6969313	·4478786							
4	*n* 8·9700681	*n* 9·1715537	9·8582565	·5580512						
5	7·8978736	*n* 9·1239255	*n* 9·1008603	9·9684291	·6401950					
6	8·3921941	*n* 7·4346157	*n* 9·1735210	*n* 8·9518914	·0505729	·7046102				
7	7·6445876	8·4636017	*n* 8·1623847	*n* 9·1850885	*n* 8·6725271	·1149881	·7567908			
8	*n* 7·6937943	9·9089835	8·4291369	*n* 8·3845639	*n* 9·1773673	*n* 7·4138186	·1671687	·8000167		
9	*n* 7·3731630	*n* 7·6775457	8·0020251	8·3541496	*n* 8·4983657	*n* 9·1578172	8·6220707	·2103946	·8364031	
10	6·7901037	*n* 7·4994597	*n* 7·5322358	8·0291877	8·2550051	*n* 8·5639242	*n* 9·1299187	8·9312956	·2467810	·8673945

(q)	m = 1	2	3	4	5	6	7	8	9	10
n = 1	·0001303									
2	9·2370227	·2946423								
3	n 9·2312672	9·5315347	·4642156							
4	n 8·8378723	n 9·3481633	9·7011080	·5826363						
5	8·4576120	n 9·0138793	n 9·3746419	9·8195288	·6730283					
6	8·3352955	8·4695705	n 9·0867351	n 9·3690813	9·9099207	·7456916				
7	n 7·5168601	8·4501863	8·3743516	n 9·1228219	n 9·3466760	9·9825840	·8061203			
8	n 7·7705274	n 7·0878538	8·4644544	8·2227843	n 9·1410580	n 9·3128307	·0430127	·8575943		
9	n 6·3115955	n 7·8432743	n 5·9069560	8·4450207	8·0047710	n 9·1491192	n 9·2694894	·0944868	·9022289	
10	7·1538481	n 6·8693683	n 7·8136222	7·1600439	8·4105971	7·6349989	n 9·1507510	n 9·2168923	·1391213	·9414684

(r)	m = 1	2	3	4	5	6	7	8	9	10
n = 1	·0000328									
2	8·9374533	·2994398								
3	n 9·2844949	9·2368604	·4739083							
4	n 8·5618346	n 9·4309563	9·4113288	·5972241						
5	8·6302807	n 8·7498820	n 9·4893541	9·5346447	·6925113					
6	8·0981356	8·7189536	n 8·8349735	n 9·5184813	9·6299318	·7700696				
7	n 7·9714021	8·2333511	8·7188622	n 8·8835000	n 9·5340634	9·7074901	·8353934			
8	n 7·5906615	n 8·0253355	8·2685974	8·6921755	n 8·9143885	n 9·5421997	9·7728140	·8917626		
9	7·2957534	n 7·6941432	n 7·9860309	8·2708526	8·6551753	n 8·9353289	n 9·5457679	9·8291831	·9412921	
10	7·0554331	7·3219969	n 7·6968573	n 7·9185981	8·2588855	8·6137820	n 8·9500739	n 9·5463056	9·8787126	·9854267

VALUES OF LOG Y^m_{-n} FROM THE FORMULA $Y^m_{-n} = r^{n-1}\, m\, (1-\mu'^2)^{-\frac{1}{2}}\, H'^m_n$.

(a)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9968790	9·2427692								
3		9·8960048	9·2396481	8·3605995							
4		9·7450647	9·1690184	8·3574785	7·4272773						
5		9·5633066	9·0556270	8·3027836	7·4241563	6·4659264					
6		9·3604822	8·9125105	8·2116685	7·3793055	6·4628054	5·4868469				
7		9·1422257	8·7474258	8·0933361	7·3029589	6·4246414	5·4837259	4·4955327			
8		8·9120879	8·5653765	7·9538227	7·2019142	6·3588101	5·4504008	4·4924116	3·4952637		
9		8·6724498	8·3697838	7·7973126	7·0809344	6·2705241	5·3924404	4·4627511	3·4921428	2·4881555	
10		8·4249852	8·1630905	7·6268297	6·9434932	6·1636137	5·3139622	4·4109037	3·4653534	2·4850344	1·4756520

(b)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9918533	9·5421104								
3		9·8833618	9·5339637	8·9592822							
4		9·7221350	9·4565834	8·9511355	8·3253013						
5		9·5273123	9·3346822	8·8901224	8·3171546	7·6632917					
6		9·3085002	9·1812523	8·7913770	8·2662446	7·6551450	6·9835534				
7		9·0711447	9·0039956	8·6640790	8·1827929	7·6110943	6·9754067	6·2915805			
8		8·8185560	8·8078474	8·5142362	8·0735825	7·5385068	6·9363181	6·2834338	5·5906530		
9		8·5528031	8·5961472	8·3459989	7·9433594	7·4425849	6·8718500	6·2481020	5·5825063	4·8828859	
10		8·2751596	8·3712384	8·1623523	7·7955779	7·3271480	6·7861130	6·1899332	5·5501176	4·8747392	4·1697238

(c)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9833918	9·7153827								
3		9·8619006	9·6987746	9·3058267							
4		9·6827653	9·6099206	9·2892185	8·8451181						
5		9·4645667	9·4733719	9·2174897	8·8285100	8·3563808					
6		9·2160885	9·3019011	9·1056844	8·7673374	8·3397726	7·8499148				
7		8·9415588	9·1029174	8·9628672	8·6717650	8·2857610	7·8333067	7·3312142			
8		8·6424341	8·8809687	8·7949002	8·5485052	8·2016733	7·7844722	7·3146060	6·8035589		
9		8·3178344	8·6388772	8·6057533	8·4022182	8·0926647	7·7089445	7·2696894	6·7869508	6·2690641	
10		7·9639959	8·3782883	8·3981841	8·2362560	7·9624986	7·6108004	7·2007902	6·7451024	6·2524559	5·7291743

(d)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9713610	9·8363652								
3		9·8309846	9·8077263	9·5477918							
4		9·6249518	9·7023120	9·5191528	9·2080657						
5		9·3698728	9·5441780	9·4320109	9·1794268	8·8403111					
6		9·0710805	9·3453673	9·3011368	9·1035247	8·8116721	8·4548276				
7		8·7262726	9·1122076	9·1352229	8·9903507	8·7433845	8·4261886	8·0571095			
8		8·3227287	8·8476020	8·9396269	8·8463944	8·6426566	8·3634014	8·0284706	7·6504368		
9		7·8199303	8·5517558	8·7176228	8·6760280	8·5144951	8·2719117	7·9698429	7·6217978	7·2369245	
10		7·0033134	8·2218469	8·4709938	8·4822307	8·3624791	8·1556912	7·8854861	7·5664262	7·2082856	6·8180173

(e)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9555611	9·9281720								
3		9·7896009	9·8837331	9·7314053							
4		9·5451856	9·7560958	9·6869664	9·4834860						
5		9·2327456	9·5680626	9·5792437	9·4390471	9·2075380					
6		8·8430946	9·3295956	9·4223254	9·3435376	9·1630990	8·9138612				
7		8·3283922	9·0433810	9·2238901	9·2065205	9·0758482	8·8694223	8·6079499			
8		7·3367529	8·7053058	8·9877479	9·0338912	8·9527000	8·7881295	8·5635110	8·2930839		
9		n 7·5150134	8·2997601	8·7146906	8·8291872	8·7982728	8·6752149	8·4867198	8·2486450	7·9713784	
10		n 7·5428247	7·7762254	8·4022353	8·5941747	8·6156299	8·5343771	8·3816733	8·1753751	7·9269396	7·6442780

(f)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9357121	0·0010898								
3		9·7361927	9·9368017	9·8772407							
4		9·4372029	9·7803987	9·8129527	9·7022392						
5		9·0291759	9·5517803	9·6787741	9·6379512	9·4992089					
6		8·4187019	9·2551683	9·4872445	9·5173452	9·4349209	9·2784499				
7		n 7·5352142	8·8800655	9·2434302	9·3490233	9·3234690	9·2141620	9·0454564			
8		n 7·9722290	8·3821854	8·9463654	9·1374091	9·1711135	9·1093024	8·9811684	8·8035081		
9		n 7·8601411	7·4813761	8·5869388	8·8836680	8·9815359	8·9686295	8·8812834	8·7392201	8·5547203	
10		n 7·5952661	n 7·4921836	8·1371281	8·5854589	8·7564043	8·7951953	8·7495393	8·6432226	8·4904323	8·3005375

VALUES OF LOG Y^m_{-n} FROM THE FORMULA $Y^m_{-n} = r^{n-1}\, m\,(1-\mu'^2)^{-\frac{1}{2}}\, H'^m_n$ (*continued*).

(*g*)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·9114309	0·0605961								
3		9·6683440	9·9720269	9·9962533							
4		9·2887462	9·7789670	9·9076842	9·8807580						
5		8·6822177	9·4947547	9·7401132	9·7921889	9·7372341					
6		n 7·8858880	9·1066996	9·5026870	9·6400922	9·6486649	9·5759813				
7		n 8·2550991	8·5398380	9·1935854	9·4310005	9·5069672	9·4874121	9·4024939			
8		n 8·1068415	n 7·0435872	8·7950245	9·1656806	9·3170088	9·3531851	9·3139248	9·2200520		
9		n 7·7784528	n 7·9552412	8·2389749	8·8380638	9·0802243	9·1770577	9·1853252	9·1314829	9·0307705	
10		n 7·2413255	n 7·8508930	6·4714612	8·4286165	8·7943705	8·9605691	9·0196792	9·0072751	8·9422014	8·8360940

(*h*)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·8821978	0·1099496								
3		9·5821483	9·9921474	0·0949606							
4		9·0708342	9·7524086	9·9771584	0·0288190						
5		n 6·9372169	9·3888038	9·7676277	9·9110168	9·9346486					
6		n 8·4636771	8·8271058	9·4680037	9·7196700	9·8168464	9·8227496				
7		n 8·3417234	n 7·5265471	9·0562688	9·4568505	9·6376568	9·7049474	9·6986160			
8		n 7·9829402	n 8·2552910	8·4275634	9·1136013	9·3989949	9·5344617	9·5808138	9·5655276		
9		n 7·2844815	n 8·1118127	n 7·6074798	8·6566008	9·0966572	9·3129514	9·4168681	9·4477253	9·4255997	
10		6·9562811	n 7·7473040	n 7·9557304	7·9314377	8·7157401	9·0383031	9·2081957	9·2888744	9·3077975	9·2802769

(*i*)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·8473026	0·1512409								
3		9·4708000	9·9985434	0·1775430							
4		8·6788764	9·6985452	0·0248456	0·1526927						
5		n 8·5631807	9·2101312	9·7621663	9·9999953	0·0998136					
6		n 8·5713280	7·9311779	9·3734998	9·7594964	9·9471161	0·0292057				
7		n 8·2379895	n 8·4796902	8·7591851	9·4232377	9·7213352	9·8765082	9·9463632			
8		n 7·4810263	n 8·3653938	n 7·8978903	8·9517985	9·4179159	9·6612123	9·7936659	9·8545662		
9		7·3283042	n 7·9657668	n 8·2530563	8·1222034	9·0167236	9·3804569	9·5862200	9·7018687	9·7559295	
10		7·3513287	n 6·9961609	n 8·0351708	n 7·9359047	8·4453749	9·0224448	9·3221071	9·5005219	9·6032321	9·6518979

(k)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·8057588	0·1858836								
3		9·3208847	9·9916424	0·2468285							
4		*n* 8·0749906	9·6113227	0·0525874	0·2566209						
5		*n* 8·7582196	8·8746706	9·7210454	0·0623797	0·2383846					
6		*n* 8·5285142	*n* 8·5857885	9·1887625	9·7592485	0·0441433	0·2024194				
7		*n* 7·8809793	*n* 8·6165469	*n* 6·4899520	9·3144332	9·7596462	0·0081782	0·1542197			
8		7·5350832	*n* 8·2449119	*n* 8·5023021	8·5432896	9·3650277	9·7368551	9·9599785	0·0970653		
9		7·6090846	*n* 7·1855937	*n* 8·3066344	*n* 8·2143761	8·7816389	9·3751990	9·6984667	9·9028241	0·0330714	
10		7·2436269	7·5048934	*n* 7·7698647	*n* 8·2426826	*n* 7·1454885	8·8801852	9·3602490	9·6489046	9·8388302	9·9636825

(l)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·7561603	0·2148719								
3		9·0995581	9·9710323	0·3048051							
4		*n* 8·7660664	9·4768065	0·0609653	0·3435856						
5		*n* 8·8004548	*n* 7·8329987	9·6361494	0·0997459	0·3543374					
6		*n* 8·3406523	*n* 8·8282361	8·7868469	9·7139449	0·1104977	0·3473606				
7		7·4666026	*n* 8·5668560	*n* 8·6683716	9·0777557	9·7497902	0·1035209	0·3281491			
8		7·8482927	*n* 7·7430267	*n* 8·5922609	*n* 8·2826642	9·2100561	9·7603354	0·0843094	0·2999830		
9		7·5185514	7·7406421	*n* 8·0891025	*n* 8·5237470	7·8286595	9·2774138	9·7540619	0·0561434	0·2649774	
10		6·1452503	7·6489002	7·2293621	*n* 8·1818686	*n* 8·3840965	8·4468633	9·3090442	9·7358446	0·0211377	0·2245768

(m)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·6964243	0·2389233								
3		8·6759455	9·9353476	0·3529078							
4		*n* 8·9535599	9·2578994	0·0493321	0·4157398						
5		*n* 8·7477070	*n* 8·8682973	9·4869560	0·1121641	0·4505431					
6		*n* 7·7823209	*n* 8·8753758	*n* 8·4505313	9·6097766	0·1469673	0·4676176				
7		8·0276408	*n* 8·3183044	*n* 8·8565548	8·3348798	9·6817342	0·1640419	0·4724575			
8		7·8316279	7·8417616	*n* 8·4763539	*n* 8·7637567	8·8229114	9·7241675	0·1688818	0·4683428		
9		6·8719430	7·9212608	6·7706985	*n* 8·5174041	*n* 8·6054186	9·0147402	9·7474467	0·1647670	0·4573885	
10		*n* 7·1165692	7·3967454	7·8424717	*n* 7·6721404	*n* 8·4990066	*n* 8·3477987	9·1204791	9 7573562	0·1538129	0·4410394

VALUES OF LOG Y^m_{-n} FROM THE FORMULA $Y^m_{-n} = r^{n-1}\, m\, (1-\mu'^2)^{-\frac{1}{2}}\, H'^m_n$ (*continued*).

(n)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1		0·0000000									
2		9·6232990	0·2585648								
3		n 8·3677773	9·8818636	0·3921907							
4		n 9·0240493	8·7868711	0·0154896	0·4746642						
5		n 8·5893912	n 9·0763721	9·2081727	0·0979631	0·5291089					
6		7·9672463	n 8·7996897	n 8·9975961	9·4073355	0·1524078	0·5658249				
7		8·1268700	6·2153626	n 8·8587167	n 8·8308022	9·5274048	0·1891239	0·5903064			
8		7·5244715	8·1644037	n 7·9984543	n 8·8551136	n 8·5201867	9·6066354	0·2136053	0·6058331		
9		n 7·2786542	7·7849710	8·0559447	n 8·2394097	n 8·8153696	6·4412700	9·6610327	0·2291320	0·6145203	
10		n 7·2109869	n 6·9621967	7·8448971	7·8517709	n 8·3368822	n 8·7492333	8·4862031	9·6988030	0·2378192	0·6178125
(o)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1		0·0000000									
2		9·5313129	0·2741854								
3		n 8·9259981	9·8054983	0·4234322							
4		n 9·0264782	n 8·7093055	9·9547451	0·5215263						
5		n 8·2057413	n 9·1430832	8·0753704	0·0528392	0·5915917					
6		8·3072409	n 8·5610551	n 9·1506554	8·9137506	0·1229045	0·6439283				
7		8·0343500	8·2722030	n 8·7017787	n 9·1138331	9·1787218	0·1752412	0·6840305			
8		n 7·1178445	8·1689000	8·0752705	n 8·7636962	n 9·0489026	9·3332080	0·2153434	0·7151779		
9		n 7·4938206	6·7880838	8·1728363	7·6739796	n 8·7860416	n 8·9593026	9·4378719	0·2464908	0·7394858	
10		n 6·8663034	n 7·5192366	7·3821515	8·1230195	n 7·1391380	n 8·7848291	n 8·8412806	9·5139730	0·2707987	0·7583987
(p)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1		0·0000000									
2		9·4101850	0·2860709								
3		n 9·1264092	9·6962559	0·4472031							
4		n 8·9691996	n 9·1706852	9·8573881	0·5571827						
5		7·8968121	n 9·1228640	n 9·0997988	9·9673677	0·6391335					
6		8·3909396	n 7·4333612	n 9·1722665	n 8·9506369	0·0493185	0·7033557				
7		7·6431401	8·4621542	n 8·1609372	n 9·1836410	n 8·6710796	0·1135407	0·7553433			
8		n 7·6921538	7·9073430	8·4274964	n 8·3829234	n 9·1757268	n 7·4121781	0·1655283	0·7983762		
9		n 7·3713294	n 7·6757121	8·0001915	8·3523160	n 8·4965321	n 9·1559836	8·6202371	0·2085611	0·8345695	
10		6·7880771	n 7·4974331	n 7·5302092	8·0271611	8·2529785	n 8·5618976	n 9·1278921	8·9292690	0·2447545	0·8653679

(q)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		9·2368056	0·2944252								
3		*n* 9·2309632	9·5312307	0·4639116							
4		*n* 8·8374815	*n* 9·3477725	9·7007172	0·5822455						
5		8·4571343	*n* 9·0134016	*n* 9·3741642	9·8190511	0·6725506					
6		8·3347310	8·4690060	*n* 9·0861706	*n* 9·3685168	9·9093562	0·7451271				
7		*n* 7·5162087	8·4495349	8·3737002	*n* 9·1221705	*n* 9·3460246	9·9819326	0·8054689			
8		*n* 7·7697892	*n* 7·0871156	8·4637162	8·2220461	*n* 9·1403198	*n* 9·3120925	0·0422745	0·8568561		
9		*n* 6·3107704	*n* 7·8424492	*n* 5·9061309	8·4441956	8·0039459	*n* 9·1482941	*n* 9·2686643	0·0936617	0·9014038	
10		7·1529362	*n* 6·8684564	*n* 7·8127103	7·1591320	8·4096852	7·6340870	*n* 9·1498391	*n* 9·2159804	0·1382094	0·9405565

(r)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$		0·0000000									
2		8·9373986	0·2993851								
3		*n* 9·2844183	9·2367838	0·4738317							
4		*n* 8·5617362	*n* 9·4308578	9·4112304	0·5971257						
5		8·6301604	*n* 8·7497617	*n* 9·4892338	9·5345244	0·6923910					
6		8·0979934	8·7188114	*n* 8·8348313	*n* 9·5183391	9·6297896	0·7699274				
7		*n* 7·9712381	8·2331871	8·7186981	*n* 8·8833360	*n* 9·5338994	9·7073261	0·8352294			
8		*n* 7·5904756	*n* 8·0251496	8·2684115	8·6919896	*n* 8·9142026	*n* 9·5420138	9·7726281	0·8915767		
9		7·2955456	*n* 7·6939354	*n* 7·9858231	8·2706448	8·6549675	*n* 8·9351211	*n* 9·5455601	9·8289753	0·9410843	
10		7·0552034	7·3217672	*n* 7·6966276	*n* 7·9183684	8·2586558	8·6135523	*n* 8·9498442	*n* 9·5460759	9·8784829	0·9851970

VALUES OF LOG X_n^m FROM THE FORMULA $X_n^m = \frac{1}{r^{n+2}}[(n-m)\,H'^{m+1}_n - m\mu'(1-\mu'^2)^{-\frac{1}{2}}\,H'^m_n]$.

(a)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n=1$	8·9475112	n ·0026511									
2	9·2483062	n 9·9990339	n 9·2483063								
3	9·3264094	n 9·8959060	n 9·2463835	n 8·3690227							
4	9·3032940	n 9·7408708	n 9·1763740	n 8·3676632	n 7·4385865						
5	9·2213320	n 9·5530586	n 9·0630114	n 8·3144423	n 7·4375085	n 6·4801217					
6	9·1005749	n 9·3420623	n 8·9193147	n 8·2245089	n 7·3945284	n 6·4792124	n 5·5039282				
7	8·9521511	n 9·1133076	n 8·7530193	n 8·1070605	n 7·3198783	n 6·4431454	n 5·5031315	n 4·5155000			
8	8·7828914	n 8·8700732	n 8·5691028	n 7·9681271	n 7·2203536	n 6·3792955	n 5·4720486	n 4·5147834	n 3·5181171		
9	8·5972918	n 8·6143848	n 8·3709548	n 7·8118852	n 7·1007143	n 6·2928738	n 5·4162481	n 4·4874661	n 3·5174609	n 2·5138949	
10	8·3984708	n 8·3474505	n 8·1609813	n 7·6413500	n 6·9644312	n 6·1877094	n 5·3398466	n 4·4379001	n 3·4930886	n 2·5132854	n 1·5042775

(b)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·2467200	n 9·9974929									
2	9·5424231	n 9·9782610	n 9·5424231								
3	9·6128427	n 9·8505645	n 9·5301993	n 8·9624147							
4	9·5793744	n 9·6603244	n 9·4461817	n 8·9525021	n 8·3312536						
5	9·4842815	n 9·4237926	n 9·3148334	n 8·8884661	n 8·3224920	n 7·6720638					
6	9·3474705	n 9·1454731	n 9·1487976	n 8·7853607	n 8·2702197	n 7·6639913	n 6·9951453				
7	9·1798815	n 8·8209587	n 8·9552944	n 8·6522622	n 8·1846229	n 7·6195192	n 6·9875316	n 6·3059922			
8	8·9881046	n 8·4311085	n 8·7385907	n 8·4950468	n 8·0724410	n 7·5459962	n 6·9486233	n 6·2987059	n 5·6078845		
9	8·7763240	n 7·9061624	n 8·5010851	n 8·3176981	n 7·9383670	n 7·4486021	n 6·8839718	n 6·2639707	n 5·6008436	n 4·9029372	
10	8·5472578	n 5·8302365	n 8·2437599	n 8·1229905	n 7·7857910	n 7·3311303	n 6·7976748	n 6·2061276	n 5·5693560	n 4·8960871	n 4·1925949

(c)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·4197761	n 9·9888153									
2	9·7069096	n 9·9421087	n 9·7069097								
3	9·7642214	n 9·7671463	n 9·6769180	n 9·3000654							
4	9·7127365	n 9·4978473	n 9·5678483	n 9·2755053	n 8·8420685						
5	9·5941596	n 9·1169111	n 9·4026314	n 9·1925493	n 8·8201990	n 8·3560429					
6	9·4275744	n 8·4696131	n 9·1913746	n 9·0657482	n 8·7518404	n 8·3357797	n 7·8522886				
7	9·2227031	8·1856569	n 8·9368843	n 8·9034632	n 8·6469444	n 8·2768458	n 7·8330931	n 7·3362997			
8	8·9842821	8·4531225	n 8·6350036	n 8·7105414	n 8·5119421	n 8·1864577	n 7·7807734	n 7·3178651	n 6·8113561		
9	8·7135466	8·4286334	n 8·2693193	n 8·4894484	n 8·3511200	n 8·0696310	n 7·7007979	n 7·2704527	n 6·7934915	n 6·2795730	
10	8·4081773	8·3098822	n 7·7841134	n 8·2406183	n 8·1673288	n 7·9299527	n 7·5971641	n 7·1983470	n 6·7498681	n 6·2621510	n 5·7423949

(d)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·5404655	n 9·9764913									
2	9·8154216	n 9·8878046	n 9·8154216								
3	9·8537016	n 9·6273138	n 9·7591688	n 9·5294133							
4	9·7751726	n 9·1446999	n 9·6106628	n 9·4834549	n 9·1922523						
5	9·6195687	8·3331395	n 9·3864407	n 9·3718386	n 9·1513511	n 8·8270628					
6	9·4025229	8·9193348	n 9·0819036	n 9·2072494	n 9·0590546	n 8·7891678	n 8·4441445				
7	9·1272268	8·9417009	n 8·6556258	n 8·9948766	n 8·9244903	n 8·7089309	n 8·4082421	n 8·0489915			
8	8·7842400	8·8349691	n 7·8161309	n 8·7337373	n 8·7528673	n 8·5931213	n 8·3362963	n 8·0145070	n 7·6448839		
9	8·3351592	8·6560579	7·9512286	n 8·4141037	n 8·5465906	n 8·4461405	n 8·2334993	n 7·9486273	n 7·6114596	n 7·2339367	
10	7·5668649	8·4223872	8·0872164	n 8·0045525	n 8·3054539	n 8·2707168	n 8·1034902	n 7·8554525	n 7·5502174	n 7·2013354	n 6·8175946

(e)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·6319112	n 9·9603303									
2	9·8908869	n 9·8101619	n 9·8908869								
3	9·9034026	n 9·3747742	n 9·7981312	n 9·6965048							
4	9·7863106	8·6722729	n 9·5887389	n 9·6213155	n 9·4509701						
5	9·5731652	9·2163023	n 9·2525261	n 9·4682099	n 9·3843046	n 9·1774067					
6	9·2650802	9·2299328	n 8·6542117	n 9·2435886	n 9·2583262	n 9·1157762	n 8·8861146				
7	8·8197091	9·1027973	8·2866002	n 8·9380719	n 9·0796216	n 9·0060382	n 8·8278087	n 8·5825879			
8	7·8884464	8·8865142	8·5849244	n 8·5056245	n 8·8488080	n 8·8536466	n 8·7289569	n 8·5266412	n 8·2701065		
9	n 8·1202440	8·5892056	8·5457893	n 7·6153853	n 8·5589591	n 8·6609216	n 8·5939669	n 8·4356204	n 8·2159209	n 7·9507856	
10	n 8·1961974	8·1888554	8·3906025	7·8281981	n 8·1867780	n 8·4270797	n 8·4253347	n 8·3131113	n 8·1308141	n 7·8979649	n 7·6260698

(f)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·7044113	n 9·9400636									
2	9·9433291	n 9·6989223	n 9·9433291								
3	9·9220768	n 8·5700247	n 9·8008628	n 9·8216558							
4	9·7502015	9·3482775	n 9·4921752	n 9·7076585	n 9·6488301						
5	9·4412603	9·4361901	n 8·8456293	n 9·4940986	n 9·5483967	n 9·4479756					
6	8·9121434	9·3225024	8·7145049	n 9·1647452	n 9·3755368	n 9·3554817	n 9·2293924				
7	n 8·0977781	9·0857673	8·9253637	n 8·6094978	n 9·1289868	n 9·2056827	n 9·1421120	n 8·9985746			
8	n 8·5949607	8·7219023	8·8508905	8·0088015	n 8·7894901	n 9·0002734	n 9·0072808	n 8·9149802	n 8·7588021		
9	n 8·5362011	8·1125061	8·6508647	8·4056480	n 8·2754812	n 8·7340926	n 8·8271674	n 8·7906893	n 8·6779518	n 8·5121901	
10	n 8·3192594	n 7·5814032	8·3463240	8·3652875	7·0751781	n 8·3885066	n 8·6005664	n 8·6279318	n 8·5615017	n 8·4334622	n 8·2601831

VALUES OF LOG X_n^m FROM THE FORMULA $X_n^m = \frac{1}{r^{n+2}}[(n-m)\,H'^{m+1}_n - m\mu'\,(1-\mu'^2)^{-\frac{1}{2}}H'^m_n]$ (*continued*).

(g)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·7634561	n 9·9153210									
2	9·9778621	n 9·5303500	n 9·9778622								
3	9·9128116	9·2092122	n 9·7671956	n 9·9154645							
4	9·6600975	9·5657834	n 9·2658956	n 9·7501971	n 9·8019142						
5	9·1524241	9·5167921	8·7984387	n 9·4432746	n 9·6576826	n 9·6603354					
6	n 8·4372208	9·2923954	9·1585807	n 8·8663704	n 9·4178710	n 9·5282528	n 9·5010277				
7	n 8·8753237	8·8874204	9·0996265	8·4584906	n 9·0536539	n 9·3234703	n 9·3768597	n 9·3294854			
8	n 8·7870032	7·9315986	8·8785521	8·7528241	n 8·3977874	n 9·0361793	n 9·1938143	n 9·2108837	n 9·1489886		
9	n 8·5117120	n 8·2193790	8·5053365	8·6750931	8·1327574	n 8·6267159	n 8·9481009	n 9·0427139	n 9·0345152	n 8·9616521	
10	n 8·0222873	n 8·2500519	7·8246307	8·4487132	8·3453754	n 7·8692864	n 8·6245511	n 8·8235538	n 8·8771935	n 8·8503628	n 8·7689207

(h)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·8123189	n 9·8855970									
2	9·9972463	n 9·2264952	n 9·9972462								
3	9·8749877	9·5398148	n 9·6894958	n 9·9839568							
4	9·4903119	9·6679729	n 8·3843494	n 9·7498886	n 9·9195148						
5	n 7·4553042	9·5098589	9·2901843	n 9·2753429	n 9·7180750	n 9·8270440					
6	n 9·0626453	9·1268762	9·3152315	8·4792569	n 9·3752207	n 9·6440659	n 9·7168446				
7	n 9·0093379	8·0243739	9·1096493	9·0031114	n 8·7170778	n 9·3603233	n 9·5457523	n 9·5944106			
8	n 8·7102463	n 8·5370399	8·6983431	8·9404516	8·4936598	n 8·9228528	n 9·2961476	n 9·4316132	n 9·4630218		
9	n 8·0646398	n 8·5202264	7·5940756	8·6881865	8·6786339	n 7·8611366	n 8·9446208	n 9·2043902	n 9·3063433	n 9·3247936	
10	7·7838965	n 8·2462347	n 8·0698229	8·2364498	8·5457268	8·2973871	n 8·3958663	n 8·9004675	n 9·0949809	n 9·1728153	n 9·1811704

(i)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·8531046	n 9·8501963									
2	0·0028841	7·8287418	n 0·0028841								
3	9·8039197	9·7029290	n 9·5436782	n 0·0306331							
4	9·1383816	9·7036871	9·1791280	n 9·7003534	n 0·0072296						
5	n 9·1210428	9·4161473	9·4792054	n 8·7308329	n 9·7290840	n 9·9557974					
6	n 9·2098183	8·6264157	9·3469328	9·1533421	n 9·1963902	n 9·7061704	n 9·8866364				
7	n 8·9448734	n 8·7271959	8·9611330	9·1797286	8·5830004	n 9·2964756	n 9·6550318	n 9·8052408			
8	n 8·2473490	n 8·7563033	7·7224059	8·9414020	8·9442790	n 8·3523126	n 9·3067199	n 9·5860665	n 9·7148906		
9	8·1472263	n 8·4566729	n 8·3779755	8·4405978	8·8225964	8·6194733	n 8·7294536	n 9·2750561	n 9·5048107	n 9·6177008	
10	8·2174552	n 7·6916169	n 8·3122208	n 7·5023785	8·4897132	8·6446695	8·0527398	n 8·8194827	n 9·2190145	n 9·4145652	n 9·5151161

(k)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·8872428	n 9·8081480									
2	9·9952262	9·2579880	n 9·9952262								
3	9·6876380	9·7962052	n 9·2497819	n 0·0573657							
4	n 8·5678771	9·6852032	9·5056131	n 9·5804820	n 0·0683526						
5	n 9·3492107	9·1878779	9·5581795	9·0342549	n 9·6815831	n 0·0513109					
6	n 9·1998812	n 8·7268504	9·2698569	9·3733771	n 8·4495653	n 9·7108141	n 0·0165403				
7	n 8·6204876	n 8·9653100	8·4253251	9·2093854	9·1326022	n 9·0391974	n 9·7043619	n 9·9695352			
8	8·3337781	n 8·7043043	n 8·5852013	8·7417520	9·0895171	8·7966320	n 9·1837990	n 9·6765101	n 9·9135754		
9	8·4601266	n 7·8754862	n 8·5595089	n 7·7825577	8·7668430	8·9292202	8·1088049	n 9·2320353	n 9·6343806	n 9·8507761	
10	8·1416210	8·0096226	n 8·1619431	n 8·3078167	8·0084752	8·7069267	8·7287589	n 8·2776395	n 9·2365420	n 9·5820417	n 9·7825818

(l)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9157426	n 9·7580611									
2	9·9738833	9·5433800	n 9·9738834								
3	9·4943227	9·8460400	8·2567331	n 0·0647669							
4	n 9·2867201	9·6090209	9·6657178	n 9·3145879	n 0·1044978						
5	n 9·4189689	8·4012142	9·5604205	9·4356335	n 9·5457563	n 0·1162000					
6	n 9·0392981	n 9·1054588	9·0292327	9·4590429	9·1134050	n 9·6416368	n 0·1101736				
7	8·2331455	n 8·9733429	n 8·6375230	9·1033387	9·3263663	8·4282064	n 9·6838452	n 0·0919125			
8	8·6737780	n 8·3075210	n 8·7937708	7·4323818	9·0780236	9·1657441	n 8·6517039	n 9·6970268	n 0·0646968		
9	8·3961395	8·2182238	n 8·4348935	n 8·5420294	8·4000448	9·0082384	8·9707127	n 8·9659299	n 9·6919840	n 0·0306415	
10	7·0695463	8·2242986	7·0775080	n 8·3680436	n 8·1911408	8·5280933	8·9108551	8·7196044	n 9·0864992	n 9·6744241	n 9·9911913

(m)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9393365	n 9·6978675									
2	9·9375125	9·7047396	n 9·9375125								
3	9·0938466	9·8622004	9·3508497	n 0·0522186							
4	n 9·4971213	9·4509348	9·7497171	7·8791680	n 0·1157722						
5	n 9·3889000	n 9·0051870	9·4853204	9·6155691	n 9·1929253	n 0·1512971					
6	n 8·5034169	n 9·2081439	7·6768628	9·4502084	9·4582516	n 9·4416291	n 0·1690933				
7	8·8164051	n 8·7796515	n 8·9764862	8·7418962	9·3844612	9·2673869	n 9·5593137	n 0·1746548			
8	8·6791058	8·2544569	n 8·7497380	n 8·6877437	8·8813989	9·3000089	9·0149366	n 9·6243483	n 0·1712617		
9	7·7712951	8·4628363	n 7·4775871	n 8·6415701	n 8·2762016	8·9093040	9·2010117	8·5957838	n 9·6607598	n 0·1610291	
10	n 8·0624004	8·0238950	8·1794445	n 8·0327630	n 8·4919410	n 6·6156681	8·8912318	9·0882059	n 8·0006297	n 9·6791460	n 0·1454016

VALUES OF LOG X_n^m FROM THE FORMULA $X_n^m = \frac{1}{r^{n+2}}[(n-m)\,H'^{m+1}_n - m\mu'(1-\mu'^2)^{-\frac{1}{2}}\,H'^m_n]$ (*continued*).

(n)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9585651	n 9·6243293									
2	9·8834092	9·8117336	n 9·8834092								
3	n 8·8044940	9·8475741	9·6312250	n 0·0175503							
4	n 9·5862199	9·1086582	9·7813081	9·3845919	n 0·1005390						
5	n 9·2489870	n 9·3039506	9·2965351	9·7023377	8·9424416	n 0·1554989					
6	8·7065386	n 9·1704472	n 8·9756303	9·3371052	9·6141845	n 8·6106458	n 0·1927301				
7	8·9336241	n 7·7183587	n 9·0424958	n 8·4966559	9·3298853	9·5170115	n 9·1651999	n 0·2177267			
8	8·3897328	8·6572703	n 8·3885588	n 8·8912112	7·7868325	9·2995559	9·4093242	n 9·3654878	n 0·2337686		
9	n 8·1955832	8·3651150	8·3377663	n 8·4582360	n 8·7210698	8·4464537	9·2554594	9·2879723	n 9·4781197	n 0·2429710	
10	n 8·1741886	n 7·4910810	8·2438395	7·9269330	n 8·4309269	n 8·5280806	8·6017565	9·2018459	9·1471403	n 9·5499403	n 0·2467784
(o)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9738298	n 9·5319873									
2	9·8065099	9·8864862	n 9·8065099								
3	n 9·3776236	9·8002458	9·7892105	n 9·9560939							
4	n 9·6033796	n 8·5589391	9·7666774	9·6781144	n 0·0545252						
5	n 8·8798899	n 9·4101266	8·7686706	9·7236676	9·5464576	n 0·1249278					
6	9·0609081	n 8·9681810	n 9·2197041	9·0271978	9·6755261	9·3809218	n 0·1776017				
7	8·8553011	8·6938041	n 8·9331752	n 9·0241887	9·1142670	9·6236874	9·1486940	n 0·2180411			
8	n 7·9971248	8·6931287	8·2422055	n 8·8618024	n 8·8122144	9·1469032	9·5686353	8·7177842	n 0·2495257		
9	n 8·4245906	7·5307334	8·5145569	7·2064633	n 8·7745333	n 8·5632609	9·1537217	9·5104294	n 8·4216388	n 0·2741708	
10	n 7·8431681	n 8·1414848	7·8590321	8·3263144	n 7·7865905	n 8·6782312	n 8·2229042	9·1458016	9·4488767	n 8·9817785	n 0·2934209
(p)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9854269	n 9·4105710									
2	9·6968349	9·9387506	n 9·6968349								
3	n 9·5893434	9·7121272	9·8895423	n 9·8581601							
4	n 9·5572655	n 9·2814031	9·7015845	9·8373655	n 9·9683327						
5	8·5819810	n 9·4135138	n 8·9429207	9·6828170	9·7816596	n 0·0504765					
6	9·1554828	n 8·1111810	n 9·2820127	n 8·4236328	9·6602051	9·7216890	n 0·1148917				
7	8·4748230	8·9199756	n 8·4897903	n 9·1591145	8·0831566	9·6353468	9·6564606	n 0·1670723			
8	n 8·5820217	8·4578587	8·6974962	n 8·5586098	n 9·0416646	8·5733536	9·6089516	9·5845886	n 0·2102982		
9	n 8·3125429	n 8·2501632	8·3779949	8·4796978	n 8·5593980	n 8·9271177	8·7351396	9·5813745	9·5040540	n 0·2466846	
10	7·7752411	n 8·1427969	n 7·9011623	8·2779044	8·2579087	n 8·5331468	n 8·8132670	8·8170738	9·5527912	9·4117511	n 0·2776760

(q)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9935689	n 9·2369793									
2	9·5314913	9·9735192	n 9·5314913								
3	n 9·7018271	9·5614940	9·9530342	n 9·7010646							
4	n 9·4333709	n 9·5017431	9·5663381	9·9320851	n 9·8194853						
5	9·1500206	n 9·3187871	n 9·3336013	9·5633844	9·9106399	n 9·9098773					
6	9·1068854	8·8380196	n 9·2186734	n 9·1811253	9·5569734	9·8886634	n 9·9825406				
7	n 8·3553967	8·9236924	8·5315774	n 9·1296272	n 9·0350655	9·5487166	9·8661163	n 0·0429693			
8	n 8·6670560	n 7·7537253	8·7605180	8·1914024	n 9·0489146	n 8·8884081	9·5393085	9·8429548	n 0·0944433		
9	n 7·2592766	n 8·4382135	7·0073611	8·6146975	7·7119681	n 8·9745738	n 8·7339552	9·5291596	9·8191300	n 0·1390779	
10	8·1472867	n 7·5474011	n 8·2297741	7·5581177	8·4796422	n 6·9535582	n 8·9051923	n 8·5617265	9·5184737	9·7945868	n 0·1783174

(r)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9983990	n 8·9374424									
2	9·2368495	9·9934836	n 9·2368494								
3	n 9·7599824	9·2754249	9·9885432	n 9·4113179							
4	n 9·1622608	n 9·6019992	9·2891287	9·9835772	n 9·5346337						
5	9·3276169	n 9·0633753	n 9·4807890	9·2951629	9·9785853	n 9·6299209					
6	8·8746531	9·1116300	n 8·9794256	n 9·3813521	9·2978330	9·9735669	n 9·7074792				
7	n 8·8148663	8·7156673	8·9316091	n 8·9069609	n 9·2962840	9·2987282	9·9685219	n 9·7728030			
8	n 8·4921177	n 8·5637879	8·5786399	8·7759146	n 8·8432449	n 9·2213768	9·2985720	9·9634498	n 9·8291722		
9	8·2483621	n 8·2983234	n 8·3440346	8·4581768	8·6376775	n 8·7863329	n 9·1540064	9·2977407	9·9583498	n 9·8787017	
10	8·0537993	7·9690117	n 8·1271219	n 8·1468621	8·3505673	8·5124878	n 8·7348311	n 9·0924229	9·2964495	9·9532219	n 9·9228363

VALUES OF LOG X^m_{-n} FROM THE FORMULA $X^m_{-n} = r^{n-1}[(n-m)\,H'^{m+1}_n - m\mu'\,(1-\mu'^2)^{-\frac{1}{2}}\,H'^m_n]$.

(a)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	8·9431821	*n* 9·9983220									
2	9·2410911	*n* 9·9918188	*n* 9·2410912								
3	9·3163082	*n* 9·8858048	*n* 9·2362823	*n* 8·3589215							
4	9·2903068	*n* 9·7278836	*n* 9·1633868	*n* 8·3546760	*n* 7·4255993						
5	9·2054587	*n* 9·5371853	*n* 9·0471381	*n* 8·2985690	*n* 7·4216352	*n* 6·4642484					
6	9·0818156	*n* 9·3233030	*n* 8·9005554	*n* 8·2057496	*n* 7·3757691	*n* 6·4604531	*n* 5·4851689				
7	8·9305058	*n* 9·0916623	*n* 8·7313740	*n* 8·0854152	*n* 7·2982330	*n* 6·4215001	*n* 5·4814862	*n* 4·4938547			
8	8·7583600	*n* 8·8455418	*n* 8·5445714	*n* 7·9435957	*n* 7·1958222	*n* 6·3547641	*n* 5·4475172	*n* 4·4902520	*n* 3·4935857		
9	8·5698744	*n* 8·5869674	*n* 8·3435374	*n* 7·7844678	*n* 7·0732969	*n* 6·2654564	*n* 5·3888307	*n* 4·4600487	*n* 3·4900435	*n* 2·4864775	
10	8·3681673	*n* 8·3171470	*n* 8·1306778	*n* 7·6110465	*n* 6·9341277	*n* 6·1574059	*n* 5·3095431	*n* 4·4075966	*n* 3·4627851	*n* 2·4829819	*n* 1·4739740

(b)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·2424903	*n* 9·9932632									
2	9·5353736	*n* 9·9712115	*n* 9·5353736								
3	9·6029734	*n* 9·8406952	*n* 9·5203300	*n* 8·9525454							
4	9·5666853	*n* 9·6476353	*n* 9·4334926	*n* 8·9398130	*n* 8·3185645						
5	9·4687726	*n* 9·4082837	*n* 9·2993245	*n* 8·8729572	*n* 8·3069831	*n* 7·6565549					
6	9·3291418	*n* 9·1271444	*n* 9·1304689	*n* 8·7670320	*n* 8·2518910	*n* 7·6456626	*n* 6·9768166				
7	9·1587330	*n* 8·7998102	*n* 8·9341459	*n* 8·6311137	*n* 8·1634744	*n* 7·5983707	*n* 6·9663831	*n* 6·2848437			
8	8·9641363	*n* 8·4071402	*n* 8·7146224	*n* 8·4710785	*n* 8·0484727	*n* 7·5220279	*n* 6·9246550	*n* 6·2747376	*n* 5·5839162		
9	8·7495359	*n* 7·8793743	*n* 8·4742970	*n* 8·2909100	*n* 7·9115789	*n* 7·4218140	*n* 6·8571837	*n* 6·2371826	*n* 5·5740555	*n* 4·8761491	
10	8·5176499	*n* 5·8006286	*n* 8·2141520	*n* 8·0933826	*n* 7·7561831	*n* 7·3015224	*n* 6·7680669	*n* 6·1765197	*n* 5·5397481	*n* 4·8664792	*n* 4·1629870

(c)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·4157085	*n* 9·9847477									
2	9·7001303	*n* 9·9353294	*n* 9·7001304								
3	9·7547304	*n* 9·7576553	*n* 9·6674270	*n* 9·2905744							
4	9·7005338	*n* 9·4856446	*n* 9·5556456	*n* 9·2633026	*n* 8·8298658						
5	9·5792452	*n* 9·1019967	*n* 9·3877170	*n* 9·1776349	*n* 8·8052846	*n* 8·3411285					
6	9·4099483	*n* 8·4519870	*n* 9·1737485	*n* 9·0481221	*n* 8·7342143	*n* 8·3181536	*n* 7·8346625				
7	9·2023653	8·1653191	*n* 8·9165465	*n* 8·8831254	*n* 8·6266066	*n* 8·2565080	*n* 7·8127553	*n* 7·3159619			
8	8·9612326	8·4300730	*n* 8·6119541	*n* 8·6874919	*n* 8·4888926	*n* 8·1634082	*n* 7·7577239	*n* 7·2948156	*n* 6·7883066		
9	8·6877854	8·4028722	*n* 8·2435581	*n* 8·4636872	*n* 8·3253588	*n* 8·0438698	*n* 7·6750367	*n* 7·2446915	*n* 6·7677303	*n* 6·2538118	
10	8·3797044	8·2814093	*n* 7·7556405	*n* 8·2121454	*n* 8·1388559	*n* 7·9014798	*n* 7·5686912	*n* 7·1698741	*n* 6·7213952	*n* 6·2336781	*n* 5·7139220

(d)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·5366178	n 9·9726436									
2	9·8090088	n 9·8813918	n 9·8090088								
3	9·8447237	n 9·6183359	n 9·7501909	n 9·5204354							
4	9·7636296	n 9·1331569	n 9·5991198	n 9·4719119	n 9·1807093						
5	9·6054606	8·3190314	n 9·3723326	n 9·3577305	n 9·1372430	n 8·8129547					
6	9·3858496	8·9026615	n 9·0652303	n 9·1905761	n 9·0423813	n 8·7724945	n 8·4274712				
7	9·1079884	8·9224625	n 8·6363874	n 8·9756382	n 8·9052519	n 8·6896925	n 8·3890037	n 8·0297531			
8	8·7624365	8·8131656	n 7·7943274	n 8·7119338	n 8·7310638	n 8·5713178	n 8·3144928	n 7·9927035	n 7·6230804		
9	8·3107906	8·6316893	7·9268600	n 8·3897351	n 8·5222220	n 8·4217719	n 8·2091307	n 7·9242587	n 7·5870910	n 7·2095681	
10	7·5399312	8·3954535	8·0602827	n 7·9776188	n 8·2785202	n 8·2437831	n 8·0765565	n 7·8285188	n 7·5232837	n 7·1744017	n 6·7906609

(e)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·6283343	n 9·9567534									
2	9·8849254	n 9·8042004	n 9·8849254								
3	9·8950565	n 9·3664281	n 9·7897851	n 9·6881587							
4	9·7755799	8·6615422	n 9·5780082	n 9·6105848	n 9·4402394						
5	9·5600499	9·2031870	n 9·2394108	n 9·4550946	n 9·3711893	n 9·1642914					
6	9·2495802	9·2144328	n 8·6387117	n 9·2280886	n 9·2428262	n 9·1002762	n 8·8706146				
7	8·8018245	9·0849127	8·2687156	n 8·9201873	n 9·0617370	n 8·9881536	n 8·8099241	n 8·5647033			
8	7·8681772	8·8662450	8·5646552	n 8·4853553	n 8·8285388	n 8·8333774	n 8·7086877	n 8·5063720	n 8·2498373		
9	n 8·0975902	8·5665518	8·5231355	n 7·5927315	n 8·5363053	n 8·6382678	n 8·5713131	n 8·4129666	n 8·1932671	n 7·9281318	
10	n 8·1711590	8·1638170	8·3655641	7·8031597	n 8·1617396	n 8·4020413	n 8·4002963	n 8·2880729	n 8·1057757	n 7·8729265	n 7·6010314

(f)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·7011476	n 9·9367999									
2	9·9378897	n 9·6934829	n 9·9378897								
3	9·9144616	n 8·5624095	n 9·7932476	n 9·8140406							
4	9·7404105	9·3384865	n 9·4823842	n 9·6978675	n 9·6390391						
5	9·4292935	9·4242233	n 8·8336625	n 9·4821318	n 9·5364299	n 9·4360088					
6	8·8980008	9·3083598	8·7003623	n 9·1506026	n 9·3613942	n 9·3413391	n 9·2152498				
7	n 8·0814598	9·0694490	8·9090454	n 8·5931795	n 9·1126685	n 9·1893644	n 9·1257937	n 8·9822563			
8	n 8·5764666	8·7034082	8·8323964	7·9903074	n 8·7709960	n 8·9817793	n 8·9887867	n 8·8964861	n 8·7403080		
9	n 8·5155312	8·0918362	8·6301948	8·3849781	n 8·2548113	n 8·7134227	n 8·8064975	n 8·7700194	n 8·6572819	n 8·4915202	
10	n 8·2964137	n 7·5585575	8·3234783	8·3424418	7·0523324	n 8·3656609	n 8·5777207	n 8·6050861	n 8·5386560	n 8·4106165	n 8·2373374

Values of Log X^{m}_{-n} from the formula $X^{m}_{-n} = r^{n-1}\left[(n-m)\,H'^{m+1}_{n} - m\mu'\,(1-\mu'^2)^{-\frac{1}{2}}\,H'^{m}_{n}\right]$ (*continued*).

(g)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·7605385	n 9·9124034									
2	9·9729994	n 9·5254873	n 9·9729995								
3	9·9060038	9·2024044	n 9·7603878	n 9·9086567							
4	9·6513447	9·5570306	n 9·2571428	n 9·7414443	n 9·7931614						
5	9·1417262	9·5060942	8·7877408	n 9·4325767	n 9·6469847	n 9·6496375					
6	n 8·4245778	9·2797524	9·1459377	n 8·8537274	n 9·4052280	n 9·5156098	n 9·4883847				
7	n 8·8607356	8·8728323	9·0850384	8·4439025	n 9·0390658	n 9·3088822	n 9·3622716	n 9·3148973			
8	n 8·7704700	7·9150654	8·8620189	8·7362909	n 8·3812542	n 9·0196461	n 9·1772811	n 9·1943505	n 9·1324554		
9	n 8·4932338	n 8·2009008	8·4868583	8·6566149	8·1142792	n 8·6082377	n 8·9296227	n 9·0242357	n 9·0160370	n 8·9431739	
10	n 8·0018640	n 8·2296286	7·8042074	8·4282899	8·3249521	n 7·8488631	n 8·6041278	n 8·8031305	n 8·8567702	n 8·8299395	n 8·7484974

(h)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·8097695	n 9·8830476									
2	9·9929973	n 9·2222462	n 9·9929972								
3	9·8690391	9·5338662	n 9·6835472	n 9·9780082							
4	9·4826637	9·6603247	n 8·3767012	n 9·7422404	n 9·9118666						
5	n 7·4459564	9·5005111	9·2808365	n 9·2659951	n 9·7087272	n 9·8176962					
6	n 9·0515979	9·1158288	9·3041841	8·4682095	n 9·3641733	n 9·6330185	n 9·7057972				
7	n 8·9965909	8·0116269	9·0969023	8·9903644	n 8·7043308	n 9·3475763	n 9·5330053	n 9·5816636			
8	n 8·6957997	n 8·5225933	8·6838965	8·9260050	8·4792132	n 8·9084062	n 9·2817010	n 9·4171666	n 9·4485752		
9	n 8·0484935	n 8·5040801	7·5779293	8·6720402	8·6624876	n 7·8449903	n 8·9284745	n 9·1882439	n 9·2901970	n 9·3086473	
10	7·7660506	n 8·2283888	n 8·0519770	8·2186039	8·5278809	8·2795412	n 8·3780204	n 8·8826216	n 9·0771350	n 9·1549694	n 9·1633245

(i)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·8509343	n 9·8480260									
2	9·9992669	7·8251246	n 9·9992669								
3	9·7988556	9·6978649	n 9·5386141	n 0·0255690							
4	9·1318707	9·6971762	9·1726171	n 9·6938425	n 0·0007187						
5	n 9·1130850	9·4081895	9·4712476	n 8·7228751	n 9·7211262	n 9·9478396					
6	n 9·2004136	8·6170110	9·3375281	9·1439374	n 9·1869855	n 9·6967657	n 9·8772317				
7	n 8·9340218	n 8·7163443	8·9502814	9·1688770	8·5721488	n 9·2856240	n 9·6441802	n 9·7943892			
8	n 8·2350506	n 8·7440049	7·7101075	8·9291036	8·9319806	n 8·3400142	n 9·2944215	n 9·5737681	n 9·7025922		
9	8·1334810	n 8·4429276	n 8·3642302	8·4268525	8·8088511	8·6057280	n 8·7157083	n 9·2613108	n 9·4910654	n 9·6039555	
10	8·2022630	n 7·6764247	n 8·2970286	n 7·4871863	8·4745210	8·6294773	8·0375476	n 8·8042905	n 9·2038223	n 9·3993730	n 9·4999239

(k)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·8854509	n 9·8063561									
2	9·9922397	9·2550015	n 9·9922397								
3	9·6834569	9·7920241	n 9·2456008	n 0·0531846							
4	n 8·5625015	9·6798276	9·5002375	n 9·5751064	n 0·0629770						
5	n 9·3426405	9·1813077	9·5516093	9·0276847	n 9·6750129	n 0·0447407					
6	n 9·1921164	n 8·7190856	9·2620921	9·3656123	n 8·4418005	n 9·7030493	n 0·0087755				
7	n 8·6115282	n 8·9563506	8·4163657	9·2004260	9·1236428	n 9·0302380	n 9·6954025	n 9·9605758			
8	8·3236241	n 8·6941503	n 8·5750473	8·7315980	9·0793631	8·7864780	n 9·1736450	n 9·6663561	n 9·9034214		
9	8·4487780	n 7·8641376	n 8·5481603	n 7·7712091	8·7554944	8·9178716	8·0974563	n 9·2206867	n 9·6230320	n 9·8394275	
10	8·1290778	7·9970794	n 8·1493999	n 8·2952735	7·9959320	8·6943835	8·7162157	n 8·2650963	n 9·2239988	n 9·5694985	n 9·7700386

(l)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9143170	n 9·7566355									
2	9·9715073	9·5410040	n 9·9715074								
3	9·4909964	9·8427137	8·2534068	n 0·0614406							
4	n 9·2824434	9·6047442	9·6614411	n 9·3103112	n 0·1002211						
5	n 9·4137418	8·3959871	9·5551934	9·4304064	n 9·5405292	n 0·1109729					
6	n 9·0331206	n 9·0992813	9·0230552	9·4528654	9·1072275	n 9·6354593	n 0·1039961				
7	8·2260176	n 8·9662150	n 8·6303951	9·0062108	9·3192384	8·4210785	n 9·6767173	n 0·0847846			
8	8·6656997	n 8·2994427	n 8·7856925	7·4243035	9·0699453	9·1576658	n 8·6436256	n 9·6889485	n 0·0566185		
9	8·3871109	8·2091952	n 8·4258649	n 8·5330008	8·3910162	8·9992098	8·9616841	n 8·9569013	n 9·6829554	n 0·0216129	
10	7·0595673	8·2143196	7·0675290	n 8·3580646	n 8·1811618	8·5181143	8·9008761	8·7096254	n 9·0765202	n 9·6644451	n 9·9812123

(m)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9382541	n 9·6967851									
2	9·9357084	9·7029355	n 9·9357084								
3	9·0913209	9·8596747	9·3483240	n 0·0496929							
4	n 9·4938740	9·4476875	9·7464698	7·8759207	n 0·1125249						
5	n 9·3849311	n 9·0012181	9·4813515	9·6116002	n 9·1889564	n 0·1473282					
6	n 8·4987263	n 9·2034533	7·6721722	9·4455178	9·4535610	n 9·4369385	n 0·1644027				
7	8·8109929	n 8·7742393	n 8·9710740	8·7364840	9·3790490	9·2619747	n 9·5539015	n 0·1692426			
8	8·6729720	8·2483231	n 8·7436042	n 8·6816099	8·8752651	9·2938751	9·0088028	n 9·6182145	n 0·1651279		
9	9·7644396	8·4559808	n 7·4707316	n 8·6347146	n 8·2693461	8·9024485	9·1941562	8·5889283	n 9·6539043	n 0·1541736	
10	n 8·0548233	8·0163179	8·1718674	n 8·0251859	n 8·4843639	n 6·6080910	8·8836547	9·0806288	n 7·9930526	n 9·6715689	n 0·1378245

VALUES OF LOG X^m_{-n} FROM THE FORMULA $X^m_{-n} = r^{n-1}[(n-m)\,H'^{m+1}_n - m\mu'(1-\mu'^2)^{-\frac{1}{2}}H'^m_n]$ (*continued*).

(n)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9577923	n 9·6235565									
2	9·8821213	9·8104457	n 9·8821213								
3	n 8·8026909	9·8457710	9·6294219	n 0·0157472							
4	n 9·5839016	9·1063399	9·7789898	9·3822736	n 0·0982207						
5	n 9·2461535	n 9·3011171	9·2937016	9·6995042	8·9396081	n 0·1526654					
6	8·7031899	n 9·1670985	n 8·9722816	9·3337565	9·6108358	n 8·6072971	n 0·1893814				
7	8·9297603	n 7·7144949	n 9·0386320	n 8·4927921	9·3260215	9·5131477	n 9·1613361	n 0·2138629			
8	8·3853538	8·6528913	n 8·3841798	n 8·8868322	7·7824535	9·2951769	9·4049452	n 9·3611088	n 0·2293896		
9	n 8·1906890	8·3602208	8·3328721	n 8·4533418	n 8·7161756	8·4415595	9·2505652	9·2830781	n 9·4732255	n 0·2380768	
10	n 8·1687792	n 7·4856716	8·2384301	7·9215236	n 8·4255175	n 8·5226712	8·5963471	9·1964365	9·1417309	n 9·5445309	n 0·2413690

(o)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9733240	n 9·5314815									
2	9·8056669	9·8856432	n 9·8056669								
3	n 9·3764434	9·7990656	9·7880303	n 9·9549137							
4	n 9·6018622	n 8·5574217	9·7651600	9·6765970	n 0·0530078						
5	n 8·8780353	n 9·4082720	8·7668160	9·7218130	9·5446030	n 0·1230732					
6	9·0587162	n 8·9659891	n 9·2175122	9·0250059	9·6733342	9·3787299	n 0·1754098				
7	8·8527720	8·6912750	n 8·9306461	n 9·0216596	9·1117379	9·6211583	9·1461649	n 0·2155120			
8	n 7·9942585	8·6902624	8·2393392	n 8·8589361	n 8·8093481	9·1440369	9·5657690	8·7149179	n 0·2466594		
9	n 8·4213871	7·5275299	8·5113534	7·2032598	n 8·7713298	n 8·5600574	9·1505182	9·5072259	n 8·4184353	n 0·2709673	
10	n 7·8396274	n 8·1379441	7·8554914	8·3227737	n 7·7830498	n 8·6746905	n 8·2193635	9·1422609	9·4453360	n 8·9782378	n 0·2898802

(p)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9851374	n 9·4102815									
2	9·6963524	9·9382681	n 9·6963524								
3	n 9·5886679	9·7114517	9·8888668	n 9·8574846							
4	n 9·5563970	n 9·2805346	9·7007160	9·8364970	n 9·9674642						
5	8·5809195	n 9·4124523	n 8·9418592	9·6817555	9·7805981	n 0·0494150					
6	9·1542283	n 8·1099265	n 9·2807582	n 8·4223783	9·6589506	9·7204345	n 0·1136372				
7	8·4733755	8·9185281	n 8·4883428	n 9·1576670	8·0817091	9·6338993	9·6550131	n 0·1656248			
8	n 8·5803812	8·4562182	8·6958557	n 8·5579693	n 9·0400241	8·5717131	9·6073111	9·5829481	n 0·2086577		
9	n 8·3107093	n 8·2483296	8·3761613	8·4778642	n 8·5575644	n 8·9252841	8·7333060	9·5795409	9·5022204	n 0·2448510	
10	7·7732145	n 8·1407703	n 7·8991357	8·2758778	8·2558821	n 8·5311202	n 8·8112404	8·8150472	9·5507646	9·4097245	n 0·2756494

(q)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·9934386	*n* 9·2368490									
2	9·5312742	9·9733021	*n* 9·5312742								
3	*n* 9·7015231	9·5611900	9·9527302	*n* 9·7007606							
4	*n* 9·4329801	*n* 9·5013523	9·5659473	9·9316943	*n* 9·8190945						
5	9·1495429	*n* 9·3183094	*n* 9·3331236	9·5629067	9·9101622	*n* 9·9093996					
6	9·1063209	8·8374551	*n* 9·2181089	*n* 9·1805608	9·5564089	9·8880989	*n* 9·9819761				
7	*n* 8·3547453	8·9230410	8·5309260	*n* 9·1289758	*n* 9·0344141	9·5480652	9·8654649	*n* 0·0423179			
8	*n* 8·6663178	*n* 7·7529871	8·7597798	8·1906642	*n* 9·0481764	*n* 8·8876699	9·5385703	9·8422166	*n* 0·0937051		
9	*n* 7·2584515	*n* 8·4373884	7·0065360	8·6138724	7·7111430	*n* 8·9737487	*n* 8·7331301	9·5283345	9·8183049	*n* 0·1382528	
10	8·1463748	*n* 7·5464892	*n* 8·2288622	7·5572058	8·4787303	*n* 6·9526463	*n* 8·9042804	*n* 8·5608146	9·5175618	9·7936749	*n* 0·1774055

(r)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	9·9983662	*n* 8·9374096									
2	9·2367948	9·9934289	*n* 9·2367947								
3	*n* 9·7599058	9·2753483	9·9884666	*n* 9·4112413							
4	*n* 9·1621624	*n* 9·6019008	9·2890303	9·9834788	*n* 9·5345353						
5	9·3274966	*n* 9·0632550	*n* 9·4806687	9·2950426	9·9784650	*n* 9·6298006					
6	8·8745109	9·1114878	*n* 8·9792834	*n* 9·3812099	9·2976908	9·9734247	*n* 9·7073370				
7	*n* 8·8147023	8·7155033	8·9314451	*n* 8·9067969	*n* 9·2961200	9·2985642	9·9683579	*n* 9·7726390			
8	*n* 8·4919318	*n* 8·5636020	8·5784540	8·7757287	*n* 8·8430590	*n* 9·2211909	9·2983861	9·9632639	*n* 9·8289863		
9	8·2481543	*n* 8·2981156	*n* 8·3438268	8·4579690	8·6374697	*n* 8·7861251	*n* 9·1537986	9·2975329	9·9581420	*n* 9·8784939	
10	8·0535696	7·9687820	*n* 8·1268922	*n* 8·1466324	8·3503376	8·5122581	*n* 8·7346014	*n* 9·0921932	9·2962198	9·9529922	*n* 9·9226066

VALUES OF LOG Z^m_n FROM THE FORMULA $Z^m_n = \frac{1}{\gamma^{n+2}}(n+1)\,H'^m_n$.

(a)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	·3036811	9·2485412									
2	·3017582	9·4243974	8·3692576								
3	·2012178	9·4513481	8·4939614	7·4388215							
4	·0497214	9·4002040	8·5231277	7·5354965	6·4803566						
5	9·8664822	9·3005132	8·4918036	7·5628689	6·5593029	5·5041630					
6	9·6612055	9·1675216	8·4185199	7·5415866	6·5842849	5·5708748	4·5157350				
7	9·4394672	9·0101431	8·3143132	7·4841322	6·5688163	5·5935888	4·5734920	3·5183521			
8	9·2047462	8·8340439	8·1863025	7·3986574	6·5218102	5·5817961	4·5942055	3·5692696	2·5141297		
9	8·9593381	8·6430493	8·0393533	7·2907908	6·4494739	5·5421536	4·5848886	3·5882526	2·5596523	1·5045125	
10	8·7047989	8·4398635	7·8769388	7·1645867	6·3563115	5·4795220	4·5506892	3·5806840	2·5771417	1·5456702	·4905303

(b)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	·2985229	9·5477500									
2	·2862992	9·7185144	8·9677415								
3	·1699309	9·7377814	9·0873533	8·3365805							
4	9·9965562	9·6762844	9·1097028	8·4281636	7·6773907						
5	9·7845879	9·5634628	9·0698027	8·4491516	7·7512451	7·0004722					
6	9·5425299	9·4144173	8·9861394	8·4201728	7·7701017	7·0620921	6·3113192				
7	9·2741347	9·2378735	8·8696944	8·3536865	7·7474617	7·0788531	6·3639842	5·6132113			
8	8·9800007	9·0392571	8·7275185	8·2578160	7·6922236	7·0602379	6·3788679	5·6590369	4·9082641		
9	8·6575874	8·8220815	8·5643956	8·1381560	7·6105778	7·0128933	6·3629771	5·6722824	4·9486947	4·1979218	
10	8·2995092	8·5886505	8·3836993	7·9987219	7·5070088	6·9416689	6·3214526	5·6583261	4·9605185	4·2339876	3·4832147

(c)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	·2898452	9·7208060									
2	·2598536	9·8830009	9·3139618								
3	·1149483	9·8891600	9·4250040	8·8559648							
4	9·8992629	9·8096465	9·4357718	8·9389784	8·3699393						
5	9·6254320	9·6733408	9·3811160	8·9491425	8·4352241	7·8661849					
6	9·2897178	9·4945211	9·2793037	8·9069957	8·4437100	7·9192352	7·3501961				
7	8·8639500	9·2806950	9·1410236	8·8248821	8·4088412	7·9259272	7·3942916	6·8252524			
8	8·2123255	9·0354346	8·9729392	8·7107794	8·3394457	7·8957038	7·3993214	6·8625085	6·2934694		
9	n 7·7429434	8·7593041	8·7793169	8·5701017	8·2416279	7·8351644	7·3722629	6·8660611	6·3253305	5·7562913	
10	n 8·0383479	8·4495700	8·5628324	8·4066369	8·1197701	7·7491027	7·3182232	6·8412663	6·3275865	5·7837875	5·2147484

(d)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	·2775213	9·8414955									
2	·2212684	9·9915129	9·5554871								
3	·0307978	9·9786403	9·6543520	9·2183262							
4	9·7372829	9·8720827	9·6484129	9·2891624	8·8531366						
5	9·3133999	9·6987500	9·5720252	9·2837668	8·9062440	8·4702183					
6	8·5033963	9·4694696	9·4427264	9·2224046	8·8998538	8·5110912	8·0750654				
7	n 8·5462898	9·1852188	9·2701238	9·1170478	8·8472369	8·5033607	8·1069835	7·6709577			
8	n 8·6778707	8·8353925	9·0592358	8·9751694	8·7569982	8·4563504	8·0979139	7·6960364	7·2600106		
9	n 8·5881046	8·3809167	8·8117122	8·8014879	8·6349544	8·3765115	8·0547468	7·6857313	7·2796942	6·8436684	
10	n 8·4055360	7·6082576	8·5257611	8·5988167	8·4851149	8·2684533	7·9824841	7·6453323	7·2682804	6·8589873	6·4229615

(e)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	·2613603	9·9329412									
2	·1686046	·0669782	9·7385591								
3	9·9060521	·0283413	9·8214435	9·4930244							
4	9·4449337	9·8832206	9·7931008	9·5478801	9·2194610						
5	n 7·7079758	9·6523465	9·6866335	9·5217233	9·2565880	8·9281689					
6	n 9·0188376	9·3320270	9·5174980	9·4341365	9·2304100	8·9530614	8·6246423				
7	n 9·0278108	8·8777011	9·2916599	9·2960777	9·1537694	8·9261871	8·6405799	8·3121608			
8	n 8·8747418	7·9395990	9·0071219	9·1134727	9·0346773	8·8565761	8·6128243	8·3212591	7·9928400		
9	n 8·6227844	n 8·1660015	8·6497182	8·8885574	8·8781153	8·7502909	8·5480517	8·2926099	7·9965431	7·6681240	
10	n 8·2739994	n 8·2375900	8·1699607	8·6198793	8·6868800	8·6114252	8·4509911	8·2313406	7·9670504	7·6674624	7·3390433

(f)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	·2410936	·0054413									
2	·0986274	·1194203	9·8837680								
3	9·7130334	·0470155	9·9465945	9·7109422							
4	8·2788099	9·8471115	9·8892773	9·7457400	9·5100878						
5	n 9·2503470	9·5204416	9·7420160	9·6929185	9·5271569	9·2915046					
6	n 9·2711749	8·9790902	9·5145266	9·5705115	9·4756735	9·2963392	9·0606869				
7	n 9·0966063	n 8·1557701	9·1995914	9·3868648	9·3675190	9·2450549	9·0565666	8·8209143			
8	n 8·7865629	n 8·6461132	8·7550396	9·1431283	9·2092333	9·1460277	9·0050353	8·8099546	8·5743023		
9	n 8·2906518	n 8·5819586	7·9021636	8·8316350	9·0034255	9·0043834	8·9122957	8·7580029	8·5579476	8·3222953	
10	6·3533005	n 8·3606521	n 7·9565396	8·4253928	8·7487849	8·8228203	8·7824300	8·6698273	8·5055186	8·3015758	8·0659235

Values of Log Z_n^m from the formula $Z_n^m = \frac{1}{\gamma^{n+2}}(n+1)\,H'^m_n$ (*continued*).

(*g*)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	·2163510	·0644861									
2	·0056844	·1539533	·0020885								
3	9·3521326	·0377503	·0404032	9·8885383							
4	*n* 9·3134468	9·7570075	9·9461983	9·8988242	9·7469593						
5	*n* 9·4599845	9·2316053	9·7431123	9·8123795	9·7395165	9·5876517					
6	*n* 9·3064964	*n* 8·5041676	9·4239492	9·6438453	9·6563118	9·5679745	9·4161096				
7	*n* 8·9655511	*n* 8·9333157	8·9170246	9·3946807	9·5071571	9·4862138	9·3874774	9·2356125			
8	*n* 8·2878179	*n* 8·8381557	*n* 7·4738714	9·0492174	9·2949348	9·3493530	9·3063480	9·2001410	9·0482762		
9	8·0588677	*n* 8·5574695	*n* 8·4332279	8·5408703	9·0150205	9·1602710	9·1779231	9·1192439	9·0074096	8·8555447	
10	8·2119682	*n* 8·0636800	*n* 8·3722175	6·8166944	8·6489110	8·9177550	9·0047723	8·9969357	8·9265396	8·8103134	8·6584485

(*h*)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	·1866270	·1133489									
2	9·8788765	·1733376	·1000594								
3	*n* 8·7049147	9·9999264	·1088955	·0356174							
4	*n* 9·5729112	9·5872219	9·9677663	·0164248	9·9431467						
5	*n* 9·5107652	*n* 7·5344855	9·6850424	9·8877750	9·9062254	9·8329472					
6	*n* 9·1951278	*n* 9·1295921	9·1919908	9·6567974	9·7835250	9·7837914	9·7105133				
7	*n* 8·4449167	*n* 9·0673299	*n* 7·9511236	9·3047540	9·5803970	9·6642933	9·6524026	9·5791245			
8	8·4288664	*n* 8·7613988	*n* 8·7327196	8·7289007	9·2899999	9·4784835	9·5347690	9·5141744	9·4408962		
9	8·4997418	*n* 8·1103973	*n* 8·6366985	*n* 7·9562743	8·8804566	9·2236030	9·3607159	9·3976859	9·3705511	9·2972730	
10	8·2636180	7·8252892	*n* 8·3152821	*n* 8·3476172	8·1983858	8·8857782	9·1291599	9·2321058	9·2547925	9·2225631	9·1492850

(*i*)	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=1$	·1512263	·1541346									
2	9·6930501	·1789753	·1818836								
3	*n* 9·4680066	9·9288584	·1555718	·1584801							
4	*n* 9·6733053	9·2352916	9·9539304	·1041395	·1070479						
5	*n* 9·4578156	*n* 9·2002240	9·5461445	9·9220883	·0349786	·0378869					
6	*n* 8·8397654	*n* 9·2767650	8·3355849	9·6018155	9·8628734	9·9535831	9·9564914				
7	8·6112811	*n* 9·0028654	*n* 8·9435361	9·0469397	9·5860536	9·7872411	9·8632328	9·8661411			
8	8·7392733	*n* 8·2985015	*n* 8·8818390	*n* 8·2382442	9·1672137	9·5364211	9·7005362	9·7660431	9·7689514		
9	8·4795722	8·1929838	*n* 8·5294164	*n* 8·6406146	8·3848230	9·1824332	9·4669852	9·6058016	9·6634583	9·6663666	
10	7·8166355	8·2588478	*n* 7·6026500	*n* 8·4655686	8·2413638	8·6539240	9·1518126	9·3845282	9·5049510	9·5565087	9·5594170

(k)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	·1091780	·1882728									
2	9·3637335	·1713174	·2504122								
3	n 9·6903110	9·8125767	·1823044	·2613992							
4	n 9·6920561	n 8·6647871	9·9000892	·1652626	·2443574						
5	n 9·2818929	n 9·4283919	9·2438129	9·9140964	·1304920	·2095869					
6	8·4956130	n 9·2668280	n 9·0230723	9·4499550	9·8955023	·0834871	·1625819				
7	8·9412817	n 8·6784796	n 9·1130172	n 6·8103310	9·5098735	9·8581765	·0275272	·1066220			
8	8·7231714	8·3849306	n 8·7937293	n 8·8750282	8·7910770	9·5159051	9·8085512	9·9647299	·0438227		
9	8·0076979	8·5058841	n 7·7813632	n 8·7263126	n 8·5091156	8·9794684	9·4938472	9·7501682	9·8965336	9·9756284	
10	n 7·9575811	8·1830137	8·1432502	n 8·2321302	n 8·5800094	n 7·3859053	9·0414207	9·4545378	9·6852014	9·8239745	9·9030693

(l)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	·0590911	·2167726									
2	n 8·3419461	·1499745	·3076561								
3	n 9·7987584	9·6192613	·1897055	·3473870							
4	n 9·6435636	n 9·3836302	9·7933403	·2014078	·3590894						
5	n 8·7902225	n 9·4981502	n 8·2296641	9·8567235	·1953813	·3530628					
6	9·0599240	n 9·1062449	n 9·2927987	9·0753182	9·8774775	·1771203	·3348019				
7	8·9811217	8·2911375	n 9·0903609	n 9·0157852	9·3002306	9·8753551	·1499045	·3075860			
8	8·3874532	8·7249305	n 8·3186345	n 8·9917774	n 8·5572420	9·3877239	9·8588219	·1158492	·2735308		
9	n 8·1644633	8·4418970	8·3629577	n 8·5353268	n 8·8450326	8·0530351	9·4226081	9·8323095	·0763990	·2340805	
10	n 8·2221717	7·1109390	8·3135589	7·7179295	n 8·5454973	n 8·6508152	8·6344007	9·4296349	9·7984433	·0325839	·1902655

(m)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9988975	·2403665									
2	n 9·4122350	·1136037	·3550727								
3	n 9·8478120	9·2187853	·1771574	·4186263							
4	n 9·5151913	n 9·5940313	9·5973408	·2126822	·4541512						
5	8·8640182	n 9·4680812	n 9·2876415	9·7302089	·2304783	·4719473					
6	9·1998716	n 8·5703636	n 9·3623885	n 8·7614527	9·7957593	·2360400	·4775090				
7	8·8167482	8·8743971	n 8·8640307	n 9·2261898	8·5795761	9·8295205	·2326469	·4741158			
8	n 8·1626486	8·7302583	8·4393620	n 8·8978630	n 9·0603271	9·0225718	9·8446466	·2224142	·4638832		
9	n 8·4571649	7·8170526	8·5653404	7·2386868	n 8·8604537	n 8·8515582	9·1816985	9·8474583	·2067866	·4482556	
10	n 8·0503017	n 8·1037930	8·0829392	8·3525742	n 8·0573042	n 8·7872604	n 8·5568712	9·2626049	9·8414900	·1867942	·4282632

VALUES OF LOG Z^m_n FROM THE FORMULA $Z^m_n = \frac{1}{r^{n+2}}(n+1)\,H'^m_n$ (*continued*).

(n)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·9253593	·2595951									
2	n 9·6731752	·0595004	·3937362								
3	n 9·8536042	n 8·9294327	·1424890	·4767248							
4	n 9·2382666	n 9·6831299	9·1449217	·1974489	·5316848						
5	9·2669711	n 9·3281683	n 9·5141192	9·4698285	·2346802	·5689160					
6	9·1818176	8·7734854	n 9·3048988	n 9·3267139	9·6115146	·2596769	·5939127				
7	8·0377320	8·9916161	6·7790787	n 9·2463415	n 9·0934883	9·6931809	·2757187	·6099545			
8	n 8·6434725	8·4408853	8·7797875	n 8·4377468	n 9·1694674	n 8·7376305	9·7448979	·2849211	·6191569		
9	n 8·3799758	n 8·2413407	8·4466275	8·5415099	n 8·6000362	n 9·0790861	6·6258052	9·7786212	·2887285	·6229643	
10	7·3813436	n 8·2155813	n 7·6657611	8·3723702	8·2543053	n 8·6425066	n 8·9756764	8·6456995	9·8003074	·2881711	·6224069

(o)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·8330173	·2748598									
2	n 9·8157178	9·9826012	·4244437								
3	n 9·8195874	n 9·5025623	·0810325	·5228751							
4	n 5·6493716	n 9·7002896	n 9·0820869	·1514352	·5932777						
5	9·4007633	n 8·9590712	n 9·5953831	8·3515790	·2041091	·6459516					
6	8·9994289	9·1278549	n 9·0806391	n 9·4941481	9·1323046	·2445485	·6863910				
7	n 8·6599867	8·9132931	8·8501161	n 9·1036005	n 9·3907162	9·3586949	·2760330	·7178756			
8	n 8·6970565	n 8·0482773	8·7983028	8·5285820	n 9·0920690	n 9·2803654	9·4854895	·3006782	·7425207		
9	n 7·6496287	n 8·4703481	7·4635813	8·6722425	8·0484471	n 9·0635991	n 9·1576788	9·5693014	·3199283	·7617708	
10	8·1335328	n 7·8845608	n 8·2364640	7·9232876	8·5392169	n 7·4584254	n 9·0249352	n 9·0144400	9·6291404	·3348136	·7766561

(p)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	9·7116010	·2864569									
2	n 9·9043084	9·8729262	·4477821								
3	n 9·7402390	n 9·7142821	9·9830988	·5579547							
4	9·2231199	n 9·6541755	n 9·5546311	·0652427	·6400986						
5	9·4169075	8·6611623	n 9·5861842	n 9·3870277	·1296579	·7045137					
6	8·2847288	9·2224296	n 7·9638212	n 9·5266352	n 9·1800669	·1818385	·7566944				
7	n 8·9140124	8·5328150	9·0507991	n 8·5734908	n 9·4712559	n 8·8617845	·2250643	·7999202			
8	n 8·4766725	n 8·6331742	8·5473334	8·8913955	n 8·7218838	n 9·4177772	n 7·5750472	·2614507	·8363066		
9	8·2350723	n 8·3583004	n 8·3616531	8·5100412	8·7372270	n 8·7845331	n 9·3648033	8·7622101	·2924421	·8672980	
10	8·1477201	7·8166338	n 8·2249598	n 8·0816446	8·4536578	8·5825652	n 8·8123030	n 9·3113508	9·0547357	·3190687	·8939246

(q)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n = 1$	9·5380093	·2945989									
2	n 9·9595522	9·7075826	·4641722								
3	n 9·5951804	n 9·8267658	9·8260033	·5825929							
4	9·4851857	n 9·5302809	n 9·7395419	9·9163953	·6729849						
5	9·3289808	9·2292018	n 9·4844391	n 9·6691104	9·9890586	·7456481					
6	n 8·8138116	9·1738321	9·0070771	n 9·4481504	n 9·6055579	·0494873	·8060769				
7	n 8·9266221	n 8·4133887	9·0456849	8·7937589	n 9·4172905	n 9·5442346	·1009613	·8575509			
8	7·6678311	n 8·7182085	n 7·9345049	8·9350142	8·5684054	n 9·3897691	n 9·4823605	·1455958	·9021854		
9	8·4376414	n 7·3050341	n 8·5356829	n 6·4232733	8·8363993	8·2992396	n 9·3644065	n 9·4178300	·1848354	·9414250	
10	7·5765744	8·1886794	n 7·6031696	n 8·3713322	7·5928152	8·7464584	7·8916789	n 9·3404843	n 9·3486336	·2197101	·9762997

(r)	$m = 0$	1	2	3	4	5	6	7	8	9	10
$n = 1$	9·2384724	·2994290									
2	n 9·9901661	9·4129408	·4738973								
3	n 9·3122151	n 9·8849211	9·5362566	·5972132							
4	9·5985579	n 9·2591708	n 9·8272624	9·6315437	·6925003						
5	9·0770285	9·4067981	n 9·2253694	n 9·7887502	9·7091021	·7700587					
6	n 9·1079687	8·9415998	9·2613878	n 9·2013164	n 9·7598855	9·7744260	·8353825				
7	n 8·7223900	n 8·8728583	8·8337773	9·1431970	n 9·1828962	n 9·7365496	9·8307950	·8917516			
8	8·5597647	n 8·5432702	n 8·6769142	8·7440848	9·0427242	n 9·1680272	n 9·7166571	9·8803247	·9412813		
9	8·3020397	8·2941196	n 8·3914794	n 8·5072758	8·6671588	8·9545715	n 9·1555438	n 9·6990361	9·9244593	·9854158	
10	n 7·9644050	8·0951920	8·0607258	n 8·2594949	n 8·3562970	8·5996744	8·8753896	n 9·1447348	n 9·6829745	9·9642290	1·0251856

VALUES OF LOG Z^m_{-n} FROM THE FORMULA $Z^m_{-n} = -nr^{n-1} H'^m_n$.

(a)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·9983220	n 8·9431821									
2	n 0·1184519	n 9·2410911	n 8·1859513								
3	n 0·0661779	n 9·3163082	n 8·3589215	n 7·3037816							
4	n 9·9398242	n 9·2903068	n 8·4132305	n 7·4255993	n 6·3704594						
5	n 9·7714277	n 9·2054587	n 8·3967491	n 7·4678144	n 6·4642484	n 5·4091085					
6	n 9·5754994	n 9·0818155	n 8·3328138	n 7·4558805	n 6·4985788	n 5·4851687	n 4·4300289				
7	n 9·3598299	n 8·9305058	n 8·2346759	n 7·4044949	n 6·4891790	n 5·5139515	n 4·4938547	n 3·4387148			
8	n 9·1290623	n 8·7583600	n 8·1106186	n 7·3229735	n 6·4461263	n 5·5061122	n 4·5185216	n 3·4935857	n 2·4384458		
9	n 8·8861631	n 8·5698743	n 7·9661783	n 7·2176158	n 6·3762989	n 5·4689786	n 4·5117136	n 3·5150776	n 2·4864773	n 1·4313375	
10	n 8·6331027	n 8·3681673	n 7·8052426	n 7·0928905	n 6·2846153	n 5·4078258	n 4·4789930	n 3·5089878	n 2·5054455	n 1·4739740	n 0·4188341

(b)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·9932632	n 9·2424903									
2	n 0·1031584	n 9·5353736	n 8·7846007								
3	n 0·0351229	n 9·6029734	n 8·9525453	n 8·2017725							
4	n 9·8869571	n 9·5666853	n 9·0001037	n 8·3185645	n 7·5677916						
5	n 9·6898977	n 9·4687726	n 8·9751125	n 8·3544614	n 7·6565549	n 6·9057820					
6	n 9·4572543	n 9·3291417	n 8·9008638	n 8·3348972	n 7·6848261	n 6·9768165	n 6·2260436				
7	n 9·1949942	n 9·1587330	n 8·7905539	n 8·2745460	n 7·6683212	n 6·9997126	n 6·2848437	n 5·5340708			
8	n 8·9048799	n 8·9641363	n 8·6523977	n 8·1826952	n 7·6171028	n 6·9851171	n 6·3037471	n 5·5839161	n 4·8331433		
9	n 8·5850418	n 8·7495359	n 8·4918500	n 8·0656104	n 7·5380322	n 6·9403477	n 6·2904315	n 5·5997368	n 4·8761491	n 4·1253762	
10	n 8·2285086	n 8·5176499	n 8·3126987	n 7·9277213	n 7·4360082	n 6·8706683	n 6·2504520	n 5·5873255	n 4·8895179	n 4·1629870	n 3·4122141

(c)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·9847477	n 9·4157085									
2	n 0·0769830	n 9·7001303	n 9·1310912								
3	n 9·9805187	n 9·7547304	n 9·2905744	n 8·7215352							
4	n 9·7901502	n 9·7005338	n 9·3266591	n 8·8298657	n 8·2608266						
5	n 9·5313364	n 9·5792452	n 9·2870204	n 8·8550469	n 8·3411285	n 7·7720893					
6	n 9·2051450	n 9·4099483	n 9·1947309	n 8·8224229	n 8·3591372	n 7·8346624	n 7·2656233				
7	n 8·7856203	n 9·2023653	n 9·0626939	n 8·7465524	n 8·3305115	n 7·8475975	n 7·3159619	n 6·7469227			
8	n 8·1381235	n 8·9612326	n 8·8987372	n 8·6365774	n 8·2652437	n 7·8215018	n 7·3251194	n 6·7883065	n 6·2192674		
9	7·6714247	n 8·6877854	n 8·7077982	n 8·4985830	n 8·1701092	n 7·7636457	n 7·3007442	n 6·7945424	n 6·2538118	n 5·6847726	
10	7·9684823	n 8·3797044	n 8·4929668	n 8·3367713	n 8·0499045	n 7·6792371	n 7·2483576	n 6·7714007	n 6·2577209	n 5·7139219	n 5·1448828

(d)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	*n* 9·9726436	*n* 9·5366178									
2	*n* 0·0387643	*n* 9·8090088	*n* 9·3729830								
3	*n* 9·8968811	*n* 9·8447236	*n* 9·5204353	*n* 9·0844095							
4	*n* 9·6288298	*n* 9·7636296	*n* 9·5399598	*n* 9·1807093	*n* 8·7446835						
5	*n* 9·2201105	*n* 9·6054606	*n* 9·4787358	*n* 9·1904774	*n* 8·8129546	*n* 8·3769289					
6	*n* 8·4197763	*n* 9·3858496	*n* 9·3591064	*n* 9·1387846	*n* 8·8162338	*n* 8·4274712	*n* 7·9914454				
7	8·4690595	*n* 9·1079885	*n* 9·1928935	*n* 9·0398175	*n* 8·7700066	*n* 8·4261304	*n* 8·0297532	*n* 7·5937274			
8	8·6049147	*n* 8·7624365	*n* 8·9862798	*n* 8·9022134	*n* 8·6840422	*n* 8·3833944	*n* 8·0249578	*n* 7·6230804	*n* 7·1870546		
9	8·5179785	*n* 8·3107906	*n* 8·7415861	*n* 8·7313618	*n* 8·5648283	*n* 8·3063854	*n* 7·9846207	*n* 7·6156052	*n* 7·2095681	*n* 6·7735423	
10	8·3372096	*n* 7·5399312	*n* 8·4574347	*n* 8·5304903	*n* 8·4167885	*n* 8·2001269	*n* 7·9141577	*n* 7·5770059	*n* 7·1999540	*n* 6·7906609	*n* 6·3546351

(e)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	*n* 9·9567534	*n* 9·6283343									
2	*n* 9·9865518	*n* 9·8849254	*n* 9·5565063								
3	*n* 9·7727672	*n* 9·8950564	*n* 9·6881586	*n* 9·3597395							
4	*n* 9·3372930	*n* 9·7755799	*n* 9·6854601	*n* 9·4402394	*n* 9·1118203						
5	7·6156792	*n* 9·5600499	*n* 9·5943369	*n* 9·4294267	*n* 9·1642914	*n* 8·8358723					
6	8·9363908	*n* 9·2495802	*n* 9·4350512	*n* 9·3516897	*n* 9·1479632	*n* 8·8706146	*n* 8·5421955				
7	8·9519343	*n* 8 8018246	*n* 9·2157834	*n* 9·2202012	*n* 9·0778929	*n* 8·8503106	*n* 8·5647034	*n* 8·2362843			
8	8·8033201	*n* 7·8681773	*n* 8·9357002	*n* 9·0420510	*n* 8·9632556	*n* 8·7851544	*n* 8·5414026	*n* 8·2498374	*n* 7·9214183		
9	8·5543732	8·0975903	*n* 8·5813070	*n* 8·8201462	*n* 8·8097041	*n* 8·6818797	*n* 8·4796405	*n* 8·2241987	*n* 7·9281319	*n* 7·5997128	
10	8·2075684	8·1711590	*n* 8·1035297	*n* 8·5534483	*n* 8·6204490	*n* 8·5449942	*n* 8·3845601	*n* 8·1649096	*n* 7·9006194	*n* 7·6010314	*n* 7·2726123

(f)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	*n* 9·9367999	*n* 9·7011476									
2	*n* 9·9170967	*n* 9·9378896	*n* 9·7022373								
3	*n* 9·5804795	*n* 9·9144616	*n* 9·8140406	*n* 9·5783883							
4	*n* 8·1721089	*n* 9·7404105	*n* 9·7825763	*n* 9·6390390	*n* 9·4033868						
5	9·1591989	*n* 9·4292935	*n* 9·6508679	*n* 9·6017704	*n* 9·4360088	*n* 9·2003565					
6	9·1900855	*n* 8·8980008	*n* 9·4334372	*n* 9·4894221	*n* 9·3945841	*n* 9·2152498	*n* 8·9795975				
7	9·0222960	8·0814598	*n* 9·1252811	*n* 9·3125545	*n* 9·2932087	*n* 9·1707446	*n* 8·9822563	*n* 8·7466040			
8	8·7169163	8·5764666	*n* 8·6853930	*n* 9·0734817	*n* 9·1395867	*n* 9·0763811	*n* 8·9353887	*n* 8·7403080	*n* 8·5046557		
9	8·2242244	8·5155312	*n* 7·8357362	*n* 8·7652076	*n* 8·9369981	*n* 8·9379560	*n* 8·8458683	*n* 8·6915755	*n* 8·4915202	*n* 8 2558679	
10	*n* 6·2890621	8·2964137	7·8923012	*n* 8·3611544	*n* 8·6845465	*n* 8·7585819	*n* 8·7181916	*n* 8 6055889	*n* 8·4412802	*n* 8·2373374	*n* 8·0016851

VALUES OF LOG Z^{m}_{-n} FROM THE FORMULA $Z^{m}_{-n} = -nr^{n-1} H'^{m}_{n}$ (*continued*).

(g)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·9124034	n 9·7605385									
2	n 9·8247305	n 9·9729994	n 9·8211346								
3	n 9·2203861	n 9·9060038	n 9·9086567	n 9·7567918							
4	9·2077840	n 9·6513447	n 9·8405355	n 9·7931614	n 9·6412965						
5	9·3701053	n 9·1417261	n 9·6532331	n 9·7225003	n 9·6496373	n 9·4977725					
6	9·2269066	8·4245778	n 9·3443594	n 9·5642555	n 9·5767220	n 9·4883847	n 9·3365198				
7	8·8929711	8·8607357	n 8·8444446	n 9·3221007	n 9·4345771	n 9·4136338	n 9·3148974	n 9·1630325			
8	8·2201322	8·7704700	7·4061857	n 8·9815317	n 9·2272491	n 9·2816673	n 9·2386623	n 9·1324553	n 8·9805905		
9	n 7·9946320	8·4932338	8·3689922	n 8·4766346	n 8·9507848	n 9·0960353	n 9·1136874	n 9·0550082	n 8·9431739	n 8·7913090	
10	n 8·1501522	8·0018640	8·3104015	n 6·7548784	n 8·5870950	n 8·8559390	n 8·9429563	n 8·9351197	n 8·8647236	n 8·7484974	n 8·5966325

(h)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·8830476	n 9·8097695									
2	n 9·6985362	n 9·9929973	n 9·9197191								
3	8·5740273	n 9·8690390	n 9·9780081	n 9·9047300							
4	9·4683530	n 9·4826637	n 9·8632081	n 9·9118666	n 9·8385885						
5	9·4222361	7·4459564	n 9·5965133	n 9·7992459	n 9·8176963	n 9·7444181					
6	9·1171335	9·0515978	n 9·1139965	n 9·5788031	n 9·7055307	n 9·7057971	n 9·6325190				
7	8·3741777	8·9965909	7·8803846	n 9·2340150	n 9·5096580	n 9·5935543	n 9·5816636	n 9·5083855			
8	n 8·3632673	8·6957997	8·6671205	n 8·6633016	n 9·2244008	n 9·4128844	n 9·4691699	n 9·4485753	n 9·3752971		
9	n 8·4378381	8·0484936	8·5747948	7·8943706	n 8·8185529	n 9·1616993	n 9·2988122	n 9·3357822	n 9·3086474	n 9·2353693	
10	n 8·2043795	n 7·7660507	8·2560436	8·2883787	n 8·1391473	n 8·8265397	n 9·0699214	n 9·1728673	n 9·1955540	n 9·1633246	n 9·0900465

(i)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·8480260	n 9·8509343									
2	n 9·5133417	n 9·9992669	n 0·0021752								
3	9·3380038	n 9·7988556	n 0·0255690	n 0·0284773							
4	9·5698844	n 9·1318707	n 9·8505095	n 0·0007186	n 0·0036270						
5	9·3706766	9·1130850	n 9·4590055	n 9·8349493	n 9·9478396	n 9·9507479					
6	8·7634140	9·2004136	n 8·2592335	n 9·5254641	n 9·7865220	n 9·8772317	n 9·8801400				
7	n 8·5424376	8·9340219	8·8746926	n 8·9780962	n 9·5172101	n 9·7183976	n 9·7943893	n 9·7972976			
8	n 8·6758223	8·2350505	8·8183880	8·1747932	n 9·1037627	n 9·4729701	n 9·6370852	n 9·7025921	n 9·7055004		
9	n 8·4200694	n 8·1334810	8·4699136	8·5811118	n 8·3253202	n 9·1229304	n 9·4074824	n 9·5462988	n 9·6039555	n 9·6068638	
10	n 7·7600507	n 8·2022630	7·5460652	8·4089838	8·1847790	n 8·5973392	n 9·0952278	n 9·3279434	n 9·4483662	n 9·4999239	n 9·5028322

(*k*)	*m* = 0	1	2	3	4	5	6	7	8	9	10
n = 1	*n* 9·8063561	*n* 9·8854509									
2	*n* 9·1846558	*n* 9·9922397	*n* 0·0713345								
3	9·5611912	*n* 9·6834569	*n* 0·0531846	*n* 0·1322794							
4	9·5897704	8·5625014	*n* 9·7978035	*n* 0·0629769	*n* 0·1420717						
5	9·1961414	9·3426404	*n* 9·1580614	*n* 9·8283449	*n* 0·0447405	*n* 0·1238354					
6	*n* 8·4209014	9·1921164	8·9483607	*n* 9·3752434	*n* 9·8207907	*n* 0·0087755	*n* 0·0878703				
7	*n* 8·8743304	8·6115283	9·0460659	6·7433797	*n* 9·4429222	*n* 9·7912252	*n* 9·9605759	*n* 0·0396707			
8	*n* 8·6618649	*n* 8·3236241	8·7324228	8·8137217	*n* 8·7297705	*n* 9·4545986	*n* 9·7472447	*n* 9·9034214	*n* 9·9825162		
9	*n* 7·9505919	*n* 8·4487781	7·7242572	8·6692066	8·4520096	*n* 8·9223624	*n* 9·4367412	*n* 9·6930622	*n* 9·8394276	*n* 9·9185224	
10	7·9036453	*n* 8·1290779	*n* 8·0893144	8·1781944	8·5260736	7·3319695	*n* 8·9874849	*n* 9·4006020	*n* 9·6312656	*n* 9·7700387	*n* 9·8491335

(*l*)	*m* = 0	1	2	3	4	5	6	7	8	9	10
n = 1	*n* 9·7566355	*n* 9·9143170									
2	8·1634789	*n* 9·9715073	*n* 0·1291889								
3	9·6704934	*n* 9·4909963	*n* 0·0614405	*n* 0·2191220							
4	9·5423768	9·2824434	*n* 9·6921535	*n* 0·1002210	*n* 0·2579026						
5	8·7058141	9·4137418	8·1452557	*n* 9·7723151	*n* 0·1109729	*n* 0·2686544					
6	*n* 8·9867997	9·0331206	9·2196744	*n* 9·0021939	*n* 9·8043532	*n* 0·1039960	*n* 0·2616776				
7	*n* 8·9160019	*n* 8·2260177	9·0252411	8·9506654	*n* 9·2351108	*n* 9·8102353	*n* 0·0847847	*n* 0·2424662			
8	*n* 8·3282224	*n* 8·6656997	8·2594037	8·9325466	8·4980112	*n* 9·3284931	*n* 9·7995911	*n* 0·0566184	*n* 0·2143000		
9	8·1096772	*n* 8·3871109	*n* 8·3081716	8·4805407	8·7902465	*n* 7·9982490	*n* 9·3678220	*n* 9·7775234	*n* 0·0216129	*n* 0·1792944	
10	8·1708000	*n* 7·0595673	*n* 8·2621872	*n* 7·6665578	8·4941256	8·5994435	*n* 8·5830290	*n* 9·3782632	*n* 9·7470716	*n* 9·9812122	*n* 0·1388938

(*m*)	*m* = 0	1	2	3	4	5	6	7	8	9	10
n = 1	*n* 9·6967851	*n* 9·9382541									
2	9·2343397	*n* 9·9357084	*n* 0·1771774								
3	9·7203475	*n* 9·0913208	*n* 0·0496929	*n* 0·2911618							
4	9·4150340	9·4938740	*n* 9·4971835	*n* 0·1125249	*n* 0·3539939						
5	*n* 8·7808681	9·3849311	9·2044914	*n* 9·6470588	*n* 0·1473282	*n* 0·3887972					
6	*n* 9·1282343	8·4987263	9·2907512	8·6898154	*n* 9·7241220	*n* 0·1644027	*n* 0·4058717				
7	*n* 8·7533441	*n* 8·8109930	8·8006266	9·1627857	*n* 8·5161720	*n* 9·7661164	*n* 0·1692428	*n* 0·4107117			
8	8·1053623	*n* 8·6729720	*n* 8·3820757	8·8405767	9·0030408	*n* 8·9652855	*n* 9·7873603	*n* 0·1651279	*n* 0·4065969		
9	8·4045520	*n* 7·7644397	*n* 8·5127275	*n* 7·1860739	8·8078408	8·7989453	*n* 9·1290856	*n* 9·7948454	*n* 0·1541737	*n* 0·3956427	
10	8·0013320	8·0548233	*n* 8·0339695	*n* 8·3036045	8·0083345	8·7382907	8·5079015	*n* 9·2136352	*n* 9·7925203	*n* 0·1378245	*n* 0·3792935

VALUES OF LOG Z^m_{-n} FROM THE FORMULA $Z^m_{-n} = -nr^{n-1}H'^m_n$ (*continued*).

(n)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·6235565	n 9·9577923									
2	9·4957960	n 9·8821212	n 0·2163570								
3	9·7268624	8·8026909	n 0·0157472	n 0·3499830							
4	9·1390383	9·5839016	n 9·0456934	n 0·0982206	n 0·4324565						
5	n 9·1849563	9·2461535	9·4321044	n 9·3878137	n 0·1526654	n 0·4869012					
6	n 9·1115221	n 8·7031899	9·2346033	9·2564184	n 9·5412191	n 0·1893814	n 0·5236172				
7	n 7·9758762	n 8·9297603	n 6·7172229	9·1844857	9·0316325	n 9·6313251	n 0·2138629	n 0·5480987			
8	8·5879410	n 8·3853538	n 8·7242560	8·3822153	9·1139359	8·6820990	n 9·6893664	n 0·2293896	n 0·5636254		
9	8·3293241	8·1906890	n 8·3959758	n 8·4908582	8·5493845	9·0284344	n 6·5751535	n 9·7279695	n 0·2380768	n 0·5723126	
10	n 7·3345415	8·1687792	7·6189590	n 8·3255681	n 8·2075032	8·5957045	8·9288743	n 8·5988974	n 9·7535053	n 0·2413690	n 0·5756048

(o)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·5314815	n 9·9733240									
2	9·6387835	n 9·8056669	n 0·2475094								
3	9·6934684	9·3764433	n 9·9549135	n 0·3967561							
4	5·5509442	9·6018622	8·9836595	n 0·0530078	n 0·4948503						
5	n 9·3197274	8·8780353	9·5143472	n 8·2705431	n 0·1230732	n 0·5649157					
6	n 8·9302902	n 9·0587162	9·0115004	9·4250094	n 9·0631659	n 0·1754098	n 0·6172523				
7	8·5994657	n 8·8527721	n 8·7895951	9·0430795	9·3301952	n 9·2981739	n 0·2155120	n 0·6573546			
8	8·6430378	7·9942586	n 8·7442841	n 8·4745633	9·0380503	9·2263467	n 9·4314708	n 0·2466595	n 0·6885020		
9	7·6006678	8·4213872	n 7·4146204	n 8·6232816	n 7·9994862	9·0146382	9·1087179	n 9·5203405	n 0·2709674	n 0·7128099	
10	n 8·0885995	7·8396275	8·1915307	n 7·8783543	n 8·4942836	7·4134921	8·9800019	8·9695067	n 9·5842071	n 0·2898803	n 0·7317228

(p)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·4102815	n 9·9851374									
2	9·7277346	n 9·6963524	n 0·2712083								
3	9·6146247	9·5886678	n 9·8574845	n 0·4323404							
4	n 9·1253414	9·5563970	9·4568526	n 9·9674642	n 0·5423201						
5	n 9·3366647	n 8·5809195	9·5059414	9·3067849	n 0·0494151	n 0·6242709					
6	n 8·2165274	n 9·1542282	7·8956198	9·4584338	9·1118655	n 0·1136371	n 0·6884930				
7	8·8545729	n 8·4733755	n 8·9913596	8·5140513	9·4118164	8·8023450	n 0·1656248	n 0·7404807			
8	8·4238795	8·5803812	n 8·4945404	n 8·8386025	8·6690908	9·3649842	7·5222542	n 0·2086577	n 0·7835136		
9	n 8·1874813	8·3107094	8·3140621	n 8·4624502	n 8·6896360	8·7369421	9·3172123	n 8·7146191	n 0·2448511	n 0·8197070	
10	n 8·1043009	n 7·7732146	8·1815406	8·0382254	n 8·4102386	n 8·5391460	8·7688838	9·2679316	n 9·0113165	n 0·2756495	n 0·8505054

(q)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 9·2368490	n 9·9934386									
2	9·7832438	n 9·5312742	n 0·2878638								
3	9·4699377	9·7015231	n 9·7007606	n 0·4573502							
4	n 9·3878849	9·4329801	9·6422411	n 9·8190945	n 0·5756841						
5	n 9·2493219	n 9·1495429	9·4047802	9·5894515	n 9·9093997	n 0·6659892					
6	8·7463003	n 9·1063208	n 8·9395658	9·3806391	9·5380466	n 9·9819760	n 0·7385656				
7	8·8679788	8·3547454	n 8·9870416	n 8·7351156	9·3586472	9·4855913	n 0·0423180	n 0·7989076			
8	n 7·6159404	8·6663178	7·8826142	n 8·8831235	n 8·3165147	9·3378784	9·4304698	n 0·0937051	n 0·8502947		
9	n 8·3910588	7·2584515	8·4891003	6·3766907	n 8·7898167	n 8·2526570	9·3178239	9·3712474	n 0·1382528	n 0·8948424	
10	n 7·5342698	n 8·1463748	7·5608650	8·3290276	n 7·5505106	n 8·7041538	n 7·8493743	9·2981797	9·3063290	n 0·1774055	n 0·9339951

(r)	m = 0	1	2	3	4	5	6	7	8	9	10
n = 1	n 8·9374096	n 9·9983662									
2	9·8140202	n 9·2367949	n 0·2977514								
3	9·1871998	9·7599058	n 9·4112413	n 0·4721979							
4	n 9·5015495	9·1621624	9·7302540	n 9·5345353	n 0·5954919						
5	n 8·9977270	n 9·3274966	9·1460679	9·7094487	n 9·6298006	n 0·6907572					
6	9·0408798	n 8·8745109	n 9·1942989	9·1342275	9·6927966	n 9·7073371	n 0·7682936				
7	8·6642340	8·8147023	n 8·7756213	n 9·0850410	9·1247402	9·6783936	n 9·7726390	n 0·8335956			
8	n 8·5084262	8·4919317	8·6255757	n 8·6927463	n 8·9913857	9·1166887	9·6653186	n 9·8289862	n 0·8899428		
9	n 8·2560744	n 8·2481543	8·3455141	8·4613105	n 8·6211935	n 8·9086062	9·1095785	9·6530708	n 9·8784940	n 0·9394505	
10	7·9227827	n 8·0535697	n 8·0191035	8·2178726	8·3146747	n 8·5580521	n 8·8337673	9·1031125	9·6413522	n 9·9226067	n 0·9835633

In these equations the expression of the Magnetic Forces in terms of the Magnetic Constants for a given belt of latitude is equated to the corresponding expression called the *absolute term* which is derived from the observations taken in that belt of latitude.

The logarithms and the signs (+ or −) of the coefficients of g_n^m or h_n^m in the equations are given in the tables. The *numerical* values of the absolute terms are given for 1845 and for 1880 for the g and h equations.

$m = 0$. n ODD.

Co-latitude	For X g_1^0	g_{-1}^0	g_3^0	g_{-3}^0	g_5^0	g_{-5}^0
(*a*) 5°	8·9532263	+ 8·9402960	+ 9·3283042	+ 9·3148815	+ 9·2224495	+ 9·2045250
(*b*) 10	9·2523050	+ 9·2396702	+ 9·6146381	+ 9·6016214	+ 9·4852763	+ 9·4679413
(*c*) 15	9·4251486	+ 9·4129962	+ 9·7658529	+ 9·7535016	+ 9·5949419	+ 9·5785911
(*d*) 20	9·5455497	+ 9·5340517	+ 9·8551072	+ 9·8436647	+ 9·6200305	+ 9·6050735
(*e*) 25	9·6366401	+ 9·6259482	+ 9·9045229	+ 9·8942119	+ 9·5731486	+ 9·5600612
(*f*) 30	9·7071397	+ 9·6989700	+ 9·9228538	+ 9·9138747	+ 9·4404450	+ 9·4299685
(*g*) 35	9·7673181	+ 9·7585912	+ 9·9131855	+ 9·9057193	+ 9·1496354	+ 9·1440324
(*h*) 40	9·8156959	+ 9·8080674	+ 9·8748884	+ 9·8691093	− 7·5935679	− 7·2832876
(*i*) 45	9·8559816	+ 9·8494850	+ 9·8032458	+ 9·7993447	− 9·1241868	− 9·1104431
(*k*) 50	9·8896198	+ 9·8842539	+ 9·6861931	+ 9·6845333	− 9·3504323	− 9·3416154
(*l*) 55	9·9176350	+ 9·9133645	+ 9·4915605	+ 9·4930529	− 9·4192876	− 9·4134723
(*m*) 60	9·9407744	+ 9·9375307	+ 9·0866507	+ 9·0966376	− 9·3885226	− 9·3852421
(*n*) 65	9·9595923	+ 9·9572757	− 8·8167641	− 8·7932528	− 9·2478255	− 9·2471165
(*o*) 70	9·9745025	+ 9·9729858	− 9·3801944	− 9·3745034	− 8·8767844	− 8·8806045
(*p*) 75	9·9858120	+ 9·9849437	− 9·5903678	− 9·5878970	+ 8·5869098	+ 8·5767679
(*q*) 80	9·9937423	+ 9·9933515	− 9·7022145	− 9·7012319	+ 9·1507683	+ 9·1489183
(*r*) 85	9·9984426	+ 9·9983442	− 9·7600721	− 9·7598384	+ 9·3277581	+ 9·3273795
(*s*) 90	0·0000000	+ 0·0000000	− 9·7781513	− 9·7781513	+ 9·3767507	+ 9·3767507
	For Z					
(*a*) 5°	0·3036698	− 9·9983442	+ 0·2011839	− 0·0662229	+ 9·8664247	− 9·7714964
(*b*) 10	0·2984783	− 9·9933515	+ 0·1697922	− 0·0353070	+ 9·7843377	− 9·6901972
(*c*) 15	0·2897462	− 9·9849437	+ 0·1146224	− 9·9809515	+ 9·6247523	− 9·5321492
(*d*) 20	0·2773486	− 9·9729859	+ 0·0301730	− 9·8977104	+ 9·3114982	− 9·2223794
(*e*) 25	0·2610968	− 9·9572757	+ 9·9049397	− 9·7742428	− 7·7829577	+ 7·5043358
(*f*) 30	0·2407250	− 9·9375306	+ 9·7109854	− 9·5831910	− 9·2522987	+ 9·1568413
(*g*) 35	0·2158664	− 9·9133645	+ 9·3471268	− 9·2269669	− 9·4606556	+ 9·3692938
(*h*) 40	0·1860190	− 9·8842539	− 8·7256091	+ 8·5447993	− 9·5107502	+ 9·4222489
(*i*) 45	0·1504913	− 9·8494851	− 9·4711444	+ 9·3337788	− 9·4571430	+ 9·3714770
(*k*) 50	0·1083162	− 9·8080675	− 9·6917288	+ 9·5592880	− 9·2802161	+ 9·1981399
(*l*) 55	0·0581067	− 9·7585914	− 9·7994329	+ 9·6695875	− 8·7843756	+ 8·7127235
(*m*) 60	9·9977982	− 9·6989700	− 9·8480317	+ 9·7200502	+ 8·8682063	− 8·7757844
(*n*) 65	9·9241563	− 9·6259483	− 9·8535035	+ 9·7269933	+ 9·2680349	− 9·1836732
(*o*) 70	9·8317250	− 9·5340516	− 9·8192493	+ 9·6939165	+ 9·4010432	− 9·3193891
(*p*) 75	9·7102363	− 9·4129963	− 9·7397259	+ 9·6153065	+ 9·4168009	− 9·3367913
(*q*) 80	9·5365913	− 9·2396702	− 9·5945458	+ 9·4707816	+ 9·3286520	− 9·2497155
(*r*) 85	9·2370218	− 8·9402960	− 9·3115087	+ 9·1881396	+ 9·0765800	− 8·9982644
(*s*) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

GAUSS' MAGNETIC CONSTANTS.

The following is the type of these equations:

$$X'^m_n g^m_n + X'^m_{-n} g^m_{-n} + X'^m_{n_1} g^m_{n_1} + X'^m_{-n_1} g^m_{-n_1} + \&c. = x'_m,$$

with similar equations for Y and Z: where

$$X'^m_n = X^m_n \cos\psi + Z^m_n \sin\psi \quad \text{and} \quad Z'^m_n = -X^m_n \sin\psi + Z^m_n \cos\psi.$$

An explanation of the mode of formation of these equations is given at the beginning of Section VIII.

$g_7{}^0$	$g_{-7}{}^0$	$g_9{}^0$	$g_{-9}{}^0$	Absolute term, $\frac{1}{2}(a_0+a'_0)$ for X 1845	1880	
+ 8·9529284	+ 8·9298243	+ 8·5978745	+ 8·5693492			(a)
+ 9·1805006	+ 9·1581900	+ 8·7767031	+ 8·7491938			(b)
+ 9·2230216	+ 9·2020853	+ 8·7134680	+ 8·6878550	1·6141		(c)
+ 9·1269798	+ 9·1082025	+ 8·3334771	+ 8·3122970	2·1097		(d)
+ 8·8179013	+ 8·8033976	− 8·1237791	− 8·0943811	2·61865	2·6096	(e)
− 8·1101781	− 8·0703080	− 8·5369154	− 8·5148839	3·1453	3·0990	(f)
− 8·8770022	− 8·8592574	− 8·5112273	− 8·4936654	3·6774	3·5780	(g)
− 9·0097260	− 8·9962465	− 8·0607174	− 8·0519890	4·1814	4·0501	(h)
− 8·9441959	− 8·9346092	+ 8·1503385	+ 8·1306562	4·6719	4·5157	(i)
− 8·6174790	− 8·6141392	+ 8·4606289	+ 8·4483209	5·1196	4·9913	(k)
+ 8·2407168	+ 8·2192787	+ 8·3953360	+ 8·3878288	5·56235	5·4432	(l)
+ 8·8176597	+ 8·8098887	+ 7·7651506	+ 7·7699044	5·99975	5·8674	(m)
+ 8·9337640	+ 8·9296353	− 8·1972780	− 8·1891552	6·3925	6·2499	(n)
+ 8·8547051	+ 8·8532910	− 8·4247461	− 8·4212452	6·7362	6·6060	(o)
+ 8·4728248	+ 8·4751154	− 8·3119353	− 8·3112542	7·01835	6·8952	(p)
− 8·3572397	− 8·3531256	− 7·2517353	− 7·2651280	7·23835	7·1009	(q)
− 8·8150696	− 8·8145242	+ 8·2486467	+ 8·2478979	7·3475	7·2224	(r)
− 8·9116107	− 8·9116107	+ 8·4135556	+ 8·4135556	7·3658	7·2546	(s)

				Absolute term, $\frac{1}{2}(a_0-a'_0)$ for Z 1845	1880	
+ 9·4393846	− 9·3599240	+ 8·9592279	− 8·8862853			(a)
+ 9·2737326	− 9·1954526	+ 8·6569308	− 8·5857688			(b)
+ 8·8622805	− 8·7875193	− 7·7497047	+ 7·6637853	12·7873		(c)
− 8·5498448	+ 8·4649586	− 8·5886267	+ 8·5173955	12·57105		(d)
− 9·0285002	+ 8·9511420	− 8·6224319	+ 8·5547616	12·26415	12·0989	(e)
− 9·0964780	+ 9·0224387	− 8·2884239	+ 8·2266827	11·85765	11·6153	(f)
− 8·9644361	+ 8·8942372	+ 8·0627293	− 7·9902960	11·33005	11·0729	(g)
− 8·4396264	+ 8·3801408	+ 8·5002652	− 8·4372506	10·7181	10·4632	(h)
+ 8·6144022	− 8·5388377	+ 8·4788928	− 8·4208180	10·0332	9·8042	(i)
+ 8·9419625	− 8·8735459	+ 8·0036200	− 7·9550734	9·23315	9·0396	(k)
+ 8·9808757	− 8·9162784	− 8·1667819	+ 8·1070819	8·2793	8·2132	(l)
+ 8·8154883	− 8·7547756	− 8·4574222	+ 8·4042576	7·24715	7·2945	(m)
+ 8·0288949	− 7·9857574	− 8·3792468	+ 8·3301297	6·18635	6·2535	(n)
− 8·6614450	+ 8·5977909	− 7·6440388	+ 7·6067935	5·03565	5·1167	(o)
− 8·9142754	+ 8·8542709	+ 8·2359374	− 8·1865168	3·8353	3·8950	(p)
− 8·9264887	+ 8·8681306	+ 8·4376739	− 8·3910220	2·5879	2·6577	(q)
− 8·7220784	+ 8·6645896	+ 8·3018171	− 8·2563214	1·2897	1·3875	(r)
0·0000000	0·0000000	0·0000000	0·0000000			(s)

The *absolute terms* of the equations for the determination of g_n^m are,

$$x'_m = \tfrac{1}{2}(a_m - a'_m), \text{ when } n-m \text{ is even,}$$

and

$$= \tfrac{1}{2}(a_m + a'_m), \text{ when } n-m \text{ is odd.}$$

$$y'_m = \tfrac{1}{2}(b_m + b'_m), \text{ when } n-m \text{ is even,}$$

and

$$= \tfrac{1}{2}(b_m - b'_m), \text{ when } n-m \text{ is odd.}$$

$$z'_m = \tfrac{1}{2}(a_m + a'_m), \text{ when } n-m \text{ is even,}$$

and

$$= \tfrac{1}{2}(a_m - a'_m), \text{ when } n-m \text{ is odd.}$$

$m = 0$. n EVEN.

Co-latitude	For X g_2^0	g_{-2}^0	g_4^0	g_{-4}^0	g_6^0	g_{-6}^0
(a) 5°	9·2511620	+ 9·2391766	+ 9·3047046	+ 9·2891747	+ 9·1014950	+ 9·0810252
(b) 10	9·5451804	+ 9·5335251	+ 9·5806760	+ 9·5656408	+ 9·3482513	+ 9·3284708
(c) 15	9·7095062	+ 9·6983896	+ 9·7138548	+ 9·6996359	+ 9·4281044	+ 9·4094924
(d) 20	9·8177999	+ 9·8074143	+ 9·7760296	+ 9·7629410	+ 9·4026401	+ 9·3857473
(e) 25	9·8929964	+ 9·8835110	+ 9·7868176	+ 9·7751714	+ 9·2644451	+ 9·2501212
(f) 30	9·9451273	+ 9·9366837	+ 9·7502423	+ 9·7403746	+ 8·9092486	+ 8·9004635
(g) 35	9·9793160	+ 9·9720238	+ 9·6594790	+ 9·6518348	− 8·4472269	− 8·4158096
(h) 40	9·9983336	+ 9·9922669	+ 9·4885729	+ 9·4840456	− 9·0645826	− 9·0499259
(i) 45	0·0035936	+ 9·9987893	+ 9·1333668	+ 9·1358370	− 9·2104357	− 9·1998792
(k) 50	9·9955581	+ 9·9920143	− 8·5865187	− 8·5469853	− 9·1995959	− 9·1923563
(l) 55	9·9738493	+ 9·9715265	− 9·2898112	− 9·2799508	− 9·0378623	− 9·0343436
(m) 60	9·9371354	+ 9·9359566	− 9·4984282	− 9·4928224	− 8·4971198	− 8·5040488
(n) 65	9·8827222	+ 9·8825763	− 9·5867171	− 9·5835009	+ 8·7098452	+ 8·7003329
(o) 70	9·8055555	+ 9·8063004	− 9·6033787	− 9·6018611	+ 9·0617156	+ 9·0580210
(p) 75	9·6956638	+ 9·6971304	− 9·5569323	− 9·5566650	+ 9·1555798	+ 9·1541440
(q) 80	9·5301605	+ 9·5321587	− 9·4328116	− 9·4334265	+ 9·1066325	+ 9·1065369
(r) 85	9·2354209	+ 9·2377445	− 9·1615728	− 9·1627118	+ 8·8742221	+ 8·8748798
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
	For Z					
(a) 5°	0·3017357	− 0·1184854	+ 0·0496759	− 9·9398809	+ 9·6611357	− 9·5755806
(b) 10	0·2862089	− 0·1032930	+ 9·9963649	− 9·8871954	+ 9·5422111	− 9·4576253
(c) 15	0·2596488	− 0·0772884	+ 9·8987874	− 9·7907424	+ 9·2887144	− 9·2063114
(d) 20	0·2208992	− 0·0393151	+ 9·7362581	− 9·6301051	+ 8·4959021	− 8·4283571
(e) 25	0·1680138	− 9·9874329	+ 9·4424753	− 9·3403435	− 9·0207999	+ 8·9340871
(f) 30	0·0977425	− 9·9184162	+ 8·2397558	− 8·2164492	− 9·2717247	+ 9·1894393
(g) 35	0·0043971	− 9·8266488	− 9·3164732	+ 9·2039662	− 9·3063093	+ 9·2271200
(h) 40	9·8769881	− 9·7013475	− 9·5740920	+ 9·4668671	− 9·1940679	+ 9·1183615
(i) 45	9·6900696	− 9·5177685	− 9·6737270	+ 9·5693513	− 8·8363402	+ 8·7673710
(k) 50	9·3575610	− 9·1937472	− 9·6919462	+ 9·5899024	+ 8·5027949	− 8·4123641
(l) 55	− 8·3968121	+ 8·0654802	− 9·6429609	+ 9·5431243	+ 9·0612216	− 8·9852764
(m) 60	− 9·4164269	+ 9·2279704	− 9·5139818	+ 9·4165371	+ 9·2001226	− 9·1279373
(n) 65	− 9·6749738	+ 9·4930805	− 9·2357813	+ 9·1421219	+ 9·1814439	− 9·1119546
(o) 70	− 9·8166286	+ 9·6374112	+ 6·9120725	+ 6·9512776	+ 8·9983525	− 8·9315405
(p) 75	− 9·9047572	+ 9·7270590	+ 9·2246814	− 9·1233803	+ 8·2793108	− 8·2227628
(q) 80	− 9·9597369	+ 9·7829659	+ 9·4856252	− 9·3873342	− 8·8147838	+ 8·7451626
(r) 85	− 9·9902104	+ 9·8139535	+ 9·5986500	− 9·5014342	− 9·1081156	+ 9·0407080
(s) 90	− 0·0000000	+ 9·8239087	+ 9·6320232	− 9·5351132	− 9·1804560	+ 9·1135093

The *absolute terms* for the determination of h_n^m are,

$$x'_m = \tfrac{1}{2}(b_m - b'_m), \text{ when } n-m \text{ is even,}$$

and
$$= \tfrac{1}{2}(b_m + b'_m), \text{ when } n-m \text{ is odd.}$$

$$y'_m = -\tfrac{1}{2}(a_m + a'_m), \text{ when } n-m \text{ is even,}$$

and
$$= -\tfrac{1}{2}(a_m - a'_m), \text{ when } n-m \text{ is odd.}$$

$$z'_m = \tfrac{1}{2}(b_m + b'_m), \text{ when } n-m \text{ is even,}$$

and
$$= \tfrac{1}{2}(b_m - b'_m), \text{ when } n-m \text{ is odd.}$$

The meaning and the formation of these expressions is given in Section VIII.

g_8^0	g_{-8}^0	g_{10}^0	g_{-10}^0	Absolute term, $\frac{1}{2}(a_0 - a'_0)$ for X		
				1845	1880	
+ 8·7835600	+ 8·7577646	+ 8·3989833	+ 8·3677007			(a)
+ 8·9885938	+ 8·9637005	+ 8·5475395	+ 8·5173932			(b)
+ 8·9844047	+ 8·9611224	+ 8·4078654	+ 8·3799866	− ·4268		(c)
+ 8·7835048	+ 8·7630870	+ 7·5603518	+ 7·5457667	− ·4561		(d)
+ 7·8774911	+ 7·8776859	− 8·1975294	− 8·1699419	− ·41595	− ·3817	(e)
− 8·5969155	− 8·5747182	− 8·3192440	− 8·2964243	− ·3567	− ·3139	(f)
− 8·7874342	− 8·7700823	− 8·0201626	− 8·0037823	− ·2584	− ·2116	(g)
− 8·7094936	− 8·6964631	+ 7·7881973	+ 7·7620985	− ·1721	− ·0810	(h)
− 8·2428092	− 8·2390420	+ 8·2180302	+ 8·2017351	− ·0635	·0582	(i)
+ 8·3372700	+ 8·3204919	+ 8·1406807	+ 8·1299262	+ ·0302	·2005	(k)
+ 8·6744813	+ 8·6650696	+ 7·0497009	+ 7·0768501	·12375	·3185	(l)
+ 8·6787210	+ 8·6733103	− 8·0636196	− 8·0537085	·17865	·3872	(m)
+ 8·3877330	+ 8·3871210	− 8·1740081	− 8·1689406	·2068	·4217	(n)
− 8·0017709	− 7·9900846	− 7·8413438	− 7·8412773	·2287	·4056	(o)
− 8·5825896	− 8·5798747	+ 7·7769464	+ 7·7716572	·20025	·3584	(p)
− 8·6670060	− 8·6663616	+ 8·1474196	+ 8·1462533	·12555	·2766	(q)
− 8·4918234	− 8·4921930	+ 8·0535943	+ 8·0537557	·0788	·1530	(r)
0·0000000	0·0000000	0·0000000	0·0000000			(s)
				Absolute term, $\frac{1}{2}(a_0 + a'_0)$ for Z		
				1845	1880	
+ 9·2046502	− 9·1291701	+ 8·7046736	− 8·6332402			(a)
+ 8·9794918	− 8·9054511	+ 8·2986254	− 8·2294781			(b)
+ 8·2079901	− 8·1429484	− 8·0400527	+ 7·9665976	− 0·4582		(c)
− 8·6790653	+ 8·6035647	− 8·4056708	+ 8·3370591	− 0·27425		(d)
− 8·8748556	+ 8·8031891	− 8·2730637	+ 8·2085924	− 0·10225	·0906	(e)
− 8·7857489	+ 8·7178264	+ 6·4566114	− 6·1369896	+ 0·05005	·2375	(f)
− 8·2834761	+ 8·2249608	+ 8·2128490	− 8·1491766	·18975	·3396	(g)
+ 8·4315944	− 8·3601725	+ 8·2631406	− 8·2048990	·2694	·3938	(h)
+ 8·7397391	− 8·6752925	+ 7·8129580	− 7·7640553	·3342	·3695	(i)
+ 8·7225847	− 8·6625188	− 7·9597598	+ 7·9012310	·36015	·3396	(k)
+ 8·3848041	− 8·3311790	− 8·2222656	+ 8·1706922	·3643	·2535	(l)
− 8·1667566	+ 8·1006900	− 8·0490052	+ 8·0027499	·28245	·1709	(m)
− 8·6440904	+ 8·5872419	+ 7·3881869	− 7·3268842	·17885	·0479	(n)
− 8·6968694	+ 8·6432461	+ 8·1340093	− 8·0880727	·06685	− ·0958	(o)
− 8·4757469	+ 8·4249172	+ 8·1474119	− 8·1046384	− ·0645	− ·2353	(p)
+ 7·6727519	− 7·6103363	+ 7·5747251	− 7·5362944	− ·1440	− ·3368	(q)
+ 8·5599799	− 8·5081837	− 7·9647140	+ 7·9224423	− ·1809	− ·4347	(r)
+ 8·6897620	− 8·6386095	− 8·1761947	+ 8·1348020	− ·2095	− ·4653	(s)

Co-latitude	g_1^1 or h_1^1	g_{-1}^1 or h_{-1}^1	g_3^1 or h_3^1	g_{-3}^1 or h_{-3}^1	g_5^1 or h_5^1	g_{-5}^1 or h_{-5}^1	g_7^1 or h_7^1
For X							
(a) 5°	− 0·0026064	− 9·9983442	− 9·8958149	− 9·8858730	− 9·5529168	− 9·5373032	− 9·1131077
(b) 10	− 9·9973154	− 9·9933515	− 9·8501792	− 9·8409834	− 9·4231036	− 9·4088565	− 8·8196534
(c) 15	− 9·9884211	− 9·9849437	− 9·7661787	− 9·7583784	− 9·1142766	− 9·1041789	+ 8·1946386
(d) 20	− 9·9758030	− 9·9729859	− 9·6252033	− 9·6199103	+ 8·3543590	+ 8·3005184	+ 8·9433386
(e) 25	− 9·9592795	− 9·9572757	− 9·3697157	− 9·3701812	+ 9·2193371	+ 9·2006391	+ 9·1034602
(f) 30	− 9·9385929	− 9·9375306	− 8·5304401	− 8·5898937	+ 9·4377171	+ 9·4229433	+ 9·0856173
(g) 35	− 9·9133858	− 9·9133645	+ 9·2183315	+ 9·1954331	+ 9·5174987	+ 9·5055004	+ 8·8858950
(h) 40	− 9·8831670	− 9·8842539	+ 9·5439265	+ 9·5307525	+ 9·5098413	+ 9·5005213	+ 8·0082573
(i) 45	− 9·8472561	− 9·8494851	+ 9·7053662	+ 9·6960238	+ 9·4152595	+ 9·4089235	− 8·7299283
(k) 50	− 9·8046170	− 9·8080675	+ 9·7976865	+ 9·7910056	+ 9·1853782	+ 9·1833754	− 8·9660464
(l) 55	− 9·7541164	− 9·7585914	+ 9·8468468	+ 9·8421039	+ 8·3838054	+ 8·4099759	− 8·9730570
(m) 60	− 9·6934589	− 9·6989700	+ 9·8624843	+ 9·8594585	− 9·0088201	− 8·9981638	− 8·7780831
(n) 65	− 9·6195016	− 9·6259483	+ 9·8474385	+ 9·8458702	− 9·3051227	− 9·3001352	− 7·6969857
(o) 70	− 9·5267979	− 9·5340516	+ 9·7997748	+ 9·7994168	− 9·4104555	− 9·4079959	+ 8·6953458
(p) 75	− 9·4050882	− 9·4129963	+ 9·7113974	+ 9·7119972	− 9·4133850	− 9·4125585	+ 8·9202722
(q) 80	− 9·2312804	− 9·2396702	+ 9·5605795	+ 9·5618741	− 9·3183832	− 9·3186451	+ 8·9235390
(r) 85	− 8·9316111	− 8·9402960	+ 9·2743995	+ 9·2761156	− 9·0628199	− 9·0637171	+ 8·7153056
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
For Y							
(a) 5°	0·0043291	+ 0·0000000	+ 9·9061060	+ 9·8960048	+ 9·5791799	+ 9·5633066	+ 9·1638710
(b) 10	0·0042297	+ 0·0000000	+ 9·8932311	+ 9·8833618	+ 9·5428212	+ 9·5273123	+ 9·0922932
(c) 15	0·0040676	+ 0·0000000	+ 9·8713916	+ 9·8619006	+ 9·4794811	+ 9·4645667	+ 8·9618966
(d) 20	0·0038477	+ 0·0000000	+ 9·8399625	+ 9·8309846	+ 9·3839809	+ 9·3698728	+ 8·7455110
(e) 25	0·0035769	+ 0·0000000	+ 9·7979470	+ 9·7896009	+ 9·2458609	+ 9·2327456	+ 8·3462768
(f) 30	0·0032637	+ 0·0000000	+ 9·7438079	+ 9·7361927	+ 9·0411427	+ 9·0291759	− 7·5515325
(g) 35	0·0029176	+ 0·0000000	+ 9·6751518	+ 9·6683440	+ 8·6929156	+ 8·6822177	− 8·2696872
(h) 40	0·0025494	+ 0·0000000	+ 9·5880969	+ 9·5821483	− 6·9465647	− 6·9372169	− 8·3544704
(i) 45	0·0021703	+ 0·0000000	+ 9·4758641	+ 9·4708000	− 8·5711385	− 8·5631807	− 8·2488411
(k) 50	0·0017919	+ 0·0000000	+ 9·3250658	+ 9·3208847	− 8·7647898	− 8·7582196	− 7·8899387
(l) 55	0·0014256	+ 0·0000000	+ 9·1028844	+ 9·0995581	− 8·8056819	− 8·8004548	+ 7·4737305
(m) 60	0·0010824	+ 0·0000000	+ 8·6784712	+ 8·6759455	− 8·7516759	− 8·7477070	+ 8·0330530
(n) 65	0·0007728	+ 0·0000000	− 8·3695804	− 8·3677773	− 8·5922247	− 8·5893912	+ 8·1307338
(o) 70	0·0005058	+ 0·0000000	− 8·9271783	− 8·9259981	− 8·2075959	− 8·2057413	+ 8·0368791
(p) 75	0·0002895	+ 0·0000000	− 9·1270847	− 9·1264092	+ 7·8978736	+ 7·8968121	+ 7·6445876
(q) 80	0·0001303	+ 0·0000000	− 9·2312672	− 9·2309632	+ 8·4576120	+ 8·4571343	− 7·5168601
(r) 85	0·0000328	+ 0·0000000	− 9·2844949	− 9·2844183	+ 8·6302807	+ 8·6301604	− 7·9714021
(s) 90	0·0000000	+ 0·0000000	− 9·3010300	− 9·3010300	+ 8·6777807	+ 8·6777807	− 8·0665127
For Z							
(a) 5°	9·2499770	− 8·9402960	+ 9·4520526	− 9·3153669	+ 9·3009660	− 9·2049145	+ 9·0104642
(b) 10	9·5491528	− 9·2396702	+ 9·7384275	− 9·6021097	+ 9·5638241	− 9·4683380	+ 9·2380642
(c) 15	9·7221549	− 9·4129962	+ 9·8897097	− 9·7539951	+ 9·6735427	− 9·5790015	+ 9·2806358
(d) 20	9·8427714	− 9·5340517	+ 9·9790564	− 9·8441658	+ 9·6987086	− 9·6055081	+ 9·1846826
(e) 25	9·9341272	− 9·6259482	+ 0·0285877	− 9·8947244	+ 9·6519359	− 9·5605391	+ 8·8758211
(f) 30	0·0065233	− 9·6989700	+ 0·0470557	− 9·9144037	+ 9·5193995	− 9·4305373	− 8·1663745
(g) 35	0·0654531	− 9·7585912	+ 0·0375450	− 9·9062722	+ 9·2289570	− 9·1448781	− 8·9345426
(h) 40	0·1141937	− 9·8080674	+ 9·9994270	− 9·8696984	− 7·6523349	+ 7·2424570	− 9·0674572
(i) 45	0·1548538	− 9·8494850	+ 9·9279908	− 9·8000042	− 9·2026059	+ 9·1102041	− 9·0020915
(k) 50	0·1888665	− 9·8842539	+ 9·8111938	− 9·6852884	− 9·4292114	+ 9·3416497	− 8·6756979
(l) 55	0·2172449	− 9·9133645	+ 9·6169523	− 9·4940512	− 9·4982573	+ 9·4136087	+ 8·2976524
(m) 60	0·2407251	− 9·9375307	+ 9·2132166	− 9·0986323	− 9·4676461	+ 9·3854486	+ 8·8754049
(n) 65	0·2598510	− 9·9572757	− 8·9385413	+ 8·7902378	− 9·3271143	+ 9·2474118	+ 8·9916740
(o) 70	0·2750273	− 9·9729858	− 9·5044077	+ 9·3739682	− 8·9564281	+ 8·8811837	+ 8·9127293
(p) 75	0·2865527	− 9·9849437	− 9·7150023	+ 9·5877042	+ 8·6652416	− 8·5759720	+ 8·5310428
(q) 80	0·2946420	− 9·9933515	− 9·8270346	+ 9·7011638	+ 9·2298104	− 9·1488108	− 8·4149909
(r) 85	0·2994398	− 9·9983442	− 9·8849829	+ 9·7598232	+ 9·4069121	− 9·3273595	− 8·8730335
(s) 90	0·3010300	− 0·0000000	− 9·9030900	+ 9·7781513	+ 9·4559320	− 9·3767507	− 8·9696027

g_{-7}^1 or h_{-7}^1	g_9^1 or h_9^1	g_{-9}^1 or h_{-9}^1	Absolute term for g, $\frac{1}{2}(a_1 - a'_1)$		Absolute term for h, $\frac{1}{2}(b_1 - b'_1)$		
			1845	1880	1845	1880	
− 9·0918370	− 8·6141140	− 8·5872108					(a)
− 8·8009486	− 7·9020314	− 7·8830583					(b)
+ 8·1573037	+ 8·4301914	+ 8·4014640	− ·8596		1·0391		(c)
+ 8·9210226	+ 8·6565540	+ 8·6312404	− ·8132		1·19255		(d)
+ 9·0843292	+ 8·5887827	+ 8·5669294	− ·7735	− ·6678	1·2846	1·4214	(e)
+ 9·0695768	+ 8·1087701	+ 8·0951679	− ·70845	− ·5691	1·3163	1·4095	(f)
+ 8·8741586	− 8·2223465	− 8·1982085	− ·6101	− ·4428	1·3127	1·3314	(g)
+ 8·0252500	− 8·5207813	− 8·5035754	− ·4737	− ·2877	1·2459	1·2358	(h)
− 8·7139347	− 8·4558774	− 8·4436378	− ·3066	− ·1143	1·17445	1·1229	(i)
− 8·9557007	− 7·8693294	− 7·8696006	− ·10965	·0759	1·0733	1·0007	(k)
− 8·9664611	+ 8·2205002	+ 8·2071322	+ ·12455	·2496	·9161	·8493	(l)
− 8·7756036	+ 8·4631186	+ 8·4557231	·29395	·3904	·77995	·6855	(m)
− 7·7323674	+ 8·3642769	+ 8·3609711	·3877	·4861	·6551	·5331	(n)
+ 8·6899196	+ 7·5225426	+ 7·5347699	·4500	·5073	·5040	·3765	(o)
+ 8·9182673	− 8·2510915	− 8·2474913	·44345	·4522	·35665	·2437	(p)
+ 8·9231747	− 8·4382497	− 8·4373553	·3747	·3415	·2212	·1411	(q)
+ 8·7158194	− 8·2980740	− 8·2983398	·19975	·1843	·0666	·0635	(r)
0·0000000	0·0000000	0·0000000					(s)
			Absolute term for g, $\frac{1}{2}(b_1 + b'_1)$		Absolute term for h, $-\frac{1}{2}(a_1 + a'_1)$		
			1845	1880	1845	1880	
+ 9·1422257	+ 8·6998672	+ 8·6724498					(a)
+ 9·0711447	+ 8·5795912	+ 8·5528031					(b)
+ 8·9415588	+ 8·3435956	+ 8·3178344	·8207		− ·90745		(c)
+ 8·7262726	+ 7·8442989	+ 7·8199303	·83065		− ·96175		(d)
+ 8·3283922	− 7·5376672	− 7·5150134	·85465	·8585	− 1·0252	− 1·0031	(e)
− 7·5352142	− 7·8808110	− 7·8601411	·8642	·8716	− 1·0830	− 1·0587	(f)
− 8·2550991	− 7·7969310	− 7·7784528	·8653	·8514	− 1·1312	− 1·1208	(g)
− 8·3417234	− 7·3006278	− 7·2844815	·8688	·8316	− 1·1902	− 1·1762	(h)
− 8·2379895	+ 7·3420495	+ 7·3283042	·85585	·8039	− 1·22185	− 1·2194	(i)
− 7·8809793	+ 7·6204332	+ 7·6090846	·8168	·7489	− 1·2460	− 1·2564	(k)
+ 7·4666026	+ 7·5275800	+ 7·5185514	·7681	·6867	− 1·27245	− 1·2875	(l)
+ 8·0276408	+ 6·8787985	+ 6·8719430	·6958	·6111	− 1·30025	− 1·3018	(m)
+ 8·1268700	− 7·2835484	− 7·2786542	·6365	·5272	− 1·3130	− 1·3102	(n)
+ 8·0343500	− 7·4970241	− 7·4938206	·5724	·4446	− 1·3219	− 1·3109	(o)
+ 7·6431401	− 7·3731630	− 7·3713294	·51615	·3718	− 1·32675	− 1·3111	(p)
− 7·5162087	− 6·3115955	− 6·3107704	·4706	·3130	− 1·3115	− 1·3032	(q)
− 7·9712381	+ 7·2957534	+ 7·2955456	·4318	·2738	− 1·3035	− 1·2977	(r)
− 8·0665127	+ 7·4593131	+ 7·4593131	·4207	·2547	− 1·2992	− 1·2872	(s)
			Absolute term for g, $\frac{1}{2}(a_1 + a'_1)$		Absolute term for h, $\frac{1}{2}(b_1 + b'_1)$		
			1845	1880	1845	1880	
− 8·9301383	+ 8·6432863	− 8·5696106					(a)
− 9·1585143	+ 8·8221417	− 8·7494683					(b)
− 9·2024316	+ 8·7589628	− 8·6881630	·64015		− ·17705		(c)
− 9·1085984	+ 8·3791463	− 8·3127472	·7899		− ·3328		(d)
− 8·8039603	− 8·1689481	+ 8·0942896	·9601	1·3320	− ·5039	− ·7795	(e)
+ 8·0690107	− 8·5823845	+ 8·5150536	1·1316	1·4437	− ·7269	− 1·0328	(f)
+ 8·8593245	− 8·5568387	+ 8·4939289	1·29415	1·5264	− 1·02455	− 1·2884	(g)
+ 8·9964401	− 8·1066975	+ 8·0525628	1·43965	1·6049	− 1·34865	− 1·5714	(h)
+ 8·9348995	+ 8·1956420	− 8·1305032	1·54325	1·6445	− 1·62515	− 1·8141	(i)
+ 8·6146806	+ 8·5062168	− 8·4484031	1·57305	1·6172	− 1·85715	− 2·0484	(k)
− 8·2184458	+ 8·4410786	− 8·3880139	1·5178	1·5301	− 2·07605	− 2·2480	(l)
− 8·8098344	+ 7·8114532	− 7·7705740	1·3861	1·3287	− 2·24505	− 2·4155	(m)
− 8·9296912	− 8·2428150	+ 8·1890422	1·19095	1·0573	− 2·35905	− 2·5400	(n)
− 8·8534134	− 8·4704542	+ 8·4212672	·9759	·7495	− 2·47385	− 2·6304	(o)
− 8·4753908	− 8·3577342	+ 8·3113363	·79105	·4812	− 2·5409	− 2·6233	(p)
+ 8·3529064	− 7·2982437	+ 7·2658733	·6456	·3016	− 2·58355	− 2·6183	(q)
+ 8·8145018	+ 8·2943736	− 8·2478717	·48485	·1487	− 2·5611	− 2·6097	(r)
+ 8·9116107	+ 8·4593131	− 8·4135556	·4470	·0976	− 2·5779	− 2·5903	(s)

Co-latitude	g_2^1 or h_2^1	g_{-2}^1 or h_{-2}^1	g_4^1 or h_4^1	g_{-4}^1 or h_{-4}^1	g_6^1 or h_6^1	g_{-6}^1 or h_{-6}^1	g_8^1 or h_8^1
	FOR X						
(a) 5°	− 9·9989663	− 9·9918637	− 9·7407551	− 9·7279760	− 9·3418927	− 9·3234481	− 8·8698399
(b) 10	− 9·9779863	− 9·9713941	− 9·6598062	− 9·6480489	− 9·1445450	− 9·1279377	− 8·4290796
(c) 15	− 9·9414712	− 9·9357529	− 9·4963491	− 9·4868384	− 8·4618206	− 8·4585556	+ 8·4559002
(d) 20	− 9·8866121	− 9·8821833	− 9·1396674	− 9·1371396	+ 8·9226472	+ 8·8998002	+ 8·8359051
(e) 25	− 9·8081392	− 9·8055413	+ 8·6900476	+ 8·6467768	+ 9·2313413	+ 9·2132193	+ 8·8866389
(f) 30	− 9·6955859	− 9·6956900	+ 9·3522360	+ 9·3352903	+ 9·3230723	+ 9·3078674	+ 8·7208397
(g) 35	− 9·5245589	− 9·5293019	+ 9·5679008	+ 9·5553252	+ 9·2921704	+ 9·2799411	+ 7·9204218
(h) 40	− 9·2136257	− 9·2306149	+ 9·6691587	+ 9·6593694	+ 9·1254295	+ 9·1170606	− 8·5394329
(i) 45	8·0717793	+ 7·5240878	+ 9·7041790	+ 9·6967780	+ 8·6198634	+ 8·6225453	− 8·7568073
(k) 50	9·2695533	+ 9·2471125	+ 9·6850642	+ 9·6799344	− 8·7317823	− 8·7148096	− 8·7036153
(l) 55	9·5488601	+ 9·5373082	+ 9·6082057	+ 9·6053917	− 9·1068219	− 9·0981056	− 8·3039354
(m) 60	9·7079491	+ 9·7007796	+ 9·4491816	+ 9·4490817	− 9·2084315	− 9·2032033	+ 8·2581992
(n) 65	9·8136943	+ 9·8091313	+ 9·1044640	+ 9·1096638	− 9·1699999	− 9·1674789	+ 8·6579432
(o) 70	9·8876469	+ 9·8848659	− 8·5716598	− 8·5469680	− 8·9668313	− 8·9671408	+ 8·6929164
(p) 75	9·9393726	+ 9·9378520	− 9·2831095	− 9·2791635	− 8·1017114	− 8·1178810	+ 8·4567712
(q) 80	9·9737875	+ 9·9731226	− 9·5022720	− 9·5009282	+ 8·8390922	+ 8·8365330	− 7·7582697
(r) 85	9·9935496	+ 9·9933847	− 9·6021134	− 9·6018092	+ 9·1118000	+ 9·1113418	− 8·5640278
(s) 90	0·0000000	+ 0·0000000	− 9·6320233	− 9·6320233	+ 9·1804561	+ 9·1804561	− 8·6897620
	FOR Y						
(a) 5°	0·0040941	+ 9·9968790	+ 9·7580519	+ 9·7450647	+ 9·3792415	+ 9·3604822	+ 8·9366193
(b) 10	9·9989028	+ 9·9918533	+ 9·7348241	+ 9·7221350	+ 9·3268289	+ 9·3085002	+ 8·8425243
(c) 15	9·9901712	+ 9·9833918	+ 9·6949680	+ 9·6827653	+ 9·2337146	+ 9·2160885	+ 8·6654836
(d) 20	9·9777738	+ 9·9713610	+ 9·6364948	+ 9·6249518	+ 9·0877538	+ 9·0710805	+ 8·3445322
(e) 25	9·9615226	+ 9·9555611	+ 9·5559163	+ 9·5451856	+ 8·8585946	+ 8·8430946	+ 7·3570221
(f) 30	9·9411515	+ 9·9357121	+ 9·4469939	+ 9·4372029	+ 8·4328445	+ 8·4187019	− 7·9907231
(g) 35	9·9162936	+ 9·9114309	+ 9·2974990	+ 9·2887462	− 7·8985310	− 7·8858880	− 8·1233747
(h) 40	9·8864468	+ 9·8821978	+ 9·0784824	+ 9·0708342	− 8·4747245	− 8·4636771	− 7·9973868
(i) 45	9·8509198	+ 9·8473026	+ 8·6853873	+ 8·6788764	− 8·5807327	− 8·5713280	− 7·4933247
(k) 50	9·8087453	+ 9·8057588	− 8·0803662	− 8·0749906	− 8·5362790	− 8·5285142	+ 7·5452372
(l) 55	9·7585363	+ 9·7561603	− 8·7703431	− 8·7660664	− 8·3468298	− 8·3406523	+ 7·8563710
(m) 60	9·6982284	+ 9·6964243	− 8·9568072	− 8·9535599	− 7·7870115	− 7·7823209	+ 7·8377617
(n) 65	9·6245869	+ 9·6232990	− 9·0263676	− 9·0240493	+ 7·9705950	+ 7·9672463	+ 7·5288505
(o) 70	9·5321559	+ 9·5313129	− 9·0279956	− 9·0264782	+ 8·3094328	+ 8·3072409	− 7·1207108
(p) 75	9·4106675	+ 9·4101850	− 8·9700681	− 8·9691996	+ 8·3921941	+ 8·3909396	− 7·6937943
(q) 80	9·2370227	+ 9·2368056	− 8·8378723	− 8·8374815	+ 8·3352955	+ 8·3347310	− 7·7705274
(r) 85	8·9374533	+ 8·9373986	− 8·5618346	− 8·5617362	+ 8·0981356	+ 8·0979934	− 7·5906615
(s) 90	0·0000000	0·0000000	0·0000000	0·0000900	0·0000000	0·0000000	0·0000000
	FOR Z						
(a) 5°	9·4253477	− 9·2396616	+ 9·4007587	− 9·2896123	+ 9·1679000	− 9·0813734	+ 8·8343190
(b) 10	9·7194205	− 9·5340101	+ 9·6767647	− 9·5660835	+ 9·4146855	− 9·3288279	+ 9·0393797
(c) 15	9·8838350	− 9·6988746	+ 9·8100014	− 9·7000884	+ 9·4945893	− 9·4098674	+ 9·0352432
(d) 20	9·9922493	− 9·8078993	+ 9·8722573	− 9·7634091	+ 9·4692045	− 9·3861566	+ 8·8344543
(e) 25	0·0675944	− 9·8839961	+ 9·8831505	− 9·7756644	+ 9·3311422	− 9·2506072	+ 7·9296043
(f) 30	0·1198972	− 9·9371686	+ 9·8467095	− 9·7409084	+ 8·9762979	− 8·9012321	− 8·6476108
(g) 35	0·1542765	− 9·9725087	+ 9·7561236	− 9·6524421	− 8·5124861	+ 8·4146622	− 8·8383231
(h) 40	0·1734971	− 9·9927519	+ 9·5854903	− 9·4848131	− 9·1310114	+ 9·0499309	− 8·7605408
(i) 45	0·1789664	− 9·9992742	+ 9·2309923	− 9·1371804	− 9·2770878	+ 9·2000316	− 8·2943062
(k) 50	0·1711402	− 9·9924993	− 8·6795328	+ 8·5433304	− 9·2664126	+ 9·1925953	+ 8·3879039
(l) 55	0·1496346	− 9·9720114	− 9·3859156	+ 9·2795649	− 9·1048782	+ 9·0347051	+ 8·7254501
(m) 60	0·1131112	− 9·9364416	− 9·5949329	+ 9·4927403	− 8·5648671	+ 8·5050483	+ 8·7298359
(n) 65	0·0588702	− 9·8830613	− 9·6834245	+ 9·5835299	+ 8·7762478	− 8·6999415	+ 8·4390506
(o) 70	9·9818523	− 9·8067853	− 9·7002213	+ 9·6019454	+ 9·1284990	− 9·0579615	− 8·0523724
(p) 75	9·8720812	− 9·6976154	− 9·6538675	+ 9·5567803	+ 9·2224851	− 9·1541621	− 8·6336576
(q) 80	9·7066667	− 9·5326437	− 9·5298161	+ 9·4335598	+ 9·1736029	− 9·1065874	− 8·7181544
(r) 85	9·4119815	− 9·2382295	− 9·2586161	+ 9·1628545	+ 8·9412272	− 8·8749449	− 8·5430062
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

g_{-8}^{1} or h_{-8}^{1}	g_{10}^{1} or h_{10}^{1}	g_{-10}^{1} or h_{-10}^{1}	Absolute term for g, $\frac{1}{2}(a_1+a'_1)$ 1845	Absolute term for g, $\frac{1}{2}(a_1+a'_1)$ 1880	Absolute term for h, $\frac{1}{2}(b_1+b'_1)$ 1845	Absolute term for h, $\frac{1}{2}(b_1+b'_1)$ 1880	
– 8·8457489	– 8·3471369	– 8·3174318					(a)
– 8·4089352	– 5·3634134	– 6·0044196					(b)
+ 8·4275878	+ 8·3108863	+ 8·2804934	1·0690		– ·0356		(c)
+ 8·8123300	+ 8·4225300	+ 8·3953218	·8064		– ·05995		(d)
+ 8·8661315	+ 8·1876033	+ 8·1649495	·5498	·4420	– ·0726	– ·2762	(e)
+ 8·7043471	– 7·5889238	– 7·5516022	·27255	·1948	– ·0683	– ·2601	(f)
+ 7·9247604	– 8·2509395	– 8·2288159	·0134	– ·0152	– ·0543	– ·2428	(g)
– 8·5204505	– 8·2456884	– 8·2288803	– ·2157	– ·2001	– ·0726	– ·2059	(h)
– 8·7435518	– 7·6862126	– 7·6812765	– ·4007	– ·3557	– ·07685	– ·1330	(i)
– 8·6947568	+ 8·0117486	+ 7·9951330	– ·55985	– ·5113	– ·0815	– ·0394	(k)
– 8·3026013	+ 8·2244016	+ 8·2142219	– ·66445	– ·6363	– ·0206	·0229	(l)
+ 8·2449659	+ 8·0223794	+ 8·0176877	– ·75885	– ·7460	·03435	·0830	(m)
+ 8·6522891	– 7·4969343	– 7·4802784	– ·8309	– .8329	·0804	·1517	(n)
+ 8·6904491	– 8·1419995	– 8·1374738	– ·8654	– ·8707	·0815	·2279	(o)
+ 8·4571815	– 8·1424541	– 8·1410806	– ·87465	– ·8548	·08465	·2659	(p)
– 7·7489066	– 7·5452247	– 7·5484577	– ·8743	– ·8473	·0893	·2763	(q)
– 8·5633885	+ 7·9693480	+ 7·9684759	– ·84975	– ·8326	·1408	·2896	(r)
– 8·6897620	+ 8·1761947	+ 8·1761947	– ·8329	– ·8319	·1720	·2984	(s)
			Absolute term for g, $\frac{1}{2}(b_1-b'_1)$ 1845	Absolute term for g, $\frac{1}{2}(b_1-b'_1)$ 1880	Absolute term for h, $-\frac{1}{2}(a_1-a'_1)$ 1845	Absolute term for h, $-\frac{1}{2}(a_1-a'_1)$ 1880	
+ 8·9120879	+ 8·4552887	+ 8·4249852					(a)
+ 8·8185560	+ 8·3047675	+ 8·2751596					(b)
+ 8·6424341	+ 7·9924688	+ 7·9639959	– 1·2928		– ·05955		(c)
+ 8·3227287	+ 7·0302471	+ 7·0033134	– 1·20185		– ·06615		(d)
+ 7·3367529	– 7·5678631	– 7·5428247	– 1·14205	– 1·1931	– ·0512	·0352	(e)
– 7·9722290	– 7·6181118	– 7·5952661	– 1·0813	– 1·1029	– ·0463	·0877	(f)
– 8·1068415	– 7·2617488	– 7·2413255	– ·9811	– 1·0130	– ·0359	·1191	(g)
– 7·9829402	+ 6·9741270	+ 6·9562811	– ·8677	– ·9155	– ·0335	·1408	(h)
– 7·4810263	+ 7·3665209	+ 7·3513287	– ·75365	– ·8130	– ·02125	·1548	(i)
+ 7·5350832	+ 7·2561701	+ 7·2436269	– ·6582	– ·6955	– ·0153	·1609	(k)
+ 7·8482927	+ 6·1552293	+ 6·1452503	– ·5503	– ·5921	+ ·00865	·1573	(l)
+ 7 8316279	– 7·1241463	– 7·1165692	– ·4442	– ·4841	·02085	·1518	(m)
+ 7·5244715	– 7·2163963	– 7·2109869	– ·3533	– ·3796	·0306	·1415	(n)
– 7·1178445	– 6·8698441	– 6·8663034	– ·2723	– ·2898	·0262	·1335	(o)
– 7·6921538	+ 6·7901037	+ 6·7880771	– ·20665	– ·2116	+ ·00555	·1099	(p)
– 7·7697892	+ 7·1538481	+ 7·1529362	– ·1385	– ·1380	– ·0041	·0727	(q)
– 7·5904756	+ 7·0554331	+ 7·0552034	– ·0625	– ·0714	– ·0056	·0383	(r)
0·0000000	0·0000000	0·0000000					(s)
			Absolute term for g, $\frac{1}{2}(a_1-a'_1)$ 1845	Absolute term for g, $\frac{1}{2}(a_1-a'_1)$ 1880	Absolute term for h, $\frac{1}{2}(b_1-b'_1)$ 1845	Absolute term for h, $\frac{1}{2}(b_1-b'_1)$ 1880	
– 8·7580501	+ 8·4400682	– 8·3679418					(a)
– 8·9639976	+ 8·5886511	– 8·5176486					(b)
– 8·9614466	+ 8·4490404	– 8·3802849	– 1·45415		·07575		(c)
– 8·7634875	+ 7·6021048	– 7·5465980	– 1·7280		·0305		(d)
– 7·8791501	– 8·2385854	+ 8·1700585	– 1·8717	– 2·2714	– ·0032	·2993	(e)
+ 8·5747718	– 8·3604405	+ 8·2966425	– 1·9373	– 2·1509	– ·0188	·2915	(f)
+ 8·7702769	– 8·0615715	+ 8·0041670	– 1·89095	– 2·0095	– ·00735	·3025	(g)
+ 8·6967579	+ 7·8290476	– 7·7618734	– 1·79565	– 1·8540	·04265	·3356	(h)
+ 8·2397173	+ 8·2592391	– 8·2018271	– 1·63245	– 1·6724	·05705	·3751	(i)
– 8·3202494	+ 8·1820501	– 8·1301304	– 1·47385	– 1·4880	·09215	·4540	(k)
– 8·6651099	+ 7·0928452	– 7·0786324	– 1·2730	– 1·2984	·11885	·4757	(l)
– 8·6734429	– 8·1048359	+ 8·0536694	– 1·0867	– 1·0935	·09135	·4747	(m)
– 8·3874056	– 8·2153702	+ 8·1690083	– ·89535	– ·9296	·05005	·4527	(n)
+ 7·9896028	– 7·8828719	+ 7·8414756	– ·6950	– ·7627	·02585	·4068	(o)
+ 8·5798356	+ 7·8181669	– 7·7715207	– ·48225	– ·5553	– ·0028	·3403	(p)
+ 8·6663780	+ 8·1887923	– 8·1462499	– ·3205	– ·3767	·03775	·2531	(q)
+ 8·4922283	+ 8·0950036	– 8·0537766	– ·13965	– ·1728	·0264	·1741	(r)
0·0000000	0·0000000	0·0000000					(s)

For X

Co-latitude	g_2^2 or h_2^2	g_{-2}^2 or h_{-2}^2	g_4^2 or h_4^2	g_{-4}^2 or h_{-4}^2	g_6^2 or h_6^2	g_{-6}^2 or h_{-6}^2	g_8^2 or h_8^2
(a) 5°	− 9·2482727	− 9·2411134	− 9·1763176	− 9·1634318	− 8·9192346	− 8·9006238	− 8·5689978
(b) 10	− 9·5422899	− 9·5354619	− 9·4459514	− 9·4336761	− 9·1484541	− 9·1307626	− 8·7381038
(c) 15	− 9·7066139	− 9·7003264	− 9·5673094	− 9·5560752	− 9·1904803	− 9·1745125	− 8·6334123
(d) 20	− 9·8149052	− 9·8093511	− 9·6096384	− 9·5999358	− 9·0797463	− 9·0670690	− 7·7994099
(e) 25	− 9·8900987	− 9·8854477	− 9·5869466	− 9·5794342	− 8·6459841	− 8·6456394	+ 8·5878643
(f) 30	− 9·9422261	− 9·9386204	− 9·4890144	− 9·4848932	+ 8·7223907	+ 8·6934837	+ 8·8518992
(g) 35	− 9·9764111	− 9·9739606	− 9·2592910	− 9·2623512	+ 9·1610916	+ 9·1437698	+ 8·8784960
(h) 40	− 9·9954243	− 9·9942035	− 8·3256443	− 8·4185432	+ 9·3163067	+ 9·3032560	+ 8·6967870
(i) 45	− 0·0006802	− 0·0007260	+ 9·1876985	+ 9·1656325	+ 9·3470720	+ 9·3374043	+ 7·7008874
(k) 50	− 9·9926404	− 9·9939511	+ 9·5091460	+ 9·4973860	+ 9·2690429	+ 9·2627841	− 8·5875061
(l) 55	− 9·9709277	− 9·9734633	+ 9·6675431	+ 9·6599716	+ 9·0267191	+ 9·0251943	− 8·7942255
(m) 60	− 9·9342098	− 9·9378933	+ 9·7505996	+ 9·7457593	+ 7·6111909	+ 7·7214993	− 8·7491205
(n) 65	− 9·8797931	− 9·8845131	+ 9·7815634	+ 9·7787829	− 8·9779950	− 8·9702419	− 8·3858124
(o) 70	− 9·8026234	− 9·8082370	+ 9·7664836	+ 9·7653132	− 9·2203795	− 9·2169306	+ 8·2455465
(p) 75	− 9·6927292	− 9·6990672	+ 9·7010668	+ 9·7011287	− 9·2820469	− 9·2807278	+ 8·6980084
(q) 80	− 9·5272241	− 9·5340954	+ 9·5655984	+ 9·5665376	− 9·2183684	− 9·2183696	+ 8·7604437
(r) 85	− 9·2324833	− 9·2396811	+ 9·2882594	+ 9·2897248	− 8·9789435	− 8·9796960	+ 8·5783241
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

For Y

Co-latitude	g_2^2 or h_2^2	g_{-2}^2 or h_{-2}^2	g_4^2 or h_4^2	g_{-4}^2 or h_{-4}^2	g_6^2 or h_6^2	g_{-6}^2 or h_{-6}^2	g_8^2 or h_8^2
(a) 5°	9·2499843	+ 9·2427692	+ 9·1820056	+ 9·1690184	+ 8·9312698	+ 8·9125105	+ 8·5899079
(b) 10	9·5491599	+ 9·5421104	+ 9·4692725	+ 9·4565834	+ 9·1995810	+ 9·1812523	+ 8·8318157
(c) 15	9·7221620	+ 9·7153827	+ 9·6221233	+ 9·6099206	+ 9·3195272	+ 9·3019011	+ 8·9040182
(d) 20	9·8427780	+ 9·8363652	+ 9·7138550	+ 9·7023120	+ 9·3620406	+ 9·3453673	+ 8·8694055
(e) 25	9·9341335	+ 9·9281720	+ 9·7668265	+ 9·7560958	+ 9·3450956	+ 9·3295956	+ 8·7255750
(f) 30	0·0065292	+ 0·0010898	+ 9·7901897	+ 9·7803987	+ 9·2693109	+ 9·2551683	+ 8·4006795
(g) 35	0·0654588	+ 0·0605961	+ 9·7877198	+ 9·7789670	+ 9·1193426	+ 9·1066996	− 7·0601204
(h) 40	0·1141986	+ 0·1099496	+ 9·7600568	+ 9·7524086	+ 8·8381532	+ 8·8271058	− 8·2697376
(i) 45	0·1548581	+ 0·1512409	+ 9·7050561	+ 9·6985452	+ 7·9405826	+ 7·9311779	− 8·3776922
(k) 50	0·1888701	+ 0·1858836	+ 9·6166983	+ 9·6113227	− 8·5935533	− 8·5857885	− 8·2550659
(l) 55	0·2172479	+ 0·2148719	+ 9·4810832	+ 9·4768065	− 8·8344136	− 8·8282361	− 7·7511050
(m) 60	0·2407274	+ 0·2389233	+ 9·2611467	+ 9·2578994	− 8·8800664	− 8·8753758	+ 7·8478954
(n) 65	0·2598527	+ 0·2585648	+ 8·7891894	+ 8·7868711	− 8·8030384	− 8·7996897	+ 8·1687827
(o) 70	0·2750284	+ 0·2741854	− 8·7108229	− 8·7093055	− 8·5632470	− 8·5610551	+ 8·1717663
(p) 75	0·2865534	+ 0·2860709	− 9·1715537	− 9·1706852	− 7·4346157	− 7·4333612	+ 7·9089835
(q) 80	0·2946423	+ 0·2944252	− 9·3481633	− 9·3477725	+ 8·4695705	+ 8·4690060	− 7·0878538
(r) 85	0·2994398	+ 0·2993851	− 9·4309562	− 9·4308578	+ 8·7189536	+ 8·7188114	− 8·0253355
(s) 90	0·3010300	+ 0·3010300	− 9·4559320	− 9·4559320	+ 8·7825161	+ 8·7825161	− 8·1456940

For Z

Co-latitude	g_2^2 or h_2^2	g_{-2}^2 or h_{-2}^2	g_4^2 or h_4^2	g_{-4}^2 or h_{-4}^2	g_6^2 or h_6^2	g_{-6}^2 or h_{-6}^2	g_8^2 or h_8^2
(a) 5°	8·3711709	− 8·1830652	+ 8·5242663	− 8·4118028	+ 8·4193217	− 8·3318763	+ 8·1869136
(b) 10	8·9696109	− 8·7817806	+ 9·1107839	− 8·9987479	+ 8·9868641	− 8·9000162	+ 8·7280297
(c) 15	9·3157596	− 9·1283789	+ 9·4367584	− 9·3254212	+ 9·2798983	− 9·1940349	+ 8·9732734
(d) 20	9·5571878	− 9·3704169	+ 9·6492701	− 9·5388835	+ 9·4431335	− 9·3586288	+ 9·0592883
(e) 25	9·7401402	− 9·5541202	+ 9·7937962	− 9·6845861	+ 9·5176495	− 9·4348713	+ 9·0066980
(f) 30	9·8852106	− 9·7000597	+ 9·8897808	− 9·7819421	+ 9·5143248	− 9·4336686	+ 8·7534620
(g) 35	0·0033782	− 9·8191873	+ 9·9464816	− 9·8401761	+ 9·4232038	− 9·3452227	− 7·5072860
(h) 40	0·1011864	− 9·9180170	+ 9·9678013	− 9·8631590	+ 9·1900814	− 9·1162087	− 8·7340391
(i) 45	0·1828431	− 0·0007259	+ 9·9536837	− 9·8508123	+ 8·3203932	− 8·2763055	− 8·8819373
(k) 50	0·2512043	− 0·0701375	+ 9·8995093	− 9·7985219	− 9·0255890	+ 8·9454008	− 8·7928405
(l) 55	0·3082864	− 0·1282364	+ 9·7923196	− 9·6934213	− 9·2935399	+ 9·2188035	− 8·3145371
(m) 60	0·3555512	− 0·1764540	+ 9·5955497	− 9·4994080	− 9·3624127	+ 9·2907192	+ 8·4419219
(n) 65	0·3940780	− 0·2158404	+ 9·1400815	− 9·0516657	− 9·3043763	+ 9·2352091	+ 8·7802374
(o) 70	0·4246673	− 0·2471712	− 9·0865725	+ 8·9779843	− 9·0793520	+ 9·0129951	+ 8·7980426
(p) 75	0·4479101	− 0·2710146	− 9·5556462	+ 9·4555791	− 7·9484683	+ 7·9128701	+ 8·5463071
(q) 80	0·4642298	− 0·2877767	− 9·7398742	+ 9·6418247	+ 9·0078831	− 8·9386229	− 7·9378133
(r) 85	0·4739118	− 0·2977294	− 9·8273352	+ 9·7301628	+ 9·2615191	− 9·1941453	− 8·6771148
(s) 90	0·4771213	− 0·3010300	− 9·8538720	+ 9·7569620	+ 9·3265841	− 9·2596374	− 8·7989065

g_{-8}^2 or h_{-8}^2	g_{10}^2 or h_{10}^2	g_{-10}^2 or h_{-10}^2	Absolute term for g, $\frac{1}{2}(a_2 - a'_2)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_2 - b'_2)$ 1845	1880	
− 8·5446646	− 8·1608495	− 8·1307975					(a)
− 8·7150542	− 8·2430704	− 8·2147773					(b)
− 8·6133625	− 7·7797096	− 7·7596044	·18865		·0767		(c)
− 7·8086668	+ 8·0897803	+ 8·0579368	·1567		·08695		(d)
+ 8·5620225	+ 8·3912723	+ 8·3649517	·1331	·3117	·1065	·0750	(e)
+ 8·8314944	+ 8·3458077	+ 8·3239437	·1290	·2884	·1420	·1805	(f)
+ 8·8620645	+ 7·8197744	+ 7·8085715	·10155	·2356	·1972	·2269	(g)
+ 8·6852705	− 8·0723348	− 8·0496762	·08435	·1688	·27495	·2503	(h)
+ 7·7283747	− 8·3125022	− 8·2967681	·05115	·0969	·30965	·2738	(i)
− 8·5729837	− 8·1605676	− 8·1506421	− ·0148	·0072	·2979	·2824	(k)
− 8·7852839	+ 7·1003967	+ 7·0456154	− ·12005	− ·0789	·30115	·2868	(l)
− 8·7441489	+ 8·1804483	+ 8·1709494	− ·20585	− ·1497	·30315	·2999	(m)
− 8·3866039	+ 8·2435443	+ 8·2386956	− ·2559	− ·1835	·28705	·3060	(n)
+ 8·2363459	+ 7·8568021	+ 7·8575069	− ·2911	− ·1989	·2467	·2832	(o)
+ 8·6953988	− 7·9026870	− 7·8977438	− ·26415	− ·1523	·20275	·2277	(p)
+ 8·7598453	− 8·2298909	− 8·2287554	− ·17835	− ·1011	·14715	·1532	(q)
+ 8·5787343	− 8·1269057	− 8·1270884	− ·1006	− ·0443	·09145	·0763	(r)
0·0000000	0·0000000	0·0000000					(s)

			Absolute term for g, $\frac{1}{2}(b_2 + b'_2)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_2 + a'_2)$ 1845	1880	
+ 8·5653765	+ 8·1933940	+ 8·1630905					(a)
+ 8·8078474	+ 8·4008463	+ 8·3712384					(b)
+ 8·8809687	+ 8·4067612	+ 8·3782883	− ·15935		− ·0876		(c)
+ 8·8476020	+ 8·2487806	+ 8·2218469	− ·1178		− ·12565		(d)
+ 8·7053058	+ 7·8012638	+ 7·7762254	− ·12655	− ·2588	− ·14905	− ·2329	(e)
+ 8·3821854	− 7·5150293	− 7·4921836	− ·14145	− ·3219	− ·1734	− ·2565	(f)
− 7·0435872	− 7·8713163	− 7·8508930	− ·15775	− ·3579	− ·19375	− ·2759	(g)
− 8·2552910	− 7·7651499	− 7·7473040	− ·17545	− ·3670	− ·2179	− ·2862	(h)
− 8·3653938	− 7·0113531	− 6·9961609	− ·1763	− ·3605	− ·25165	− ·3048	(i)
− 8·2449119	+ 7·5174366	+ 7·5048934	− ·14775	− ·3300	− ·2874	− ·3322	(k)
− 7·7430267	+ 7·6588792	+ 7·6489002	− ·1254	− ·2893	− ·3205	− ·3662	(l)
+ 7·8417616	+ 7·4043225	+ 7·3967454	− ·0951	− ·2511	− ·3739	− ·4064	(m)
+ 8·1644037	− 6·9676061	− 6·9621967	− ·04715	− ·2106	− ·42475	− ·4520	(n)
+ 8·1689000	− 7·5227773	− 7·5192366	·0002	− ·1651	− ·4752	− ·4959	(o)
+ 7·9073430	− 7·4994597	− 7·4974331	·04245	− ·1170	− ·5145	− ·5333	(p)
− 7·0871156	− 6·8693683	− 6·8684564	·0848	− ·0766	− ·54455	− ·5585	(q)
− 8·0251496	+ 7·3219969	+ 7·3217672	·12275	− ·0380	− ·56095	− ·5809	(r)
− 8·1456940	+ 7·5229822	+ 7·5229822	·1291	− ·0236	− ·5581	− ·5817	(s)

			Absolute term for g, $\frac{1}{2}(a_2 + a'_2)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_2 + b'_2)$ 1845	1880	
− 8·1099298	+ 7·8774257	− 7·8047062					(a)
− 8·6518212	+ 8·3840604	− 8·3123006					(b)
− 8·8983596	+ 8·5629532	− 8·4928326	·0069		·03535		(c)
− 8·9862186	+ 8·5254186	− 8·4578091	− ·0911		·0162		(d)
− 8·9361736	+ 8·1680999	− 8·1055645	− ·17605	− ·2210	− ·02575	− ·0841	(e)
− 8·6871572	− 7·9596218	+ 7·8888815	− ·2134	− ·3281	− ·0781	− ·1735	(f)
+ 7·3652342	− 8·3726029	+ 8·3099726	− ·24135	− ·4386	− ·1005	− ·2606	(g)
+ 8·6656262	− 8·3144647	+ 8·2569360	− ·2543	− ·5546	− ·1048	− ·3310	(h)
+ 8·8182723	− 7·5951324	+ 7·5541820	− ·2740	− ·6121	− ·1715	− ·3926	(i)
+ 8·7334155	+ 8·1447394	− 8·0876652	− ·28095	− ·6193	− ·2366	− ·4266	(k)
+ 8·2639631	+ 8·3134775	− 8·2622722	− ·2580	− ·5686	− ·3483	− ·4648	(l)
− 8·3791738	+ 8·0813645	− 8·0356914	− ·19035	− ·4586	− ·45925	− ·5556	(m)
− 8·7237463	− 7·6699468	+ 7·6143046	− ·1343	− ·3594	− ·62115	− ·6763	(n)
− 8·7445745	− 8·2368538	+ 8·1910994	− ·04155	− ·2779	− ·7481	− ·7634	(o)
− 8·4956908	− 8·2246151	+ 8·1819182	·0489	− ·2191	− ·8157	− ·8256	(p)
+ 7·8788613	− 7·6010657	+ 7·5631670	·15845	− ·1442	− ·8959	− ·9074	(q)
+ 8·6253497	+ 8·0610188	− 8·0187807	·2331	− ·0719	− ·9507	− ·9644	(r)
+ 8·7477540	+ 8·2633449	− 8·2219522	·2491	− ·0301	− ·9587	− 1·0085	(s)

Co-latitude	g_3^2 or h_3^2	g_{-3}^2 or h_{-3}^2	g_5^2 or h_5^2	g_{-5}^2 or h_{-5}^2	g_7^2 or h_7^2	g_{-7}^2 or h_{-7}^2
	For X					
(a) 5°	− 9·2463386	− 9·2363156	− 9·0629433	− 9·0471947	− 8·7529269	− 8·7314546
(b) 10	− 9·5300190	− 9·5204646	− 9·3145492	− 9·2995606	− 8·9548842	− 8·9345039
(c) 15	− 9·6765089	− 9·6677324	− 9·4019363	− 9·3882943	− 8·9357154	− 8·9175656
(d) 20	− 9·7584310	− 9·7507417	− 9·3850006	− 9·3735273	− 8·6517501	− 8·6397486
(e) 25	− 9·7969503	− 9·7906662	− 9·2494808	− 9·2419298	+ 8·2977489	+ 8·2587174
(f) 30	− 9·7990930	− 9·7945671	− 8·8355749	− 8·8418637	+ 8·9277268	+ 8·9069632
(g) 35	− 9·7646195	− 9·7623061	+ 8·8103166	+ 8·7775841	+ 9·1005218	+ 9·0842493
(h) 40	− 9·6857133	− 9·6863585	+ 9·2937244	+ 9·2778598	+ 9·1095474	+ 9·0969869
(i) 45	− 9·5376848	− 9·5430514	+ 9·4808962	+ 9·4698292	+ 8·9597319	+ 8·9514992
(k) 50	− 9·2373505	− 9·2546922	+ 9·5588705	+ 9·5510281	+ 8·4182933	+ 8·4224220
(l) 55	+ 8·3603074	+ 8·1554068	+ 9·5603547	+ 9·5552444	− 8·6413758	− 8·6269916
(m) 60	+ 9·3591952	+ 9·3419547	+ 9·4845203	+ 9·4820138	− 8·9774537	− 8·9702222
(n) 65	+ 9·6348162	+ 9·6267064	+ 9·2946953	+ 9·2952262	− 9·0424883	− 9·0386359
(o) 70	+ 9·7910313	+ 9·7866580	+ 8·7623683	+ 8·7719971	− 8·9324035	− 8·9313184
(p) 75	+ 9·8904400	+ 9·8881912	− 8·9460970	− 8·9391933	− 8·4871433	− 8·4906447
(q) 80	+ 9·9534038	+ 9·9524523	− 9·3343021	− 9·3325381	+ 8·5331937	+ 8·5295062
(r) 85	+ 9·9886320	+ 9·9883999	− 9·4809287	− 9·4805521	+ 8·9318099	+ 8·9312691
(s) 90	+ 0·0000000	+ 0·0000000	− 9·5228788	− 9·5228788	+ 9·0207552	+ 9·0207552
	For Y					
(a) 5°	9·2497493	+ 9·2396481	+ 9·0715003	+ 9·0556270	+ 8·7690711	+ 8·7474258
(b) 10	9·5438330	+ 9·5339637	+ 9·3501911	+ 9·3346822	+ 9·0251441	+ 9·0039956
(c) 15	9·7082656	+ 9·6987746	+ 9·4882863	+ 9·4733719	+ 9·1232552	+ 9·1029174
(d) 20	9·8167042	+ 9·8077263	+ 9·5582861	+ 9·5441780	+ 9·1314460	+ 9·1122076
(e) 25	9·8920792	+ 9·8837331	+ 9·5811779	+ 9·5680626	+ 9·0612656	+ 9·0433810
(f) 30	9·9444169	+ 9·9368017	+ 9·5637471	+ 9·5517803	+ 8·8963838	+ 8·8800655
(g) 35	9·9788347	+ 9·9720269	+ 9·5054526	+ 9·4947547	+ 8·5544261	+ 8·5398380
(h) 40	9·9980960	+ 9·9921474	+ 9·3981516	+ 9·3888038	− 7·5392941	− 7·5265471
(i) 45	0·0036075	+ 9·9985434	+ 9·2180890	+ 9·2101312	− 8·4905418	− 8·4796902
(k) 50	9·9958235	+ 9·9916424	+ 8·8812408	+ 8·8746706	− 8·6255063	− 8·6165469
(l) 55	9·9743586	+ 9·9710323	− 7·8382258	− 7·8329987	− 8·5739839	− 8·5668560
(m) 60	9·9378733	+ 9·9353476	− 8·8722662	− 8·8682973	− 8·3237166	− 8·3183044
(n) 65	9·8836667	+ 9·8818636	− 9·0792056	− 9·0763721	+ 6·2192264	+ 6·2153626
(o) 70	9·8066785	+ 9·8054983	− 9·1449378	− 9·1430832	+ 8·2747321	+ 8·2722030
(p) 75	9·6969314	+ 9·6962559	− 9·1239255	− 9·1228640	+ 8·4636017	+ 8·4621542
(q) 80	9·5315347	+ 9·5312307	− 9·0138793	− 9·0134016	+ 8·4501863	+ 8·4495349
(r) 85	9·2368604	+ 9·2367838	− 8·7498820	− 8·7497617	+ 8·2333511	+ 8·2331871
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
	For Z					
(a) 5°	8·4953916	− 8·3570069	+ 8·4927464	− 8·3956148	+ 8·3150082	− 8·2338799
(b) 10	9·0887340	− 8·9506968	+ 9·0706787	− 8·9740584	+ 8·8703013	− 8·7898586
(c) 15	9·4263039	− 9·2888337	+ 9·3818809	− 9·2860993	+ 9·1414785	− 9·0621722
(d) 20	9·6555423	− 9·5188408	+ 9·5726350	− 9·4780007	+ 9·2703504	− 9·1926322
(e) 25	9·8224988	− 9·6867442	+ 9·6870428	− 9·5938422	+ 9·2915481	− 9·2159081
(f) 30	9·9474936	− 9·8128346	+ 9·7421743	− 9·6506739	+ 9·1989182	− 9·1260453
(g) 35	0·0411297	− 9·9076811	+ 9·7429547	− 9·6534173	+ 8·9149343	− 8·8468165
(h) 40	0·1094383	− 9·9772777	+ 9·6844624	− 9·5972030	− 7·9712874	+ 7·8561267
(i) 45	0·1559247	− 0·0250925	+ 9·5448938	− 9·4604962	− 8·9450454	+ 8·8729561
(k) 50	0·1824692	− 0·0529592	+ 9·2408487	− 9·1615868	− 9·1133085	+ 9·0457276
(l) 55	0·1896875	− 0·0614597	− 8·2579525	+ 8·1086771	− 9·0898774	+ 9·0257885
(m) 60	0·1769680	− 0·0499411	− 9·2896172	+ 9·2021046	− 8·8623970	+ 8·8024823
(n) 65	0·1421450	− 0·0162022	− 9·5147908	+ 9·4312940	+ 6·9462258	− 6·3832808
(o) 70	0·0805551	− 9·9555470	− 9·5955210	+ 9·5141794	+ 8·8512425	− 8·7883021
(p) 75	9·9825133	− 9·8582625	− 9·5860187	+ 9·5061386	+ 9·0509977	− 8·9911312
(q) 80	9·8253382	− 9·7016451	− 9·4840883	+ 9·4052001	+ 9·0455328	− 8·9872148
(r) 85	9·5355428	− 9·4121910	− 9·2249159	+ 9·1466113	+ 8·8334618	− 8·7759814
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

g_9^2 or h_9^2	g_{-9}^2 or h_{-9}^2	Absolute term for g, $\frac{1}{2}(a_2+a'_2)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_2+b'_2)$ 1845	1880	
− 8·3708366	− 8·3436435					(a)
− 8·5013083	− 8·4748159					(b)
− 8·2669526	− 8·2456760	·33235		·0256		(c)
+ 7·9579715	+ 7·9206987	·3773		·03405		(d)
+ 8·5472037	+ 8·5218558	·4027	·3660	·0453	·0409	(e)
+ 8·6510878	+ 8·6299905	·3928	·3347	·0387	·0145	(f)
+ 8·5041740	+ 8·4878977	·28865	·2458	·0082	·0015	(g)
+ 7·5779716	+ 7·5919250	·16555	·1284	·01215	− ·0022	(h)
− 8·3800291	− 8·3623690	·02355	·0130	·01535	+ ·0060	(i)
− 8·5597450	− 8·5479432	− ·1508	− ·0996	·0260	− ·0007	(k)
− 8·4337333	− 8·4269025	− ·26015	− ·2107	·00955	− ·0091	(l)
− 7·4619184	− 7·4843625	− ·34615	− ·3158	− ·03405	− ·0332	(m)
+ 8·3391907	+ 8·3315834	− ·4146	− ·3929	− ·04005	− ·0543	(n)
+ 8·5146388	+ 8·5112778	− ·4279	− ·4292	− ·0852	− ·0472	(o)
+ 8·3772955	+ 8·3767886	− ·42615	− ·4345	− ·12515	− ·0394	(p)
+ 6·9902948	+ 7·0213420	− ·41725	− ·4247	− ·13995	− ·0572	(q)
− 8·3443152	− 8·3435740	− ·4000	− ·4124	− ·17875	− ·0859	(r)
− 8·5007057	− 8·5007057	− ·3703	− ·4155	− ·1795	− ·0961	(s)

		Absolute term for g, $\frac{1}{2}(b_2-b'_2)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_2-a'_2)$ 1845	1880	
+ 8·3972012	+ 8·3697838					(a)
+ 8·6229353	+ 8·5961472					(b)
+ 8·6646384	+ 8·6388772	− ·39805		− ·0192		(c)
+ 8·5761244	+ 8·5517558	− ·5969		− ·04065		(d)
+ 8·3224139	+ 8·2997601	− ·72555	− ·6796	− ·05405	·0551	(e)
+ 7·5020460	+ 7·4813761	− ·79135	− ·7220	− ·0498	·0335	(f)
− 7·9737194	− 7·9552412	− ·80775	− ·7393	− ·05495	·0178	(g)
− 8·1279590	− 8·1118127	− ·77055	− ·7331	− ·0579	− ·0056	(h)
− 7·9795121	− 7·9657668	− ·7181	− ·7017	− ·05885	− ·0270	(i)
− 7·1969423	− 7·1855937	− ·63295	− ·6432	− ·0606	− ·0335	(k)
+ 7·7496707	+ 7·7406421	− ·5331	− ·5693	− ·0857	− ·0290	(l)
+ 7·9281163	+ 7·9212608	− ·4369	− ·4787	− ·0780	− ·0157	(m)
+ 7·7898652	+ 7·7849710	− ·34635	− ·3837	− ·05725	− ·0009	(n)
+ 6·7912873	+ 6·7880838	− ·2597	− ·2811	− ·0365	·0132	(o)
− 7·6775457	− 7·6757121	− ·19605	− ·1877	− ·0218	·0149	(p)
− 7·8432743	− 7·8424492	− ·1206	− ·1153	+ ·00025	·0088	(q)
− 7·6941432	− 7·6939354	− ·06785	− ·0578	+ ·00555	·0019	(r)
0·0000000	0·0000000					(s)

		Absolute term for g, $\frac{1}{2}(a_2-a'_2)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_2-b'_2)$ 1845	1880	
+ 8·0398965	− 7·9655737					(a)
+ 8·5648263	− 8·4913703					(b)
+ 8·7795416	− 8·7075471	− ·2181		− ·11775		(c)
+ 8·8115820	− 8·7417287	− ·4678		− ·1517		(d)
+ 8·6488371	− 8·5822810	− ·72125	− ·8818	− ·16855	− ·0385	(e)
+ 7·8950310	− 7·8435225	− ·9164	− ·9700	− ·1604	·0332	(f)
− 8·4348379	+ 8·3671917	− 1·02245	− 1·0530	− ·1438	·0556	(g)
− 8·6368259	+ 8·5746481	− 1·0824	− 1·0624	− ·1697	·0891	(h)
− 8·5283867	+ 8·4710498	− 1·0969	− 1·0350	− ·1537	·1044	(i)
− 7·7726874	+ 7·7336930	− 1·05975	− 1·0002	− ·1268	·1032	(k)
+ 8·3645635	− 8·3063760	− ·9882	− ·9254	− ·0750	·0957	(l)
+ 8·5654413	− 8·5126115	− ·81215	− ·8183	− ·06525	·0659	(m)
+ 8·4457601	− 8·3969346	− ·6274	− ·6831	− ·09885	·0201	(n)
+ 7·4529672	− 7·4261153	− ·46855	− ·5522	− ·0853	− ·0296	(o)
− 8·3624046	+ 8·3132242	− ·3370	− ·4118	− ·0804	− ·0223	(p)
− 8·5356973	+ 8·4890837	− ·22325	− ·2850	− ·0381	− ·0221	(q)
− 8·3912536	+ 8·3457646	− ·0918	− ·1534	− ·0156	− ·0209	(r)
0·0000000	0·0000000					(s)

Co-latitude	g_3^3 or h_3^3	g_{-3}^3 or h_{-3}^3	g_5^3 or h_5^3	g_{-5}^3 or h_{-5}^3	g_7^3 or h_7^3	g_{-7}^3 or h_{-7}^3
	FOR X					
(a) 5°	− 8·3689928	− 8·3589437	− 8·3143973	− 8·2986063	− 8·1070000	− 8·0854679
(b) 10	− 8·9622963	− 8·9526337	− 8·8882843	− 8·8731081	− 8·6520109	− 8·6313328
(c) 15	− 9·2998025	− 9·2907704	− 9·1921322	− 9·1779811	− 8·9028537	− 8·8836569
(d) 20	− 9·5289542	− 9·5207777	− 9·3710718	− 9·3583666	− 8·9936323	− 8·9767222
(e) 25	− 9·6958041	− 9·6886810	− 9·4669439	− 9·4561442	− 8·9355173	− 8·9224077
(f) 30	− 9·8206753	− 9·8147713	− 9·4920984	− 9·4837884	− 8·6018744	− 8·5997386
(g) 35	− 9·9141746	− 9·9096178	− 9·4400601	− 9·4352334	+ 8·4701418	+ 8·4334406
(h) 40	− 9·9823375	− 9·9792145	− 9·2694303	− 9·2708572	+ 9·0059694	+ 8·9878436
(i) 45	− 0·0286744	− 0·0270281	− 8·7076342	− 8·7412993	+ 9·1807960	+ 9·1679364
(k) 50	− 0·0550677	− 0·0548960	+ 9·0449723	+ 9·0185424	+ 9·2093773	+ 9·2004286
(l) 55	− 0·0621404	− 0·0633965	+ 9·4392157	+ 9·4273946	+ 9·1022195	+ 9·0971839
(m) 60	− 0·0492839	− 0·0518778	+ 9·6172013	+ 9·6102321	+ 8·7380428	+ 8·7398245
(n) 65	− 0·0143373	− 0·0181390	+ 9·7029866	+ 9·6989602	− 8·5028563	− 8·4872905
(o) 70	− 9·9526408	− 9·9574838	+ 9·7237062	+ 9·7217791	− 9·0253056	− 9·0206780
(p) 75	− 9·8545124	− 9·8601994	+ 9·6824494	+ 9·6820605	− 9·1593021	− 9·1575017
(q) 80	− 9·6972735	− 9·7035818	+ 9·5627512	+ 9·5634331	− 9·1293981	− 9·1291756
(r) 85	− 9·4074391	− 9·4141277	+ 9·2943779	+ 9·2956955	− 8·9065270	− 8·9071761
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
	FOR Y					
(a) 5°	8·3707007	+ 8·3605995	+ 8·3186569	+ 8·3027836	+ 8·1149814	+ 8·0933361
(b) 10	8·9691515	+ 8·9592822	+ 8·9056313	+ 8·8901224	+ 8·6852275	+ 8·6640790
(c) 15	9·3153177	+ 9·3058267	+ 9·2324041	+ 9·2174897	+ 8·9832050	+ 8·9628672
(d) 20	9·5567697	+ 9·5477918	+ 9·4461190	+ 9·4320109	+ 9·1544613	+ 9·1352229
(e) 25	9·7397514	+ 9·7314053	+ 9·5923590	+ 9·5792437	+ 9·2417747	+ 9·2238901
(f) 30	9·8848559	+ 9·8772407	+ 9·6907409	+ 9·6787741	+ 9·2597485	+ 9·2434302
(g) 35	0·0030611	+ 9·9962533	+ 9·7508111	+ 9·7401132	+ 9·2081735	+ 9·1935854
(h) 40	0·1009092	+ 0·0949606	+ 9·7769755	+ 9·7676277	+ 9·0690158	+ 9·0562688
(i) 45	0·1826071	+ 0·1775430	+ 9·7701241	+ 9·7621663	+ 8·7700367	+ 8·7591851
(k) 50	0·2510096	+ 0·2468285	+ 9·7276156	+ 9·7210454	− 6·4989114	− 6·4899520
(l) 55	0·3081314	+ 0·3048051	+ 9·6413765	+ 9·6361494	− 8·6754995	− 8·6683716
(m) 60	0·3554335	+ 0·3529078	+ 9·4909249	+ 9·4869560	− 8·8619670	− 8·8565548
(n) 65	0·3939938	+ 0·3921907	+ 9·2110062	+ 9·2081727	− 8·8625805	− 8·8587167
(o) 70	0·4246124	+ 0·4234322	+ 8·0772250	+ 8·0753704	− 8·7043078	− 8·7017787
(p) 75	0·4478786	+ 0·4472031	− 9·1008603	− 9·0997988	− 8·1623847	− 8·1609372
(q) 80	0·4642156	+ 0·4639116	− 9·3746419	− 9·3741642	+ 8·3743516	+ 8·3737002
(r) 85	0·4739083	+ 0·4738317	− 9·4893541	− 9·4892338	+ 8·7188621	+ 8·7186981
(s) 90	0·4771213	+ 0·4771213	− 9·5228788	− 9·5228788	+ 8·7989065	+ 8·7989065
	FOR Z					
(a) 5°	7·4409734	− 7·3008955	+ 7·5642963	− 7·4660951	+ 7·4851941	− 7·4032779
(b) 10	8·3386831	− 8·1989524	+ 8·4505211	− 8·3528116	+ 8·3546773	− 8·2734102
(c) 15	8·8579869	− 8·7188229	+ 8·9504172	− 8·8535109	+ 8·8257545	− 8·7455519
(d) 20	9·2202391	− 9·0818434	+ 9·2849121	− 9·1890969	+ 9·1177535	− 9·0390074
(e) 25	9·4948029	− 9·3573534	+ 9·5227077	− 9·4282393	+ 9·2965656	− 9·2196399
(f) 30	9·7125651	− 9·5762107	+ 9·6937141	− 9·6008098	+ 9·3870736	− 9·3123119
(g) 35	9·8899892	− 9·7548445	+ 9·8129617	− 9·7217957	+ 9·3945200	− 9·3222795
(h) 40	0·0368854	− 9·9030279	+ 9·8881222	− 9·7988236	+ 9·3040356	− 9·2348295
(i) 45	0·1595597	− 0·0270280	+ 9·9221795	− 9·8348345	+ 9·0449585	− 8·9803443
(k) 50	0·2622906	− 0·1310824	+ 9·9139052	− 9·8285689	− 7·0718319	+ 5·4914727
(l) 55	0·3480963	− 0·2181695	+ 9·8562035	− 9·7729337	− 9·0174497	+ 8·9487508
(m) 60	0·4191649	− 0·2904384	+ 9·7292405	− 9·6482142	− 9·2266000	+ 9·1623125
(n) 65	0·4771094	− 0·3494664	+ 9·4679243	− 9·3900847	− 9·2461422	+ 9·1847103
(o) 70	0·5231268	− 0·3964179	+ 8·3290419	− 8·2961271	− 9·1028223	+ 9·0439651
(p) 75	0·5580988	− 0·4321467	− 9·3884573	+ 9·3050617	− 8·5706889	+ 8·5172302
(q) 80	0·5826577	− 0·4572631	− 9·6694986	+ 9·5889845	+ 8·7948317	− 8·7338857
(r) 85	0·5972295	− 0·4721759	− 9·7888309	+ 9·7093516	+ 9·1433430	− 9·0848739
(s) 90	0·6020600	− 0·4771213	− 9·8239088	+ 9·7447275	+ 9·2248752	− 9·1668832

g_9^3 or h_9^3	g_{-9}^3 or h_{-9}^3	Absolute term for g, $\frac{1}{2}(a_3 - a'_3)$ 1845	Absolute term for g, $\frac{1}{2}(a_3 - a'_3)$ 1880	Absolute term for h, $\frac{1}{2}(b_3 - b'_3)$ 1845	Absolute term for h, $\frac{1}{2}(b_3 - b'_3)$ 1880	
− 7·8118088	− 7·7845364					(a)
− 8·3173677	− 8·2912066					(b)
− 8·4885689	− 8·4644761	− ·0206		·00875		(c)
− 8·4118101	− 8·3917871	− ·00915		·0443		(d)
− 7·5939222	− 7·6111787	+ ·0228	·0386	·0753	·0447	(e)
+ 8·4089972	+ 8·3819381	·01895	·0643	·1136	·0462	(f)
+ 8·6760941	+ 8·6557079	·0200	·0816	·1189	·0634	(g)
+ 8·6879184	+ 8·6722768	·0064	·0956	·12015	·0784	(h)
+ 8·4382845	+ 8·4289194	·03605	·0922	·1290	·0967	(i)
− 7·7949506	− 7·7597401	·0334	·1001	·1343	·1063	(k)
− 8·5433692	− 8·5317874	− ·0069	·0894	·17955	·1177	(l)
− 8·6415186	− 8·6347575	− ·01245	·0734	·20405	·1382	(m)
− 8·4568860	− 8·4545506	− ·0001	·0658	·2097	·1533	(n)
+ 7·2328906	+ 7·1780151	+ ·01505	·0510	·18555	·1420	(o)
+ 8·4804740	+ 8·4771634	·04055	·0249	·13805	·1231	(p)
+ 8·6146940	+ 8·6138750	·03195	·0184	·09445	·0841	(q)
+ 8·4578948	+ 8·4582224	·00275	·0006	·04715	·0236	(r)
0·0000000	0·0000000					(s)

		Absolute term for g, $\frac{1}{2}(b_3 + b'_3)$ 1845	Absolute term for g, $\frac{1}{2}(b_3 + b'_3)$ 1880	Absolute term for h, $-\frac{1}{2}(a_3 + a'_3)$ 1845	Absolute term for h, $-\frac{1}{2}(a_3 + a'_3)$ 1880	
+ 7·8247300	+ 7·7973126					(a)
+ 8·3727870	+ 8·3459989					(b)
+ 8·6315145	+ 8·6057533	− ·0477		− ·0526		(c)
+ 8·7419914	+ 8·7176228	− ·04765		− ·06375		(d)
+ 8·7373444	+ 8·7146906	− ·0420	− ·0941	− ·0946	− ·0854	(e)
+ 8·6076087	+ 8·5869388	− ·0444	− ·1119	− ·1349	− ·1265	(f)
+ 8·2574531	+ 8·2389749	− ·0488	− ·1073	− ·18735	− ·1484	(g)
− 7·6236261	− 7·6074798	− ·06365	− ·1160	− ·21205	− ·1651	(h)
− 8·2668016	− 8·2530563	− ·0659	− ·1294	− ·2416	− ·1813	(i)
− 8·3179830	− 8·3066344	− ·0807	− ·1418	− ·2649	− ·1994	(k)
− 8·0981311	− 8·0891025	− ·0928	− ·1549	− ·2801	− ·2156	(l)
+ 6·7775540	+ 6·7706985	− ·0941	− ·1688	− ·31075	− ·2341	(m)
+ 8·0608389	+ 8·0559447	− ·09755	− ·1833	− ·3424	− ·2575	(n)
+ 8·1760398	+ 8·1728363	− ·1051	− ·1980	− ·38235	− ·2947	(o)
+ 8·0020251	+ 8·0001915	− ·1057	− ·2125	− ·4144	− ·3315	(p)
− 5·9069560	− 5·9061309	− ·10205	− ·2281	− ·4497	− ·3570	(q)
− 7·9860309	− 7·9858231	− ·11255	− ·2396	− ·4592	− ·3767	(r)
− 8·1327290	− 8·1327290	− ·1150	− ·2446	− ·4522	− ·3852	(s)

		Absolute term for g, $\frac{1}{2}(a_3 + a'_3)$ 1845	Absolute term for g, $\frac{1}{2}(a_3 + a'_3)$ 1880	Absolute term for h, $\frac{1}{2}(b_3 + b'_3)$ 1845	Absolute term for h, $\frac{1}{2}(b_3 + b'_3)$ 1880	
+ 7·2916310	− 7·2166802					(a)
+ 8·1389094	− 8·0647711					(b)
+ 8·5707063	− 8·4979090	− ·0688		− ·0885		(c)
+ 8·8018708	− 8·7309339	+ ·0005		− ·1157		(d)
+ 8·8886155	− 8·8200787	·01885	− ·0582	− ·1253	− ·1114	(e)
+ 8·8311599	− 8·7657310	− ·0124	− ·0382	− ·1204	− ·1008	(f)
+ 8·5390006	− 8·4786980	− ·0253	− ·0385	− ·13315	− ·1064	(g)
− 7·9639331	+ 7·8856942	− ·0884	− ·0538	− ·1674	− ·1118	(h)
− 8·6415287	+ 8·5800888	− ·0935	− ·0703	− ·18585	− ·1298	(i)
− 8·7261473	+ 8·6693852	− ·07945	− ·0665	− ·2134	− ·1924	(k)
− 8·5339363	+ 8·4820760	− ·1155	− ·1078	− ·32525	− ·2154	(l)
+ 7·2693662	− 7·1492293	− ·19265	− ·1427	− ·38395	− ·2192	(m)
+ 8·5424251	− 8·4898360	− ·18555	− ·1518	− ·45875	− ·2565	(n)
+ 8·6722096	− 8·6233160	− ·1768	− ·1718	− ·5166	− ·3190	(o)
+ 8·5099729	− 8·4632001	− ·14995	− ·1691	− ·5194	− ·3744	(p)
− 6·4941956	+ 6·2813483	− ·18135	− ·1850	− ·54805	− ·3984	(q)
− 8·5075004	+ 8·4610614	− ·18635	− ·2187	− ·55125	− ·4369	(r)
− 8·6556077	+ 8·6098502	− ·2045	− ·2435	− ·5638	− ·4832	(s)

For X

Co-latitude	g_4^3 or h_4^3	g_{-4}^3 or h_{-4}^3	g_6^3 or h_6^3	g_{-6}^3 or h_{-6}^3	g_8^3 or h_8^3	g_{-8}^3 or h_{-8}^3
(a) 5°	− 8·3676258	− 8·3547057	− 8·2244562	− 8·2057946	− 7·9680587	− 7·9436563
(b) 10	− 8·9523526	− 8·9399320	− 8·7851451	− 8·7672161	− 8·4947575	− 8·4713349
(c) 15	− 9·2751686	− 9·2635707	− 9·0652414	− 9·0485550	− 8·7098106	− 8·6881393
(d) 20	− 9·4828544	− 9·4723899	− 9·2062769	− 9·1914061	− 8·7320993	− 8·7133828
(e) 25	− 9·6203704	− 9·6113369	− 9·2418525	− 9·2295686	− 8·5010740	− 8·4893580
(f) 30	− 9·7062774	− 9·6989660	− 9·1615205	− 9·1533444	+ 8·0256553	+ 7·9747530
(g) 35	− 9·7482643	− 9·7429804	− 8·8580944	− 8·8606937	+ 8·7555205	+ 8·7338758
(h) 40	− 9·7472311	− 9·7443505	+ 8·5003017	+ 8·4493174	+ 8·9413287	+ 8·9252194
(i) 45	− 9·6966506	− 9·6967779	+ 9·1574047	+ 9·1404202	+ 8·9411115	+ 8·9293572
(k) 50	− 9·5749428	− 9·5794834	+ 9·3750786	+ 9·3641441	+ 8·7397999	+ 8·7333213
(l) 55	− 9·3039385	− 9·3186429	+ 9·4596046	+ 9·4523795	+ 7·3798377	+ 7·4662029
(m) 60	8·0897391	+ 7·5756172	+ 9·4499491	+ 9·4457366	− 8·6897767	− 8·6797913
(n) 65	9·3917554	+ 9·3764540	+ 9·3360172	+ 9·3346843	− 8·8916009	− 8·8864829
(o) 70	9·6808772	+ 9·6743723	+ 9·0244559	+ 9·0273406	− 8·8613683	− 8·8593197
(p) 75	9·8385884	+ 9·8355151	− 8·4327266	− 8·4144280	− 8·5570464	− 8·5593500
(q) 80	9·9325627	+ 9·9313113	− 9·1820409	− 9·1797738	+ 8·1941408	+ 8·1882149
(r) 85	9·9836887	+ 9·9833892	− 9·3815182	− 9·3810672	+ 8·7761483	+ 8·7755206
(s) 90	0·0000000	+ 0·0000000	− 9·4357286	− 9·4357286	+ 8·8860566	+ 8·8860566

For Y

Co-latitude	g_4^3 or h_4^3	g_{-4}^3 or h_{-4}^3	g_6^3 or h_6^3	g_{-6}^3 or h_{-6}^3	g_8^3 or h_8^3	g_{-8}^3 or h_{-8}^3
(a) 5°	8·3704657	+ 8·3574785	+ 8·2304278	+ 8·2116685	+ 7·9783541	+ 7·9538227
(b) 10	8·9638246	+ 8·9511355	+ 8·8097057	+ 8·7913770	+ 8·5382045	+ 8·5142362
(c) 15	9·3014212	+ 9·2892185	+ 9·1233105	+ 9·1056844	+ 8·8179497	+ 8·7949002
(d) 20	9·5306958	+ 9·5191528	+ 9·3178101	+ 9·3011368	+ 8·9614304	+ 8·9396269
(e) 25	9·6976971	+ 9·6869664	+ 9·4378254	+ 9·4223254	+ 9·0080171	+ 8·9877479
(f) 30	9·8227437	+ 9·8129527	+ 9·5013871	+ 9·4872445	+ 8·9648595	+ 8·9463654
(g) 35	9·9164370	+ 9·9076842	+ 9·5153300	+ 9·5026870	+ 8·8115577	+ 8·7950245
(h) 40	9·9848066	+ 9·9771584	+ 9·4790511	+ 9·4680037	+ 8·4420100	+ 8·4275634
(i) 45	0·0313565	+ 0·0248456	+ 9·3829045	+ 9·3734998	− 7·9101887	− 7·8978903
(k) 50	0·0579630	+ 0·0525874	+ 9·1965273	+ 9·1887625	− 8·5124561	− 8·5023021
(l) 55	0·0652420	+ 0·0609653	+ 8·7930244	+ 8·7868469	− 8·6003392	− 8·5922609
(m) 60	0·0525794	+ 0·0493321	− 8·4552219	− 8·4505313	− 8·4824877	− 8·4763539
(n) 65	0·0178079	+ 0·0154896	− 9·0009448	− 8·9975961	− 8·0028333	− 7·9984543
(o) 70	9·9562625	+ 9·9547451	− 9·1528473	− 9·1506554	+ 8·0781368	+ 8·0752705
(p) 75	9·8582566	+ 9·8573881	− 9·1735210	− 9·1722665	+ 8·4291369	+ 8·4274964
(q) 80	9·7011080	+ 9·7007172	− 9·0867351	− 9·0861706	+ 8·4644544	+ 8·4637162
(r) 85	9·4113288	+ 9·4112304	− 8·8349735	− 8·8348313	+ 8·2685974	+ 8·2684115
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

For Z

Co-latitude	g_4^3 or h_4^3	g_{-4}^3 or h_{-4}^3	g_6^3 or h_6^3	g_{-6}^3 or h_{-6}^3	g_8^3 or h_8^3	g_{-8}^3 or h_{-8}^3
(a) 5°	7·5372144	− 7·4234420	+ 7·5428056	− 7·4544538	+ 7·3995964	− 7·3219144
(b) 10	8·4298288	− 8·3164733	+ 8·4213277	− 8·3335453	+ 8·2586764	− 8·1817246
(c) 15	8·9405575	− 8·8278823	+ 8·9080449	− 8·8211944	+ 8·7115070	− 8·6357561
(d) 20	9·2906248	− 9·1788721	+ 9·2233076	− 9·1377265	+ 8·9757056	− 8·9016073
(e) 25	9·5491987	− 9·4385823	+ 9·4348544	− 9·3508476	+ 9·1137466	− 9·0417397
(f) 30	9·7468922	− 9·6375903	+ 9·5710050	− 9·4888417	+ 9·1430339	− 9·0735841
(g) 35	9·8997925	− 9·7919431	+ 9·6440714	− 9·5639867	+ 9·0485233	− 8·9823066
(h) 40	0·0171974	− 9·9108935	+ 9·6566998	− 9·5789118	+ 8·7265598	− 8·6659150
(i) 45	0·1047107	− 9·9999983	+ 9·6012950	− 9·5260653	− 8·2455217	+ 8·1664523
(k) 50	0·1656323	− 0·0625088	+ 9·4487511	− 9·3766386	− 8·8760776	+ 8·8125330
(l) 55	0·2015828	− 0·0999974	+ 9·0720011	− 9·0060276	− 8·9918129	+ 8·9325022
(m) 60	0·2126746	− 0·1125304	− 8·7675480	+ 8·6825904	− 8·8970855	+ 8·8414459
(n) 65	0·1972764	− 0·0984329	− 9·3278495	+ 9·2550867	− 8·4345759	+ 8·3857521
(o) 70	0·1511206	− 0·0533985	− 9·4944651	+ 9·4246370	+ 8·5305845	− 8·4722973
(p) 75	0·0648129	− 9·9679995	− 9·5265774	+ 9·4584999	+ 8·8917317	− 8·8382218
(q) 80	9·9158807	− 9·8197363	− 9·4478819	+ 9·3809515	+ 8·9349244	− 8·8832238
(r) 85	9·6309771	− 9·5352422	− 9·2009352	+ 9·1346717	+ 8·7438138	− 8·6930507
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

$g_{10}{}^3$ or $h_{10}{}^3$	$g_{-10}{}^3$ or $h_{-10}{}^3$	Absolute term for g, $\frac{1}{2}(a_3+a'_3)$ 1845	Absolute term for g, $\frac{1}{2}(a_3+a'_3)$ 1880	Absolute term for h, $\frac{1}{2}(b_3+b'_3)$ 1845	Absolute term for h, $\frac{1}{2}(b_3+b'_3)$ 1880	
− 7·6412654	− 7·6111233					(*a*)
− 8·1226153	− 8·0937229					(*b*)
− 8·2395477	− 8·2131153	− ·0383		− ·01755		(*c*)
− 8·0008540	− 7·9809521	+ ·00875		− ·0450		(*d*)
+ 7·8350524	+ 7·7968305	− ·0079	− ·0795	− ·0574	− ·0393	(*e*)
+ 8·3667320	+ 8·3411210	− ·05645	− ·0737	− ·0746	− ·0266	(*f*)
+ 8·4487429	+ 8·4282587	− ·0846	− ·0640	− ·0878	− ·0197	(*g*)
+ 8·2345925	+ 8·2202809	− ·0791	− ·0435	− ·08195	− ·0169	(*h*)
− 7·5155347	− 7·4748654	− ·04725	− ·0247	− ·0737	− ·0047	(*i*)
− 8·3090150	− 8·2941767	− ·0217	·0058	− ·0673	·0195	(*k*)
− 8·3677359	− 8·3583401	+ ·0269	·0161	− ·04065	·0591	(*l*)
− 8·0301276	− 8·0275645	·05595	·0217	− 0·2125	·0745	(*m*)
+ 7·9300207	+ 7·9186947	·0360	·0196	− ·0165	·0690	(*n*)
+ 8·3266818	+ 8·3224375	·02805	·0246	+ ·01945	·0841	(*o*)
+ 8·2774422	+ 8·2762964	·03385	·0350	·03195	·0909	(*p*)
+ 7·5548805	+ 7·5601274	·03455	·0424	·02605	·0818	(*q*)
− 8·1471881	− 8·1463357	·02545	·0679	·03015	·0661	(*r*)
− 8·3358955	− 8·3358955	·0217	·0722	·0060	·0680	(*s*)

		Absolute term for g, $\frac{1}{2}(b_3-b'_3)$ 1845	Absolute term for g, $\frac{1}{2}(b_3-b'_3)$ 1880	Absolute term for h, $-\frac{1}{2}(a_3-a'_3)$ 1845	Absolute term for h, $-\frac{1}{2}(a_3-a'_3)$ 1880	
+ 7·6571332	+ 7·6268297					(*a*)
+ 8·1919602	+ 8·1623523					(*b*)
+ 8·4266570	+ 8·3981841	·1062		− ·0016		(*c*)
+ 8·4979275	+ 8·4709938	·13005		+ ·01775		(*d*)
+ 8·4272737	+ 8·4022353	·1328	·1793	·0153	·0310	(*e*)
+ 8·1599738	+ 8·1371281	·1514	·2137	·0426	·0793	(*f*)
+ 6·4918845	+ 6·4714612	·1697	·2364	·07185	·1110	(*g*)
− 7·9735763	− 7·9557304	·17385	·2347	·10225	·1228	(*h*)
− 8·0503630	− 8·0351708	·1754	·2220	·1141	·1259	(*i*)
− 7·7824079	− 7·7698647	·1674	·1976	·1211	·1151	(*k*)
+ 7·2393411	+ 7·2293621	·1339	·1655	·1004	·1074	(*l*)
+ 7·8500488	+ 7·8424717	·0951	·1299	·09715	·1010	(*m*)
+ 7·8503065	+ 7·8448971	·06415	·0988	·0836	·0933	(*n*)
+ 7·3856922	+ 7·3821515	·0358	·0711	·06955	·0827	(*o*)
− 7·5322358	− 7·5302092	·0206	·0569	·0442	·0576	(*p*)
− 7·8136222	− 7·8127103	·00615	·0487	·0170	·0289	(*q*)
− 7·6968573	− 7·6966276	·00625	·0243	·0071	·0134	(*r*)
0·0000000	0·0000000					(*s*)

		Absolute term for g, $\frac{1}{2}(a_3-a'_3)$ 1845	Absolute term for g, $\frac{1}{2}(a_3-a'_3)$ 1880	Absolute term for h, $\frac{1}{2}(b_3-b'_3)$ 1845	Absolute term for h, $\frac{1}{2}(b_3-b'_3)$ 1880	
+ 7·1653454	− 7·0920542					(*a*)
+ 7·9993853	− 7·9269898					(*b*)
+ 8·4071335	− 8·3362230	·0308		·0133		(*c*)
+ 8·5990542	− 8·5302269	·1361		·0636		(*d*)
+ 8·6196975	− 8·5536453	·19265	·3207	·1201	·0423	(*e*)
+ 8·4242912	− 8·3623592	·2151	·3252	·1490	·0955	(*f*)
+ 6·7537082	− 6·8150073	·1827	·3338	·17615	·1357	(*g*)
− 8·3487227	+ 8·2871543	·1923	·3146	·2004	·1657	(*h*)
− 8·4654079	+ 8·4091555	·1649	·3049	·20005	·1767	(*i*)
− 8·2304207	+ 8·1800620	·12775	·2824	·1557	·1631	(*k*)
+ 7·7239834	− 7·6597951	·1142	·2221	·15945	·1618	(*l*)
+ 8·3531740	− 8·3029399	·04955	·1627	·13235	·1315	(*m*)
+ 8·3719700	− 8·3260049	·04525	·0902	·09835	·1021	(*n*)
+ 7·9209219	− 7·8809397	·0555	·0119	·0832	·0703	(*o*)
− 8·0827817	+ 8·0369698	·02025	− ·0209	·0285	·0805	(*p*)
− 8·3714081	+ 8·3289434	− ·03035	− ·0151	·03215	·1134	(*q*)
− 8·2593006	+ 8·2180860	− ·02025	·0004	·01215	·0987	(*r*)
0·0000000	0·0000000					(*s*)

Co-latitude	For X g_4^4 or h_4^4	g_{-4}^4 or h_{-4}^4	g_6^4 or h_6^4	g_{-6}^4 or h_{-6}^4	g_8^4 or h_8^4	g_{-8}^4 or h_{-8}^4
(a) 5°	− 7·4385585	− 7·4256215	− 7·3944891	− 7·3758027	− 7·2203028	− 7·1958672
(b) 10	− 8·3311426	− 8·3186528	− 8·2700616	− 8·2520259	− 8·0722328	− 8·0486572
(c) 15	− 8·8418220	− 8·8300618	− 8·7514810	− 8·7345210	− 8·5114510	− 8·4893275
(d) 20	− 9·1918219	− 9·1810516	− 9·0584036	− 9·0429367	− 8·7519192	− 8·7319030
(e) 25	− 9·4503131	− 9·4407617	− 9·2572767	− 9·2437212	− 8·8470905	− 8·8300571
(f) 30	− 9·6479108	− 9·6397698	− 9·3739435	− 9·3627519	− 8·7861595	− 8·7739318
(g) 35	− 9·8007049	− 9·7941225	− 9·4154934	− 9·4072515	− 8·3868531	− 8·3907436
(h) 40	− 9·9179967	− 9·9130729	− 9·3715339	− 9·3673042	+ 8·5025313	+ 8·4711677
(i) 45	− 0·0053934	− 0·0021778	− 9·1895880	− 9·1927279	+ 8·9466995	+ 8·9298131
(k) 50	− 0·0661984	− 0·0646884	− 8·4076406	− 8·4747679	+ 9·0902340	+ 9·0787203
(l) 55	− 0·1020358	− 0·1021770	+ 9·1212597	+ 9·1003761	+ 9·0776099	+ 9·0703088
(m) 60	− 0·1130214	− 0·1147098	+ 9·4609760	+ 9·4512088	+ 8·8794947	+ 8·8769474
(n) 65	− 0·0975275	− 0·1006125	+ 9·6152864	+ 9·6098864	+ 7·7591398	+ 7·8056666
(o) 70	− 0·0512887	− 0·0555779	+ 9·6757919	+ 9·6731044	− 8·8139857	− 8·8077656
(p) 75	− 9·9649139	− 9·9701790	+ 9·6599644	+ 9·6591556	− 9·0420110	− 9·0397148
(q) 80	− 9·8159321	− 9·8219157	+ 9·5564183	+ 9·5568836	− 9·0487503	− 9·0482681
(r) 85	− 9·5309983	− 9·5374217	+ 9·2971030	+ 9·2983154	− 8·8428462	− 8·8434129
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
	For Y					
(a) 5°	7·4402645	+ 7·4272773	+ 7·3980648	+ 7·3793055	+ 7·2264456	+ 7·2019142
(b) 10	8·3379904	+ 8·3253013	+ 8·2845733	+ 8·2662446	+ 8·0975508	+ 8·0735825
(c) 15	8·8573208	+ 8·8451181	+ 8·7849635	+ 8·7673374	+ 8·5715547	+ 8·5485052
(d) 20	9·2196087	+ 9·2080657	+ 9·1201980	+ 9·1035247	+ 8·8681979	+ 8·8463944
(e) 25	9·4942167	+ 9·4834860	+ 9·3590376	+ 9·3435376	+ 9·0541604	+ 9·0338912
(f) 30	9·7120302	+ 9·7022392	+ 9·5314878	+ 9·5173452	+ 9·1559032	+ 9·1374091
(g) 35	9·8895108	+ 9·8807580	+ 9·6527352	+ 9·6400922	+ 9·1822138	+ 9·1656806
(h) 40	0·0364672	+ 0·0288190	+ 9·7307174	+ 9·7196700	+ 9·1280479	+ 9·1136013
(i) 45	0·1592036	+ 0·1526927	+ 9·7689011	+ 9·7594964	+ 8·9640969	+ 8·9517985
(k) 50	0·2619965	+ 0·2566209	+ 9·7670133	+ 9·7592485	+ 8·5534436	+ 8·5432896
(l) 55	0·3478623	+ 0·3435856	+ 9·7201224	+ 9·7139449	− 8·2907425	− 8·2826642
(m) 60	0·4189871	+ 0·4157398	+ 9·6144672	+ 9·6097766	− 8·7698905	− 8·7637567
(n) 65	0·4769825	+ 0·4746642	+ 9·4106842	+ 9·4073355	− 8·8594926	− 8·8551136
(o) 70	0·5230437	+ 0·5215263	+ 8·9159425	+ 8·9137506	− 8·7665625	− 8·7636962
(p) 75	0·5580512	+ 0·5571827	− 8·9518914	− 8·9506369	− 8·3845639	− 8·3829234
(q) 80	0·5826363	+ 0·5822455	− 9·3690813	− 9·3685168	+ 8·2227843	+ 8·2220461
(r) 85	0·5972241	+ 0·5971257	− 9·5184813	− 9·5183391	+ 8·6921755	+ 8·6919896
(s) 90	0·6020600	+ 0·6020600	− 9·5606673	− 9·5606673	+ 8·7891466	+ 8·7891466
	For Z					
(a) 5°	6·4826516	− 6·3675733	+ 6·5859184	− 6·4966650	+ 6·5230738	− 6·4447002
(b) 10	7·6796331	− 7·5649715	+ 7·7716767	− 7·6829807	+ 7·6934191	− 7·6157533
(c) 15	8·3720959	− 8·2581143	+ 8·4451894	− 8·3574036	+ 8·3405285	− 8·2640210
(d) 20	8·8551768	− 8·7421174	+ 8·9012028	− 8·8146525	+ 8·7579245	− 8·6829957
(e) 25	9·2213579	− 9·1094342	+ 9·2315974	− 9·1465707	+ 9·0354030	− 8·9624347
(f) 30	9·5118188	− 9·4012092	+ 9·4766723	− 9·3934121	+ 9·2097111	− 9·1390448
(g) 35	9·7485069	− 9·6393492	+ 9·6570990	− 9·5757971	+ 9·2951060	− 9·2270518
(h) 40	9·9444992	− 9·8368864	+ 9·7840819	− 9·7048749	+ 9·2897684	− 9·2246559
(i) 45	0·1081995	− 0·0021777	+ 9·8631843	− 9·7861537	+ 9·1663401	− 9·1047383
(k) 50	0·2453083	− 0·1408747	+ 9·8955512	− 9·8207285	+ 8·7882198	− 8·7329575
(l) 55	0·3598459	− 0·2569501	+ 9·8772404	− 9·8046251	− 8·5617442	+ 8·4928852
(m) 60	0·4547258	− 0·3532705	+ 9·7951791	− 9·7247940	− 9·0611572	+ 9·0021012
(n) 65	0·5320951	− 0·4319399	+ 9·6103933	− 9·5425206	− 9·1695121	+ 9·1138827
(o) 70	0·5935463	− 0·4945121	+ 9·1290345	− 9·0669476	− 9·0915783	+ 9·0385996
(p) 75	0·6402523	− 0·5421264	− 9·1822509	+ 9·1093021	− 8·7203666	+ 8·6707901
(q) 80	0·6730541	− 0·5755970	− 9·6060007	+ 9·5375288	+ 8·5699016	− 8·3138333
(r) 85	0·6925177	− 0·5954699	− 9·7599722	+ 9·6926952	+ 9·0428830	− 8·9912067
(s) 90	0·6989700	− 0·6020600	− 9·8037053	+ 9·7367586	+ 9·1413291	− 9·0901766

g_{10}^4 or h_{10}^4	g_{-10}^4 or h_{-10}^4	Absolute term for g, $\frac{1}{2}(a_4 - a'_4)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_4 - b'_4)$ 1845	1880	
− 6·9643686	− 6·9341844					(a)
− 7·7855280	− 7·7564214					(b)
− 8·1666742	− 8·1394491	·02605		− ·0087		(c)
− 8·3040333	− 8·2798057	− ·00115		·0076		(d)
− 8·1832311	− 8·1649364	− ·0134	·0055	− ·0032	·0170	(e)
+ 7·1309340	+ 6·9945675	·0113	·0221	− ·01355	− ·0199	(f)
+ 8·3481165	+ 8·3224409	·0402	·0194	− ·00195	− ·0357	(g)
+ 8·5463679	+ 8·5272926	·05005	·0178	·01115	− ·0450	(h)
+ 8·4888892	+ 8·4752642	·04985	·0152	·0142	− ·0529	(i)
+ 8·0031033	+ 8·0007540	·0366	·0236	− ·0070	− ·0442	(k)
− 8·1942142	− 8·1783447	·02755	·0320	− ·00515	− ·0448	(l)
− 8·4924011	− 8·4839418	·0405	·0461	− ·0062	− ·0468	(m)
− 8·4301847	− 8·4261884	·0539	·0399	− ·01015	− ·0362	(n)
− 7·7812825	− 7·7878177	·0368	·0382	·0143	− ·0275	(o)
+ 8·2590445	+ 8·2548458	·0438	·0383	·01795	− ·0099	(p)
+ 8·4797062	+ 8·4786715	·0141	·0591	− ·01345	·0032	(q)
+ 8·3503121	+ 8·3505693	·02635	·0363	·0048	·0014	(r)
0·0000000	0·0000000					(s)

		Absolute term for g, $\frac{1}{2}(b_4 + b'_4)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_4 + a'_4)$ 1845	1880	
+ 6·9737967	+ 6·9434932					(a)
+ 7·8251858	+ 7·7955779					(b)
+ 8·2647289	+ 8·2362560	·0217		− ·00135		(c)
+ 8·5091644	+ 8·4822307	− ·0094		− ·00285		(d)
+ 8·6192131	+ 8·5941747	− ·00885	− ·0130	·01855	·0177	(e)
+ 8·6083046	+ 8·5854589	·01105	·0054	·04145	·0433	(f)
+ 8·4490398	+ 8·4286165	·0119	·0120	·0692	·0628	(g)
+ 7·9492836	+ 7·9314377	·0123	·0147	·0790	·0758	(h)
− 7·9510969	− 7·9359047	·0222	·0184	·08305	·0888	(i)
− 8·2552258	− 8·2426826	·02855	·0192	·0680	·0957	(k)
− 8·1918476	− 8·1818686	·02795	·0086	·0509	·1058	(l)
− 7·6797175	− 7·6721404	·03725	− ·0040	·0613	·1077	(m)
+ 7·8571803	+ 7·8517709	·03355	− ·0228	·0735	·0952	(n)
+ 8·1265602	+ 8·1230195	·0326	− ·0392	·0809	·0890	(o)
+ 8·0291877	+ 8·0271611	·0161	− ·0678	·0757	·0882	(p)
+ 7·1600439	+ 7·1591320	− ·01125	− ·0935	·07555	·0894	(q)
− 7·9185981	− 7·9183684	− ·0224	− ·1089	·07905	·1016	(r)
− 8·0928575	− 8·0928575	− ·0292	− ·1083	·0800	·1076	(s)

		Absolute term for g, $\frac{1}{2}(a_4 + a'_4)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_4 + b'_4)$ 1845	1880	
+ 6·3573378	− 6·2834833					(a)
+ 7·5079555	− 7·4349639					(b)
+ 8·1205823	− 8·0490080	·0016		·04125		(c)
+ 8·4857331	− 8·4161053	− ·02675		·0197		(d)
+ 8·6872314	− 8·6200592	·0176	·0347	·01605	·0146	(e)
+ 8·7487564	− 8·6845742	·08845	·0348	− ·00065	− ·0090	(f)
+ 8·6482282	− 8·5878403	·0891	·0344	·00585	− ·0176	(g)
+ 8·1951833	− 8·1426380	·1012	·0254	·00825	− ·0058	(h)
− 8·2439299	+ 8·1819336	·09555	·0076	·0423	·0165	(i)
− 8·5803907	+ 8·5256487	·0883	·0096	·0572	·0410	(k)
− 8·5448912	+ 8·4947869	·0331	·0085	·1484	·0741	(l)
− 8·0538686	+ 8·0120788	·00995	− ·0317	·16135	·0601	(m)
+ 8·2559701	− 8·2056616	·02365	− ·0681	·1273	·0690	(n)
+ 8·5393807	− 8·4941013	·02125	− ·0987	·10505	·0653	(o)
+ 8·4536110	− 8·4107458	·0431	− ·1411	·0756	·0436	(p)
+ 7·5889774	− 7·5546927	·00465	− ·1615	·03255	·0359	(q)
− 8·3565452	+ 8·3144013	− ·00095	− ·1853	·0787	·0457	(r)
− 8·5321902	+ 8·4907975	− ·0161	− ·1822	·0779	·0275	(s)

FOR X

Co-latitude	g_5^4 or h_5^4	g_{-5}^4 or h_{-5}^4	g_7^4 or h_7^4	g_{-7}^4 or h_{-7}^4	g_9^4 or h_9^4
(a) 5°	− 7·4374749	− 7·4216630	− 7·3198333	− 7·2982722	− 7·1006576
(b) 10	− 8·3223578	− 8·3070944	− 8·1844402	− 8·1636336	− 7·9381321
(c) 15	− 8·8198978	− 8·8055402	− 8·6465221	− 8·6269746	− 8·3505523
(d) 20	− 9·1508168	− 9·1376859	− 8·9237042	− 8·9059367	− 8·5454395
(e) 25	− 9·3834705	− 9·3718806	− 9·0782940	− 9·0628928	− 8 5566235
(f) 30	− 9·5471920	− 9·5374280	− 9·1267947	− 9·1145742	− 8·2686840
(g) 35	− 9·6560255	− 9·6483567	− 9·0497473	− 9·0424514	+ 8·1430638
(h) 40	− 9·7158571	− 9·7105625	− 8·7064872	− 8·7133859	+ 8·6809061
(i) 45	− 9·7261305	− 9·7235679	+ 8·5974008	+ 8·5591398	+ 8·8231244
(k) 50	− 9·6775383	− 9·6783507	+ 9 1359983	+ 9·1206448	+ 8·7660492
(l) 55	− 9·5396195	− 9·5455740	+ 9·3276475	+ 9·3181103	+ 8·3962232
(m) 60	− 9·1789945	− 9·2002303	+ 9·3846564	+ 9·3788747	− 8·2810001
(n) 65	+ 8·9636973	+ 8·9210614	+ 9·3292384	+ 9·3265841	− 8·7219088
(o) 70	+ 9·5506745	+ 9·5410554	+ 9·1125004	+ 9·1132759	− 8·7743571
(p) 75	+ 9·7832716	+ 9·7792491	+ 8·0650659	+ 8·0969419	− 8·5583042
(q) 80	+ 9·9112331	+ 9·9096667	− 9·0362590	− 9·0333665	+ 7·7185211
(r) 85	+ 9·9787205	+ 9·9783521	− 9·2964777	− 9·2959502	+ 8·6379467
(s) 90	+ 0·0000000	+ 0·0000000	− 9·3631779	− 9·3631779	+ 8·7695511

FOR Y

Co-latitude	g_5^4 or h_5^4	g_{-5}^4 or h_{-5}^4	g_7^4 or h_7^4	g_{-7}^4 or h_{-7}^4	g_9^4 or h_9^4
(a) 5°	7·4400296	+ 7·4241563	+ 7·3246042	+ 7·3029589	+ 7·1083518
(b) 10	8·3326635	+ 8·3171546	+ 8·2039414	+ 8·1827929	+ 7·9701475
(c) 15	8·8434244	+ 8·8285100	+ 8·6921028	+ 8·6717650	+ 8·4279794
(d) 20	9·1935349	+ 9·1794268	+ 9·0095891	+ 8·9903507	+ 8·7003966
(e) 25	9·4521624	+ 9·4390471	+ 9·2244051	+ 9·2065205	+ 8·8518410
(f) 30	9·6499180	+ 9·6379512	+ 9·3653414	+ 9·3490233	+ 8·9043379
(g) 35	9·8028868	+ 9·7921889	+ 9 4455886	+ 9·4310005	+ 8·8565420
(h) 40	9·9203646	+ 9 9110168	+ 9·4695975	+ 9·4568505	+ 8·6727471
(i) 45	0·0079531	+ 9·9999953	+ 9·4340893	+ 9·4232377	+ 8·1359487
(k) 50	0·0689499	+ 0·0623797	+ 9·3233926	+ 9·3144332	− 8·2257247
(l) 55	0·1049730	+ 0·0997459	+ 9·0848836	+ 9·0777557	− 8·5327756
(m) 60	0·1161330	+ 0·1121641	+ 8·3402920	+ 8·3348798	− 8·5242596
(n) 65	0·1007966	+ 0 0979631	− 8·8346660	− 8·8308022	− 8·2443039
(o) 70	0·0546938	+ 0·0528392	− 9·1163622	− 9·1138331	+ 7·6771831
(p) 75	9·9684292	+ 9·9673677	9·1850885	− 9·1836410	+ 8·3541496
(q) 80	9·8195288	+ 9 8190511	− 9·1228219	− 9·1221705	+ 8·4450207
(r) 85	9·5346447	+ 9·5345244	− 8·8835000	− 8 8833360	+ 8 2708526
(s) 90	0·0000000	0·0000000	0·0000000	0 0000000	0·0000000

FOR Z

Co-latitude	g_5^4 or h_5^4	g_{-5}^4 or h_{-5}^4	g_7^4 or h_7^4	g_{-7}^4 or h_{-7}^4	g_9^4 or h_9^4
(a) 5°	6·5612125	− 6·4619455	+ 6·5702420	− 6·4875437	+ 6·4506073
(b) 10	7·7530998	− 7·6543180	+ 7·7488245	− 7·6667578	+ 7·6116375
(c) 15	8·4369893	− 8·3389994	+ 8·4101005	− 8·3290666	+ 8·2425646
(d) 20	8·9078877	− 8·8109717	+ 8·8483540	− 8·7687242	+ 8·6357173
(e) 25	9·2580821	− 9·1624886	+ 9·1547081	− 9·0768145	+ 8·8786489
(f) 30	9·5284778	− 9·4344144	+ 9·3682450	− 9·2923736	+ 9·0036597
(g) 35	9·7406459	− 9·6482732	+ 9 5076361	− 9·4340242	+ 9·0148389
(h) 40	9 9071511	− 9·8165775	+ 9·5805908	− 9·5094312	+ 8·8795529
(i) 45	0·0356946	− 9·9469735	+ 9·5859068	− 9·5173727	+ 8·3808176
(k) 50	0·1309984	− 0·0441268	+ 9·5092703	− 9·4436054	− 8·5116964
(l) 55	0·1956849	− 0·1106037	+ 9·2987765	− 9·2367622	− 8·8455202
(m) 60	0·2305918	− 0·1471880	+ 8·5714769	− 8·5252433	− 8·8601243
(n) 65	0·2346221	− 0·1527320	− 9·0953984	+ 9·0294361	− 8·5985635
(o) 70	0·2039030	− 0·1233182	− 9·3912082	+ 9·3296300	+ 8·0533800
(p) 75	0·1293318	− 0·0498048	− 9·4712850	+ 9·4117812	+ 8·7377075
(q) 80	9·9886442	− 9·9098958	− 9·4170845	+ 9·3588818	+ 8·8363618
(r) 85	9·7086336	− 9·6303619	− 9·1825692	+ 9·1251134	+ 8·6669235
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

g_{-9}^{4} or h_{-9}^{4}	Absolute term for g, $\frac{1}{2}(a_4+a'_4)$		Absolute term for h, $\frac{1}{2}(b_4+b'_4)$		
	1845	1880	1845	1880	
− 7·0733477					(a)
− 7·9117897					(b)
− 8·3258680	− ·02015		·0011		(c)
− 8·5232535	− ·02215		·0052		(d)
− 8·5383939	− ·0144	·0022	− ·0193	− ·0102	(e)
− 8·2608357	− ·0354	− ·0129	− ·06365	− ·0050	(f)
+ 8·1047853	− ·0342	− ·0198	− ·07305	− ·0048	(g)
+ 8·6604278	− ·02225	− ·0080	− ·07195	·0014	(h)
+ 8·8083708	− ·00585	− ·0006	− ·0524	·0008	(i)
+ 8·7562196	·0118	·0098	− ·0167	− ·0040	(k)
+ 8·3944231	·01075	·0309	− ·02145	− ·0127	(l)
− 8·2649781	·0328	·0505	− ·0091	− ·0154	(m)
− 8·7154165	·0709	·0520	− ·01575	− ·0250	(n)
− 8·7714865	·0935	·0340	− ·0195	− ·0240	(o)
− 8·5585454	·0973	·0167	− ·02255	·0083	(p)
+ 7·7051590	·0899	·0256	− ·02635	·0328	(q)
+ 8·6372271	·04955	·0300	− ·0453	·0481	(r)
+ 8·7695511	·0270	·0285	− ·0410	·0609	(s)

	Absolute term for g, $\frac{1}{2}(b_4-b'_4)$		Absolute term for h, $-\frac{1}{2}(a_4-a'_4)$		
	1845	1880	1845	1880	
+ 7·0809344					(a)
+ 7·9433594					(b)
+ 8·4022182	− ·0006		·01355		(c)
+ 8·6760280	·0372		·02115		(d)
+ 8·8291872	·05615	·0272	·03695	·0247	(e)
+ 8·8836680	·05715	·0259	·04355	·0388	(f)
+ 8·8380638	·0570	·0382	·0493	·0461	(g)
+ 8·6566008	·0410	·0542	·0467	·0562	(h)
+ 8·1222034	·0307	·0600	·04745	·0656	(i)
− 8·2143761	·01755	·0604	·0425	·0731	(k)
− 8·5237470	·00315	·0514	·0557	·0766	(l)
− 8·5174041	·00315	·0321	·0462	·0779	(m)
− 8·2394097	− ·00555	·0204	·0384	·0775	(n)
+ 7·6739796	− ·0128	·0102	·0321	·0633	(o)
+ 8·3523160	− ·0190	·0087	·0165	·0423	(p)
+ 8·4441956	− ·01385	·0075	·01025	·0205	(q)
+ 8·2706448	− ·0126	·0023	·00525	·0029	(r)
0·0000000					(s)

	Absolute term for g, $\frac{1}{2}(a_4-a'_4)$		Absolute term for h, $\frac{1}{2}(b_4-b'_4)$		
	1845	1880	1845	1880	
− 6·3750359					(a)
− 7·5368512					(b)
− 8·1690648	− ·0162		·03565		(c)
− 8·5639769	− ·05695		− ·0022		(d)
− 8·8091074	− ·0362	− ·0715	·00565	·0428	(e)
− 8·9367340	− ·01035	− ·0253	·01505	·0342	(f)
− 8·9509819	·0336	− ·0218	·01385	·0328	(g)
− 8·8195498	·0114	− ·0281	·04345	·0539	(h)
− 8·3297227	·01355	− ·0297	·0785	·0576	(i)
+ 8·4491189	·0178	·0015	·0953	·0532	(k)
+ 8·7896968	·0836	·0041	·0662	·0441	(l)
+ 8·8082027	·08095	− ·0158	·00535	·0157	(m)
+ 8·5510121	·06465	− ·0304	− ·0074	·0020	(n)
− 7·9939366	·06355	− ·0366	− ·03135	− ·0212	(o)
− 8·6891002	·0075	− ·0449	− ·0358	− ·0628	(p)
− 8·7898578	− ·00075	− ·0111	·00755	− ·0525	(q)
− 8·6214546	− ·00705	− ·0060	− ·0036	− ·0340	(r)
0·0000000					(s)

Co-latitude	For X g_5^5 or h_5^5	g_{-5}^5 or h_{-5}^5	g_7^5 or h_7^5	g_{-7}^5 or h_{-7}^5	g_9^5 or h_9^5
(a) 5°	− 6·4800948	− 6·4642706	− 6·4431095	− 6·4215314	− 6·2928287
(b) 10	− 7·6719572	− 7·6566432	− 7·6193752	− 7·5984961	− 7·4484188
(c) 15	− 8·3558062	− 8·3413245	− 8·2765200	− 8·2567917	− 8·0692052
(d) 20	− 8·8266496	− 8·8132970	− 8·7083458	− 8·6902020	− 8·4453404
(e) 25	− 9·1767760	− 9·1648137	− 9·0051069	− 8·9889642	− 8·6595466
(f) 30	− 9·4470931	− 9·4367395	− 9·2042975	− 9·1905694	− 8·7317339
(g) 35	− 9·6591744	− 9·6505986	− 9·3214736	− 9·3106202	− 8·6220147
(h) 40	− 9·8255866	− 9·8189025	− 9·3574260	− 9·3500912	− 7·8267988
(i) 45	− 9·9540347	− 9·9492987	− 9·2919479	− 9·2895429	+ 8 6247545
(k) 50	− 0·0492430	− 0·0464521	− 9·0296571	− 9·0384128	+ 8·9308217
(l) 55	− 0·1138366	− 0 1129288	+ 8·4648321	+ 8·3862792	+ 9·0083876
(m) 60	− 0·1486566	− 0·1495131	+ 9·2719483	+ 9·2579404	+ 8·9082001
(n) 65	− 0·1526082	− 0·1550572	+ 9·5186745	+ 9·5116846	+ 8·4417651
(o) 70	− 0·1218212	− 0·1256433	+ 9 6241926	+ 9 6207139	− 8·5662006
(p) 75	− 0·0471949	− 0·0521298	+ 9·6352241	+ 9·6340055	− 8·9276389
(q) 80	− 9·9064668	− 9·9122208	+ 9·5482253	+ 9·5484940	− 8·9744688
(r) 85	− 9·6264315	− 9·6326870	+ 9·2980378	+ 9·2991672	− 8·7859619
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
	For Y				
(a) 5°	6·4817997	+ 6·4659264	+ 6·4462867	+ 6·4246414	+ 6·2979415
(b) 10	7·6788006	+ 7·6632917	+ 7·6322428	+ 7·6110943	+ 7·4693730
(c) 15	8·3712952	+ 8·3563808	+ 8·3060988	+ 8·2857610	+ 8·1184259
(d) 20	8·8544192	+ 8·8403111	+ 8·7626229	+ 8·7433845	+ 8·5388637
(e) 25	9·2206533	+ 9·2075380	+ 9·0937328	+ 9·0758482	+ 8·8209266
(f) 30	9·5111757	+ 9·4992089	+ 9·3397873	+ 9·3234690	+ 9·0022058
(g) 35	9·7479320	+ 9·7372341	+ 9·5215553	+ 9·5069672	+ 9·0987025
(h) 40	9·9439964	+ 9·9346486	+ 9·6504038	+ 9·6376568	+ 9·1128035
(i) 45	0·1077714	+ 0·0998136	+ 9·7321868	+ 9·7213352	+ 9·0304689
(k) 50	0·2449548	+ 0·2383846	+ 9·7686056	+ 9·7596462	+ 8·7929875
(l) 55	0·3595645	+ 0·3543374	+ 9·7569181	+ 9·7497902	+ 7 8376881
(m) 60	0·4545120	+ 0·4505431	+ 9·6871464	+ 9·6817342	− 8·6122741
(n) 65	0 5319424	+ 0·5291089	+ 9·5312686	+ 9·5274048	− 8·8202638
(o) 70	0·5934463	+ 0·5915917	+ 9·1812509	+ 9·1787218	− 8·7892451
(p) 75	0·6401950	+ 0·6391335	− 8·6725271	− 8·6710796	− 8·4983657
(q) 80	0·6730283	+ 0·6725506	− 9·3466760	− 9·3460246	+ 8·0047710
(r) 85	0·6925113	+ 0 6923910	− 9·5340634	− 9·5338994	+ 8·6551753
(s) 90	0·6989700	+ 0·6989700	− 9·5850266	− 9·5850266	+ 8·7695511
	For Z				
(a) 5°	5·5065534	− 5·4062224	+ 5·5953767	− 5·5118988	+ 5·5435782
(b) 10	7·0028078	− 6·9029619	+ 7·0805820	− 6 9977276	+ 7·0142515
(c) 15	7·8684312	− 7·7693770	+ 7·9275594	− 7·8457233	+ 7·8364131
(d) 20	8·4723434	− 8·3743628	+ 8·5048616	− 8·4244066	+ 8·3776092
(e) 25	8·9301447	− 8·8334862	+ 8·9275253	− 8·8487731	+ 8·7511973
(f) 30	9·2933076	− 9·1981789	+ 9·2462037	− 9·1694241	+ 9·0050581
(g) 35	9·5892638	− 9·4958252	+ 9·4871511	− 9·4125554	+ 9·1606691
(h) 40	9·8343560	− 9·7427160	+ 9·6650019	− 9·5927379	+ 9·2236628
(i) 45	0·0390866	− 9·9492986	+ 9·7877082	− 9·7178580	+ 9·1820328
(k) 50	0·2105775	− 0·1226384	+ 9·8583912	− 9·7909746	+ 8·9781893
(l) 55	0·3538511	− 0·2677019	+ 9·8753042	− 9·8102889	+ 8·0405437
(m) 60	0·4725460	− 0·3880738	+ 9·8291740	− 9·7665082	− 8·8529901
(n) 65	0·5693434	− 0·4863846	+ 9·6924380	− 9·6321697	− 9·0793436
(o) 70	0·6462314	− 0·5645775	+ 9·3569743	− 9·3001299	− 9·0633034
(p) 75	0·7046739	− 0·6240772	− 8·8660669	+ 8·7973972	− 8·7835246
(q) 80	0·7457201	− 0·6659021	− 9·5447349	+ 9·4850181	+ 8·3015806
(r) 85	0·7700768	− 0·6907352	− 9·7366414	+ 9·6782885	+ 8·9547422
(s) 90	0·7781513	− 0·6989700	− 9·7891466	+ 9·7311546	+ 9·0705811

g_{-9}^5 or h_{-9}^5	Absolute term for g, $\frac{1}{2}(a_5 - a'_5)$ 1845	Absolute term for g, $\frac{1}{2}(a_5 - a'_5)$ 1880	Absolute term for h, $\frac{1}{2}(b_5 - b'_5)$ 1845	Absolute term for h, $\frac{1}{2}(b_5 - b'_5)$ 1880	
− 6·2654967					(a)
− 7·4219783					(b)
− 8·0442516	− ·0171		·01635		(c)
− 8·4224888	·01355		·00065		(d)
− 8·6394989	·01115	·0127	− ·01665	·0011	(e)
− 8·7155323	·0065	·0000	− ·0113	− ·0020	(f)
− 8·6124215	·0118	− ·0079	− ·0083	·0288	(g)
− 7·8737275	·00635	− ·0096	− ·00885	·0337	(h)
+ 8·6009147	·01105	− ·0009	− ·01895	·0377	(i)
+ 8·9164206	·02165	·0071	− ·02815	·0323	(k)
+ 8·9990714	·0287	·0216	− ·00955	·0179	(l)
+ 8·9034362	·03455	·0340	·0070	·0078	(m)
+ 8·4458305	·0584	·0205	·0183	− ·0004	(n)
− 8·5573927	·0554	·0133	·02945	·0022	(o)
− 8·9248133	·04715	·0085	·0147	·0075	(p)
− 8·9738426	·0459	·0110	·02475	·0071	(q)
− 8·7864585	·01875	·0141	·0038	·0005	(r)
0·0000000					(s)

	Absolute term for g, $\frac{1}{2}(b_5 + b'_5)$ 1845	Absolute term for g, $\frac{1}{2}(b_5 + b'_5)$ 1880	Absolute term for h, $-\frac{1}{2}(a_5 + a'_5)$ 1845	Absolute term for h, $-\frac{1}{2}(a_5 + a'_5)$ 1880	
+ 6·2705241					(a)
+ 7·4425849					(b)
+ 8·0926647	·01375		− ·01635		(c)
+ 8·5144951	·0325		− ·00795		(d)
+ 8·7982728	·0103	·0458	− ·00835	− ·0146	(e)
+ 8·9815359	·0100	·0298	− ·01025	− ·0111	(f)
+ 9·0802243	·0100	·0195	·0026	− ·0179	(g)
+ 9·0966572	·00545	·0077	·01255	− ·0107	(h)
+ 9·0167236	·0048	·0052	·0027	− ·0048	(i)
+ 8·7816389	·0020	− ·0069	− ·0035	− ·0059	(k)
+ 7·8286595	− ·01945	− ·0135	− ·00505	− ·0039	(l)
− 8·6054186	− ·0369	− ·0224	− ·01875	− ·0056	(m)
− 8·8153696	− ·0435	− ·0303	− ·0268	− ·0092	(n)
− 8·7860416	− ·0447	− ·0462	− ·0373	− ·0172	(o)
− 8·4965321	− ·0480	− ·0641	− ·0485	− ·0244	(p)
+ 8·0039459	− ·0487	− ·0758	− ·05205	− ·0357	(q)
+ 8·6549675	− ·05545	− ·0753	− ·05695	− ·0372	(r)
+ 8·7695511	− ·0621	− ·0750	− ·0602	− ·0431	(s)

	Absolute term for g, $\frac{1}{2}(a_5 + a'_5)$ 1845	Absolute term for g, $\frac{1}{2}(a_5 + a'_5)$ 1880	Absolute term for h, $\frac{1}{2}(b_5 + b'_5)$ 1845	Absolute term for h, $\frac{1}{2}(b_5 + b'_5)$ 1880	
− 5·4673900					(a)
− 6·9388329					(b)
− 7·7622528	·0288		·01275		(c)
− 8·3051603	·01985		·0170		(d)
− 8·6808674	− ·01255	− ·0107	·0247	·0171	(e)
− 8·9372014	− ·03975	·0086	·0357	·0208	(f)
− 9·0955879	− ·0116	·0223	·01555	·0033	(g)
− 9·1616278	− ·01125	·0285	·0202	− ·0224	(h)
− 9·1233697	− ·01055	·0432	·0043	− ·0166	(i)
− 8·9237742	− ·0360	·0109	− ·02845	− ·0104	(k)
− 8·0117153	− ·0520	− ·0188	− ·0561	− ·0175	(l)
+ 8·7973449	− ·01035	− ·0365	− ·0712	− ·0213	(m)
+ 9·0281451	·01745	− ·0888	− ·0516	− ·0142	(n)
+ 9·0149644	− ·01545	− ·1001	− ·0541	− ·0256	(o)
+ 8·7380586	− ·04795	− ·0958	− ·0637	− ·0102	(p)
− 8·2500403	− ·15245	− ·0855	− ·0272	·0030	(q)
− 8·9084162	− ·1541	− ·0836	− ·01235	− ·0072	(r)
− 9·0248236	− ·1616	− ·0659	− ·0009	+ ·0000	(s)

Co-latitude	g_6^5 or h_6^5	g_{-6}^5 or h_{-6}^5	g_8^5 or h_8^5	g_{-8}^5 or h_{-8}^5	g_{10}^5 or h_{10}^5
	For X				
(a) 5°	− 6·4791810	− 6·4604798	− 6·3792550	− 6·3547999	− 6·1876597
(b) 10	− 7·6638662	− 7·6457692	− 7·5458328	− 7·5221725	− 7·3309264
(c) 15	− 8·3354996	− 8·3183924	− 8·1860836	− 8·1637393	− 7·9294710
(d) 20	− 8·7886725	− 8·7729167	− 8·5924358	− 8·5719243	− 8·2697825
(e) 25	− 9·1150066	− 9·1009321	− 8·8525198	− 8·8343739	− 8·4253683
(f) 30	− 9·3543776	− 9·3422799	− 8·9985035	− 8·9833432	− 8·3850620
(g) 35	− 9·5267489	− 9·5168907	− 9·0333543	− 9·0221378	− 7·8537127
(h) 40	− 9·6420823	− 9·6347071	− 8·9176691	− 8·9129581	+ 8·3029036
(i) 45	− 9·7035898	− 9·6989611	− 8·3295026	− 8·3593261	+ 8·6461501
(k) 50	− 9·7074226	− 9·7059310	+ 8·8040644	+ 8·7797584	+ 8·7068560
(l) 55	− 9·6369256	− 9·6394534	+ 9·1680116	+ 9·1556363	+ 8·5262768
(m) 60	− 9·4337246	− 9·4435979	+ 9·3006703	+ 9·2932829	− 6·7709302
(n) 65	− 8·5580423	− 8·6478099	+ 9·2992496	+ 9·2954463	− 8·5295234
(o) 70	9·3876787	+ 9·3728515	+ 9·1456326	+ 9·1451614	− 8·6782864
(p) 75	9·7237750	+ 9·7186373	+ 8·5682562	+ 8·5761934	− 8·5323331
(q) 80	9·8893805	+ 9·8874827	− 8·8899778	− 8·8862693	− 6·9216376
(r) 85	9·9737259	+ 9·9732882	− 9·2215993	− 9·2209929	+ 8·5127952
(s) 90	0·0000000	+ 0·0000000	− 9·3010300	− 9·3010300	+ 8·6668887
	For Y				
(a) 5°	6·4815647	+ 6·4628054	+ 6·3833415	+ 6·3588101	+ 6·1939172
(b) 10	7·6734737	+ 7·6551450	+ 7·5624751	+ 7·5385068	+ 7·3567559
(c) 15	8·3573987	+ 8·3397726	+ 8·2247228	+ 8·2016733	+ 7·9909715
(d) 20	8·8283454	+ 8·8116721	+ 8·6644601	+ 8·6426566	+ 8·3894128
(e) 25	9·1785990	+ 9·1630990	+ 8·9729692	+ 8·9527000	+ 8·6406683
(f) 30	9·4490635	+ 9·4349209	+ 9·1896076	+ 9·1711135	+ 8·7792500
(g) 35	9·6613079	+ 9·6486649	+ 9·3335420	+ 9·3170088	+ 8·8147938
(h) 40	9·8278938	+ 9·8168464	+ 9·4134415	+ 9·3989949	+ 8·7335860
(i) 45	9·9565208	+ 9·9471161	+ 9·4302143	+ 9·4179159	+ 8·4605671
(k) 50	0·0519081	+ 0·0441433	+ 9·3751817	+ 9·3650277	− 7·1580317
(l) 55	0·1166752	+ 0·1104977	+ 9·2181344	+ 9·2100561	− 8·3940755
(m) 60	0·1516579	+ 0·1469673	+ 8·8290452	+ 8·8229114	− 8·5065837
(n) 65	0·1557565	+ 0·1524078	− 8·5245657	− 8·5201867	− 8·3422916
(o) 70	0·1250964	+ 0·1229045	− 9·0517689	− 9·0489026	− 7·1426787
(p) 75	0·0505730	+ 0·0493185	− 9·1773673	− 9·1757268	+ 8·2550051
(q) 80	9·9099207	+ 9·9093562	− 9·1410580	− 9·1403198	+ 8·4105971
(r) 85	9·6299318	+ 9·6297896	− 8·9143885	− 8·9142026	+ 8·2588855
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
	For Z				
(a) 5°	5·5729213	− 5·4827686	+ 5·5833825	− 5·5043204	+ 5·4808139
(b) 10	7·0640822	− 6·9744824	+ 7·0617617	− 6·9833957	+ 6·9428901
(c) 15	7·9211333	− 7·8324362	+ 7·8971252	− 7·8198959	+ 7·7502066
(d) 20	8·5128644	− 8·4253911	+ 8·4576316	− 8·3819465	+ 8·2693933
(e) 25	8·9546807	− 8·8687147	+ 8·8576821	− 8·7839040	+ 8·6121535
(f) 30	9·2977805	− 9·2135583	+ 9·1469268	− 9·0753635	+ 8·8232823
(g) 35	9·5692190	− 9·4869235	+ 9·3500154	− 9·2809162	+ 8·9178752
(h) 40	9·7848265	− 9·7045812	+ 9·4788795	− 9·4124333	+ 8·8854060
(i) 45	9·9544026	− 9·8762684	+ 9·5365138	− 9·4728606	+ 8·6524957
(k) 50	0·0840911	− 0·0080647	+ 9·5156294	− 9·4549034	− 7·4148872
(l) 55	0·1775158	− 0·1035297	+ 9·3869023	− 9·3294110	− 8·6518408
(m) 60	0·2362400	− 0·1641654	+ 9·0201819	− 8·9679547	− 8·7872501
(n) 65	0·2597004	− 0·1893509	− 8·7416641	+ 8·6775130	− 8·6416502
(o) 70	0·2444199	− 0·1755576	− 9·2810495	+ 9·2255737	− 7·4426763
(p) 75	0·1815865	− 0·1139295	− 9·4178803	+ 9·3648669	+ 8·5832111
(q) 80	0·0491445	− 9·9823749	− 9·3896125	+ 9·3380539	+ 8·7464661
(r) 85	9·7740276	− 9·7078012	− 9·1677424	+ 9·1170086	+ 8·5994693
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

g_{-10}^5 or h_{-10}^5	Absolute term for g, $\frac{1}{2}(a_5+a'_5)$ 1845	Absolute term for g, $\frac{1}{2}(a_5+a'_5)$ 1880	Absolute term for h, $\frac{1}{2}(b_5+b'_5)$ 1845	Absolute term for h, $\frac{1}{2}(b_5+b'_5)$ 1880	
− 6·1574509					(a)
− 7·3017071					(b)
− 7·9019161	− ·0187		− ·01325		(c)
− 8·2446288	− ·02425		− ·00655		(d)
− 8·4035887	− ·02285	− ·0298	− ·00135	+ ·0192	(e)
− 8·3687660	− ·0075	− ·0116	·0074	·0034	(f)
− 7·8625481	− ·0095	·0076	·0072	·0119	(g)
+ 8·2744601	− ·00855	·0111	− ·00615	·0059	(h)
+ 8·6281224	− ·00975	·0025	− ·01305	− ·0021	(i)
+ 8·6944433	·00285	− ·0033	− ·01665	− ·0079	(k)
+ 8·5197551	− ·0318	− ·0083	− ·03815	− ·0195	(l)
− 6·3929252	− ·03345	− ·0165	− ·0221	− ·0216	(m)
− 8·5213527	− ·0225	− ·0166	− ·0440	− ·0076	(n)
− 8·6746384	− ·0270	− ·0144	− ·02565	− ·0007	(o)
− 8·5318575	− ·02255	− ·0044	− ·0284	·0029	(p)
− 6·9797594	− ·0227	·0066	− ·06315	·0073	(q)
+ 8·5119782	− ·01835	·0044	− ·0557	·0006	(r)
+ 8·6668887	− ·0103	·0066	− ·0473	·0045	(s)
	Absolute term for g, $\frac{1}{2}(b_5-b'_5)$ 1845	Absolute term for g, $\frac{1}{2}(b_5-b'_5)$ 1880	Absolute term for h, $-\frac{1}{2}(a_5-a'_5)$ 1845	Absolute term for h, $-\frac{1}{2}(a_5-a'_5)$ 1880	
+ 6·1636137					(a)
+ 7·3271480					(b)
+ 7·9624986	− ·00515		− ·01295		(c)
+ 8·3624791	− ·0255		− ·00175		(d)
+ 8·6156299	− ·0238	− ·0316	·00825	·0148	(e)
+ 8·7564043	− ·0166	− ·0251	− ·01245	− ·0041	(f)
+ 8·7943705	− ·0193	− ·0252	− ·0296	− ·0206	(g)
+ 8·7157401	− ·01415	− ·0247	− ·04555	− ·0404	(h)
+ 8·4453749	− ·0180	− ·0228	− ·0570	− ·0514	(i)
− 7·1454885	− ·0306	− ·0408	− ·0445	− .0561	(k)
− 8·3840965	− ·02455	− ·0405	− ·04865	− ·0586	(l)
− 8·4990066	− ·0095	− ·0433	− ·05205	− ·0584	(m)
− 8·3368822	− ·0024	− ·0479	− ·0624	− ·0551	(n)
− 7·1391380	·0019	− ·0448	− ·0569	− ·0452	(o)
+ 8·2529785	− ·0070	− ·0362	− ·0528	− ·0313	(p)
+ 8·4096852	− ·0166	− ·0262	− ·03045	− ·0194	(q)
+ 8·2586558	− ·01275	− ·0127	− ·01395	− ·0096	(r)
0·0000000					(s)
	Absolute term for g, $\frac{1}{2}(a_5-a'_5)$ 1845	Absolute term for g, $\frac{1}{2}(a_5-a'_5)$ 1880	Absolute term for h, $\frac{1}{2}(b_5-b'_5)$ 1845	Absolute term for h, $\frac{1}{2}(b_5-b'_5)$ 1880	
− 5·4064001					(a)
− 6·8693204					(b)
− 7·6780183	·0408		− ·00415		(c)
− 8·1990884	·01545		− ·0121		(d)
− 8·5441889	− ·01225	·0235	− ·0171	·0080	(e)
− 8·7580693	− ·01995	·0219	− ·0294	·0212	(f)
− 8·8558022	− ·0199	·0386	− ·04425	·0081	(g)
− 8·8269438	·01505	·0444	− ·0926	− ·0286	(h)
− 8·5988999	·01735	·0590	− ·0687	− ·0504	(i)
+ 7·2976776	− ·0187	·0078	− ·05175	− ·0622	(k)
+ 8·5983080	− ·0635	− ·0211	− ·0777	− ·0471	(l)
+ 8·7382982	− ·03815	− ·0287	− ·1001	− ·0410	(m)
+ 8·5966416	− ·00995	− ·0552	− ·0688	− ·0406	(n)
+ 7·4301789	·01305	− ·0678	− ·0865	− ·0531	(o)
− 8·5384331	− ·01485	− ·0473	− ·0815	− ·0385	(p)
− 8·7041446	·01615	− ·0414	− ·1068	− ·0284	(q)
− 8·5582784	·0033	·0099	− ·04445	− ·0312	(r)
0·0000000					(s)

For X

Co-latitude	g_6^6 or h_6^6	g_{-6}^6 or h_{-6}^6	g_8^6 or h_8^6	g_{-8}^6 or h_{-8}^6	g_{10}^6 or h_{10}^6
(a) 5°	− 5·5039021	− 5·4851911	− 5·4720149	− 5·4475469	− 5·3398053
(b) 10	− 6·9950417	− 6·9769049	− 6·9484886	− 6·9247741	− 6·7975078
(c) 15	− 7·8520585	− 7·8348585	− 7·7804697	− 7·7579925	− 7·5967797
(d) 20	− 8·4437427	− 8·4278135	− 8·3357537	− 8·3149726	− 8·1027794
(e) 25	− 8·8855013	− 8·8711369	− 8·7281002	− 8·7094451	− 8·4241474
(f) 30	− 9·2285344	− 9·2159805	− 9·0060222	− 8·9898990	− 8·5986428
(g) 35	− 9·4998990	− 9·4893458	− 9·1920358	− 9·1788518	− 8·6212530
(h) 40	− 9·7154277	− 9·7070035	− 9·2936558	− 9·2838904	− 8·3880400
(i) 45	− 9·8849227	− 9·8786908	− 9·3031014	− 9·2976083	+ 8·0706316
(k) 50	− 0·0145299	− 0·0104869	− 9·1777222	− 9·1789717	+ 8·7316868
(l) 55	− 0·1078760	− 0·1059520	− 8·6291357	− 8·6627430	+ 8·9115746
(m) 60	− 0·1665263	− 0·1665876	+ 9·0233472	+ 9·0011840	+ 8·8906474
(n) 65	− 0·1899199	− 0·1917732	+ 9·4117234	+ 9·4027988	+ 8·5991177
(o) 70	− 0·1745817	− 0·1779799	+ 9·5694035	+ 9·5650831	− 8·2287735
(p) 75	− 0·1117016	− 0·1163520	+ 9·6089443	+ 9·6073165	− 8·8139892
(q) 80	− 9·9792252	− 9·9847973	+ 9·5388731	+ 9·5389564	− 8·9051439
(r) 85	− 9·7040871	− 9·7102234	+ 9·2979123	+ 9·2989715	− 8·7344830
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

For Y

Co-latitude	g_6^6 or h_6^6	g_{-6}^6 or h_{-6}^6	g_8^6 or h_8^6	g_{-8}^6 or h_{-8}^6	g_{10}^6 or h_{10}^6
(a) 5°	5·5056062	+ 5·4868469	+ 5·4749322	+ 5·4504008	+ 5·3442657
(b) 10	7·0018821	+ 6·9835534	+ 6·9602864	+ 6·9363181	+ 6·8157209
(c) 15	7·8675409	+ 7·8499148	+ 7·8075217	+ 7·7844722	+ 7·6392733
(d) 20	8·4715009	+ 8·4548276	+ 8·3852049	+ 8·3634014	+ 8·1826249
(e) 25	8·9293612	+ 8·9138612	+ 8·8083987	+ 8·7881295	+ 8·5594155
(f) 30	9·2925925	+ 9·2784499	+ 9·1277965	+ 9·1093024	+ 8·8180410
(g) 35	9·5886243	+ 9·5759813	+ 9·3697183	+ 9·3531851	+ 8·9809924
(h) 40	9·8337970	+ 9·8227496	+ 9·5489083	+ 9·5344617	+ 9·0561490
(i) 45	0·0386104	+ 0·0292057	+ 9·6735107	+ 9·6612123	+ 9·0376370
(k) 50	0·2101842	+ 0·2024194	+ 9·7470091	+ 9·7368551	+ 8·8927284
(l) 55	0·3535381	+ 0·3473606	+ 9·7684137	+ 9·7603354	+ 8·4568423
(m) 60	0·4723082	+ 0·4676176	+ 9·7303013	+ 9·7241675	− 8·3553758
(n) 65	0·5691736	+ 0·5658249	+ 9·6110144	+ 9·6066354	− 8·7546427
(o) 70	0·6461202	+ 0·6439283	+ 9·3360743	+ 9·3332080	− 8·7883698
(p) 75	0·7046102	+ 0·7033557	− 7·4138186	− 7·4121781	− 8·5639242
(q) 80	0·7456916	+ 0·7451271	− 9·3128307	− 9·3120925	+ 7·6349989
(r) 85	0·7700696	+ 0·7699274	− 9·5421997	− 9·5420138	+ 8·6137820
(s) 90	0·7781513	+ 0·7781513	− 9·6020600	− 9·6020600	+ 8·7460700

For Z

Co-latitude	g_6^6 or h_6^6	g_{-6}^6 or h_{-6}^6	g_8^6 or h_8^6	g_{-8}^6 or h_{-8}^6	g_{10}^6 or h_{10}^6
(a) 5°	4·5181991	− 4·4271428	+ 4·5961135	− 4·5163648	+ 4·5522454
(b) 10	6·3137214	− 6·2232235	+ 6·3807163	− 6·3016576	+ 6·3229434
(c) 15	7·3525064	− 7·2629110	+ 7·4010724	− 7·3231398	+ 7·3196065
(d) 20	8·0772511	− 7·9888793	+ 8·0995324	− 8·0231276	+ 7·9837196
(e) 25	8·6266745	− 8·5398094	+ 8·6142790	− 8·5397572	+ 8·4520408
(f) 30	9·0625414	− 8·9774199	+ 9·0062997	− 8·9339580	+ 8·7832573
(g) 35	9·4177677	− 9·3345725	+ 9·3074003	− 9·2374707	+ 9·0053397
(h) 40	9·7119624	− 9·6308169	+ 9·5355929	− 9·4682359	+ 9·1294222
(i) 45	9·9577253	− 9·8786907	+ 9·7011206	− 9·6364216	+ 9·1516944
(k) 50	0·1636008	− 0·0866733	+ 9·8088882	− 9·7468601	+ 9·0407210
(l) 55	0·3356127	− 0·2607251	+ 9·8589045	− 9·7994937	+ 8·6318113
(m) 60	0·4781248	− 0·4051483	+ 9·8444587	− 9·7875678	− 8·5595760
(n) 65	0·5943524	− 0·5231006	+ 9·7443829	− 9·6899420	− 8·9761447
(o) 70	0·6866788	− 0·6169141	+ 9·4843580	− 9·4327381	− 9·0247871
(p) 75	0·7568592	− 0·6882993	− 7·6471083	+ 7·4236851	− 8·8115752
(q) 80	0·8061510	− 0·7384785	− 9·4829251	+ 9·4298331	+ 7·8967634
(r) 85	0·8354012	− 0·7682716	− 9·7167531	+ 9·6652104	+ 8·8755716
(s) 90	0·8450980	− 0·7781513	− 9·7781512	+ 9·7269987	+ 9·0093114

g_{-10}^6 or h_{-10}^6	Absolute term for g, $\frac{1}{2}(a_6 - a'_6)$		Absolute term for h, $\frac{1}{2}(b_6 - b'_6)$		
	1845	1880	1845	1880	
− 5·3095804					(a)
− 6·7682181					(b)
− 7·5690392	− ·0261		− ·0117		(c)
− 8·0771998	− ·02285		·01245		(d)
− 8·4013702	− ·00025	− ·0130	·01375	·0027	(e)
− 8·5794586	·01365	− ·0064	·01135	·0113	(f)
− 8·6071003	·00955	·0041	− ·00665	− ·0041	(g)
− 8·3850103	·00135	·0025	− ·00915	− ·0152	(h)
+ 8·0206121	·00575	− ·0029	− ·00385	− ·0096	(i)
+ 8·7135321	·0047	− ·0037	·0030	− ·0008	(k)
+ 8·9002170	·0015	·0073	− ·0100	− ·0007	(l)
+ 8·8841819	·02885	·0101	− ·0082	− ·0018	(m)
+ 8·5987295	·01265	·0166	− ·00315	·0019	(n)
− 8·2139561	·0143	·0207	·0106	− ·0008	(o)
− 8·8105817	·03825	·0039	·00345	− ·0053	(p)
− 8·9043238	·03635	·0034	− ·00225	·0056	(q)
− 8·7349175	·0211	− ·0037	·0069	·0109	(r)
0·0000000					(s)
	Absolute term for g, $\frac{1}{2}(b_6 + b'_6)$		Absolute term for h, $-\frac{1}{2}(a_6 + a'_6)$		
	1845	1880	1845	1880	
+ 5·3139622					(a)
+ 6·7861130					(b)
+ 7·6108004	·0066		·00845		(c)
+ 8·1556912	·0164		·00765		(d)
+ 8·5343771	·02275	·0145	·0229	·0083	(e)
+ 8·7951953	·0101	·0125	·01995	·0060	(f)
+ 8·9605691	·00295	·0020	·00795	− ·0049	(g)
+ 9·0383031	·00535	− ·0014	·00065	− ·0065	(h)
+ 9·0224448	·01115	·0012	·0092	·0040	(i)
+ 8·8801852	·0020	·0044	− ·0009	·0062	(k)
+ 8·4468633	·00435	·0062	− ·00925	·0073	(l)
− 8·3477987	·00825	·0110	− ·02065	·0139	(m)
− 8·7492333	·00635	·0176	− ·0328	·0179	(n)
− 8·7848291	·0031	·0167	− ·0339	·0228	(o)
− 8·5618976	− ·0060	·0127	− ·03525	·0205	(p)
+ 7·6340870	− ·0125	·0082	− ·0372	·0151	(q)
+ 8·6135523	− ·0219	− ·0017	− ·0372	·0185	(r)
+ 8·7460700	− ·0256	− ·0036	− ·0437	·0191	(s)
	Absolute term for g, $\frac{1}{2}(a_6 + a'_6)$		Absolute term for h, $\frac{1}{2}(b_6 + b'_6)$		
	1845	1880	1845	1880	
− 4·4772745					(a)
− 6·2488055					(b)
− 7·2468296	·00615		− ·03925		(c)
− 7·9127925	·0388		− ·0231		(d)
− 8·3833996	·0377	·0042	− ·02075	− ·0361	(e)
− 8·7172759	·04455	− ·0320	− ·0041	− ·0241	(f)
− 8·9423267	·0083	− ·0452	·00095	− ·0180	(g)
− 9·0696276	− ·0306	− ·0340	− ·0154	− ·0157	(h)
− 9·0953527	− ·01065	− ·0161	− ·02315	− ·0180	(i)
− 8·9882482	·00795	− ·0146	− ·02605	− ·0109	(k)
− 8·5858553	·02075	·0049	·0247	− ·0036	(l)
+ 8·5049028	·04705	·0285	·01185	·0229	(m)
+ 8·9283558	·0684	·0090	·0109	·0204	(n)
+ 8·9801627	·0737	− ·0236	− ·00685	·0481	(o)
+ 8·7696817	·0595	− ·0427	− ·06395	·0277	(p)
− 7·8437111	·0282	− ·0181	− ·04285	·0385	(q)
− 8·8335668	·03235	− ·0048	− ·04125	·0132	(r)
− 8·9679187	·0181	·0194	− ·0380	·0096	(s)

Co-latitude	For X: g_7^6 or h_7^6	g_{-7}^6 or h_{-7}^6	g_9^6 or h_9^6	g_{-9}^6 or h_{-9}^6	Absolute term for g, $\frac{1}{2}(a_6+a'_6)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_6+b'_6)$ 1845	1880
(a) 5°	− 5·5031016	− 5·4815122	− 5·4162106	− 5·3888642				
(b) 10	− 6·9874126	− 6·9664866	− 6·8838211	− 6·8573187				
(c) 15	− 7·8328269	− 7·8129869	− 7·7004549	− 7·6753440	·0272		− ·0151	
(d) 20	− 8·4077725	− 8·3894123	− 8·2328769	− 8·2096882	·01245		− ·01925	
(e) 25	− 8·8270813	− 8·8105570	− 8·5929600	− 8·5722147	·00255	·0064	− ·00805	− ·0183
(f) 30	− 9·1410730	− 9·1266974	− 8·8256283	− 8·8078747	·00745	·0088	·00555	− ·0054
(g) 35	− 9·3754533	− 9·3634943	− 8·9457701	− 8·9317049	·00405	·0139	·01165	− ·0057
(h) 40	− 9·5439143	− 9·5346027	− 8·9408670	− 8·9318208	·01155	·0099	·01375	− ·0065
(i) 45	− 9·6526744	− 9·6462280	− 8·7214316	− 8·7227990	·01255	·0124	·00985	− ·0046
(k) 50	− 9·7013368	− 9·6980278	+ 8·1422176	+ 8·0650109	·0081	·0038	·0212	·0007
(l) 55	− 9·6798329	− 9·6801940	+ 8·9745573	+ 8·9581907	− ·0342	·0004	·0263	− ·0062
(m) 60	− 9·5533452	− 9·5590541	+ 9·2022108	+ 9·1930708	− ·01005	− ·0021	·0171	− ·0051
(n) 65	− 9·1506196	− 9·1737017	+ 9·2554606	+ 9·2505615	·00005	− ·0010	·03105	− ·0020
(o) 70	+ 9·1610165	+ 9·1350863	+ 9·1527788	+ 9·1513632	·0268	− ·0014	·0392	·0080
(p) 75	+ 9·6591367	+ 9·6526568	+ 8·7320377	+ 8·7360777	·05735	− ·0104	·01875	·0196
(q) 80	+ 9·8669666	+ 9·8647190	− 8·7360670	− 8·7312202	·05295	− ·0022	·02885	·0297
(r) 85	+ 9·9687050	+ 9·9681974	− 9·1542588	− 9·1535711	·0605	− ·0092	·0050	·0073
(s) 90	+ 0·0000000	+ 0·0000000	− 9·2466724	− 9·2466724	·0590	− ·0086	·0022	·0294

Co-latitude	For Y				Absolute term for g, $\frac{1}{2}(b_6-b'_6)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_6-a'_6)$ 1845	1880
(a) 5°	5·5053712	+ 5·4837259	+ 5·4198578	+ 5·3924404				
(b) 10	6·9965552	+ 6·9754067	+ 6·8986381	+ 6·8718500				
(c) 15	7·8536445	+ 7·8333067	+ 7·7347057	+ 7·7089445	− ·0101		− ·00155	
(d) 20	8·4454270	+ 8·4261886	+ 8·2962803	+ 8·2719117	− ·0062		− ·02015	
(e) 25	8·8873069	+ 8·8694223	+ 8·6978687	+ 8·6752149	− ·01655	− ·0046	− ·0202	− ·0036
(f) 30	9·2304803	+ 9·2141620	+ 8·9892994	+ 8·9686295	− ·0222	·0014	− ·01235	·0086
(g) 35	9·5020002	+ 9·4874121	+ 9·1955359	+ 9·1770577	− ·01725	·0011	− ·01725	·0133
(h) 40	9·7176944	+ 9·7049474	+ 9·3290977	+ 9·3129514	− ·01345	·0010	− ·02005	·0119
(i) 45	9·8873598	+ 9·8765082	+ 9·3942022	+ 9·3804569	− ·00595	·0026	− ·0084	− ·0001
(k) 50	0·0171376	+ 0·0081782	+ 9·3865476	+ 9·3751990	− ·0122	·0060	− ·0302	− ·0002
(l) 55	0·1106488	+ 0·1035209	+ 9·2864424	+ 9·2774138	− ·00965	·0065	− ·02625	− ·0059
(m) 60	0·1694541	+ 0·1640419	+ 9·0215957	+ 9·0147402	·00235	·0113	− ·01505	− ·0051
(n) 65	0·1929877	+ 0·1891239	+ 6·4461642	+ 6·4412700	·01525	·0112	− ·0077	·0073
(o) 70	0·1777703	+ 0·1752412	− 8·9625061	− 8·9593026	·0136	·0086	− ·0076	·0226
(p) 75	0·1149882	+ 0·1135407	− 9·1578172	− 9·1559836	·0124	·0101	− ·01915	·0253
(q) 80	9·9825840	+ 9·9819326	− 9·1491192	− 9·1482941	·0117	·0096	− ·0175	·0257
(r) 85	9·7074901	+ 9·7073261	− 8·9353289	− 8·9351211	·0104	·0021	− ·0122	·0121
(s) 90	0·0000000	0·0000000	0·0000000	0 0000000				

Co-latitude	For Z				Absolute term for g, $\frac{1}{2}(a_6-a'_6)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_6-b'_6)$ 1845	1880
(a) 5°	4·5756411	− 4·4913852	+ 4·5866034	− 4·5098001				
(b) 10	6·3660757	− 6·2824402	+ 6·3646295	− 6·2885874				
(c) 15	7·3962893	− 7·3136663	+ 7·3738133	− 7·2990138	− ·00055		− ·02725	
(d) 20	8·1088537	− 8·0276037	+ 8·0561578	− 7·9830453	·0254		− ·0088	
(e) 25	8·6422932	− 8·5627340	+ 8·5492894	− 8·4782582	·0221	·0153	·01355	·0072
(f) 30	9·0580982	− 8·9804955	+ 8·9133296	− 8·8447126	·01435	·0014	·0391	·0048
(g) 35	9·3888082	− 9·3133668	+ 9·1787260	− 9·1127888	− ·0115	− ·0077	·02765	− ·0012
(h) 40	9·6535196	− 9·5803782	+ 9·3612628	− 9·2981986	− ·0057	− ·0007	·0427	− ·0061
(i) 45	9·8641298	− 9·7933567	+ 9·4672489	− 9·4071842	·00145	− ·0008	·03285	·0036
(k) 50	0·0282043	− 9·9597956	+ 9·4937858	− 9·4368043	− ·00735	·0285	·01995	·0205
(l) 55	0·1503689	− 0·0842489	+ 9·4221235	− 9·3683553	·00755	·0285	·0121	·0132
(m) 60	0·2329118	− 0·1689361	+ 9·1803803	− 9·1305418	− ·00245	·0382	·02215	·0012
(n) 65	0·2758034	− 0·2137630	− 5·5846275	− 6·9199709	− ·0070	·0146	·0241	− ·0084
(o) 70	0·2759625	− 0·2155905	− 9·1586007	+ 9·1076892	− ·0069	− ·0139	·00035	·0072
(p) 75	0·2248679	− 0·1658479	− 9·3649728	+ 9·3170227	− ·0238	− ·0205	− ·01065	·0085
(q) 80	0·1006722	− 0·0426475	− 9·3642901	+ 9·3179525	− ·0197	− ·0153	− ·02925	− ·0048
(r) 85	9·8304492	− 9·7730337	− 9·1552928	+ 9·1098570	− ·02375	− ·0216	− ·00165	·0036
(s) 90	0·0000000	0·0000000	0·0000000	0 0000000				

For X

Co-latitude	$g_7{}^7$ or $h_7{}^7$	$g_{-7}{}^7$ or $h_{-7}{}^7$	$g_9{}^7$ or $h_9{}^7$	$g_{-9}{}^7$ or $h_{-9}{}^7$	Absolute term for g, $\frac{1}{2}(a_7-a'_7)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_7-b'_7)$ 1845	1880
(a) 5°	− 4·5154744	− 4·4938769	− 4·4874340	− 4·4600774				
(b) 10	− 6·3058907	− 6·2849320	− 6·2638426	− 6·2372972				
(c) 15	− 7·3360742	− 7·3161579	− 7·2701646	− 7·2449494	·01095		·01515	
(d) 20	− 8·0485979	− 8·0300954	− 7·9481144	− 7·9247179	− ·0124		− ·0016	
(e) 25	− 8·5819872	− 8·5652256	− 8·4348151	− 8·4136874	− ·00655	− ·0132	− ·0023	·0021
(f) 30	− 8·9977340	− 8·9829870	− 8·7895159	− 8·7710694	− ·0052	− ·0035	·00055	·0037
(g) 35	− 9·3283797	− 9·3158584	− 9·0410770	− 9·0256995	·00505	·0005	− ·0055	·0048
(h) 40	− 9·5930226	− 9·5828699	− 9·2021458	− 9·1902495	·00105	·0019	− ·0082	·0023
(i) 45	− 9·8035621	− 9·7958483	− 9·2719282	− 9·2641022	− ·0095	·0102	− ·01645	− ·0023
(k) 50	− 9·9675659	− 9·9622872	− 9·2272881	− 9·2249108	− ·0155	·0083	− ·04395	− ·0048
(l) 55	− 0·0896619	− 0·0867405	− 8·9557758	− 8·9658374	− ·0137	·0110	− ·02295	− ·0073
(m) 60	− 0·1721403	− 0·1714275	+ 8·6176650	+ 8·5682423	− ·0225	·0030	·01115	− ·0080
(n) 65	− 0·2149740	− 0·2162547	+ 9·2913975	+ 9·2799694	− ·0157	·0051	·0013	− ·0054
(o) 70	− 0·2150829	− 0·2180821	+ 9·5114948	+ 9·5062629	− ·01115	·0112	− ·02085	− ·0042
(p) 75	− 0·1639475	− 0·1683396	+ 9·5814838	+ 9·5794414	·0130	·0072	− ·0143	− ·0041
(q) 80	− 0·0397218	− 0·0451391	+ 9·5287754	+ 9·5286793	− ·0011	·0060	− ·0037	·0001
(r) 85	− 9·7694804	− 9·7755254	+ 9·2971060	+ 9·2981031	·0029	·0002	− ·0128	− ·0011
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000				

For Y

Co-latitude	$g_7{}^7$ or $h_7{}^7$	$g_{-7}{}^7$ or $h_{-7}{}^7$	$g_9{}^7$ or $h_9{}^7$	$g_{-9}{}^7$ or $h_{-9}{}^7$	Absolute term for g, $\frac{1}{2}(b_7+b'_7)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_7+a'_7)$ 1845	1880
(a) 5°	4·5171780	+ 4·4955327	+ 4·4901685	+ 4·4627511				
(b) 10	6·3127290	+ 6·2915805	+ 6·2748901	+ 6·2481020				
(c) 15	7·3515520	+ 7·3312142	+ 7·2954506	+ 7·2696894	− ·01815		·00245	
(d) 20	8·0763479	+ 8·0571095	+ 7·9942115	+ 7·9698429	− ·00825		− ·0018	
(e) 25	8·6258345	+ 8·6079499	+ 8·5093736	+ 8·4867198	− ·0033	− ·0234	·00205	− ·0132
(f) 30	9·0617747	+ 9·0454564	+ 8·9019533	+ 8·8812834	− ·0065	− ·0125	·00335	− ·0064
(g) 35	9·4170820	+ 9·4024939	+ 9·2038034	+ 9·1853252	− ·01215	− ·0148	·0067	− ·0074
(h) 40	9·7113630	+ 9·6986160	+ 9·4330144	+ 9·4168681	− ·0153	− ·0142	·00345	− ·0099
(i) 45	9·9572148	+ 9·9463632	+ 9·5999653	+ 9·5862200	− ·0236	− ·0214	·0062	− ·0127
(k) 50	0·1631791	+ 0·1542197	+ 9·7098153	+ 9·6984667	− ·01735	− ·0238	·0159	− ·0078
(l) 55	0·3352770	+ 0·3281491	+ 9·7630905	+ 9·7540619	− ·0212	− ·0142	·0243	− ·0015
(m) 60	0·4778697	+ 0·4724575	+ 9·7543022	+ 9·7474467	− ·0192	− ·0133	·00825	·0066
(n) 65	0·5941702	+ 0·5903064	+ 9·6659269	+ 9·6610327	− ·01845	− ·0157	− ·00315	·0172
(o) 70	0·6865596	+ 0·6840305	+ 9·4410754	+ 9·4378719	− ·0215	− ·0157	− ·0026	·0264
(p) 75	0·7567908	+ 0·7553433	+ 8·6220707	+ 8·6202371	− ·03345	− ·0209	− ·00495	·0340
(q) 80	0·8061203	+ 0·8054689	− 9·2694894	− 9·2686643	− ·0285	− ·0254	− ·0022	·0363
(r) 85	0·8353934	+ 0·8352294	− 9·5457679	− 9·5455601	− ·0333	− ·0292	− ·0047	·0509
(s) 90	0·8450980	+ 0·8450980	− 9·6146491	− 9·6146491	− ·0369	− ·0261	− ·0093	·0519

For Z

Co-latitude	$g_7{}^7$ or $h_7{}^7$	$g_{-7}{}^7$ or $h_{-7}{}^7$	$g_9{}^7$ or $h_9{}^7$	$g_{-9}{}^7$ or $h_{-9}{}^7$	Absolute term for g, $\frac{1}{2}(a_7+a'_7)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_7+b'_7)$ 1845	1880
(a) 5°	3·5208617	− 3·4358287	+ 3·5902567	− 3·5128398				
(b) 10	5·6156634	− 5·5312507	+ 5·6742263	− 5·5976660				
(c) 15	6·8276107	− 6·7442104	+ 6·8678471	− 6·7924810	− ·0089		·0131	
(d) 20	7·6731889	− 7·5911613	+ 7·6874436	− 7·6136926	− ·0290		− ·00165	
(e) 25	8·3142353	− 8·2338982	+ 8·2941575	− 8·2224697	− ·00625	·0195	·0190	·0188
(f) 30	8·8228074	− 8·7444264	+ 8·7593590	− 8·6900599	·02255	·0268	·0075	·0172
(g) 35	9·2373051	− 9·1610852	+ 9·1203872	− 9·0537296	·03995	·0440	·0122	·0225
(h) 40	9·5806038	− 9·5066834	+ 9·3986007	− 9·3347584	·0407	·0280	·0028	− ·0014
(i) 45	9·8674008	− 9·7958483	+ 9·6064776	− 9·5455413	·00105	·0078	− ·0132	− ·0131
(k) 50	0·1076621	− 0·0384737	+ 9·7505997	− 9·6925772	− ·0114	− ·0118	− ·00885	·0077
(l) 55	0·3084138	− 0·2415137	+ 9·8324930	− 9·7773149	− ·0243	·0105	− ·0178	·0180
(m) 60	0·4747444	− 0·4099883	+ 9·8473861	− 9·7949218	− ·01905	·0121	− ·0104	·0233
(n) 65	0·6104034	− 0·5475821	+ 9·7782605	− 9·7283670	− ·0215	·0061	− ·00515	·0023
(o) 70	0·7181695	− 0·6570164	+ 9·5684855	− 9·5212431	− ·02335	·0049	− ·0102	·0038
(p) 75	0·8000884	− 0·7402870	+ 8·7574023	− 8·7198982	− ·05035	− ·0007	− ·01965	·0069
(q) 80	0·8576266	− 0·7988205	− 9·4184699	+ 9·3705347	− ·0556	·0144	·0326	·0045
(r) 85	0·8917707	− 0·8335736	− 9·6991359	+ 9·6529597	− ·08415	·0023	·0468	− ·0157
(s) 90	0·9030900	− 0·8450980	− 9·7695511	+ 9·7237936	− ·0899	− ·0092	·0418	− ·0258

Co-latitude	For X: $g_8{}^7$ or $h_8{}^7$	$g_{-8}{}^7$ or $h_{-8}{}^7$	$g_{10}{}^7$ or $h_{10}{}^7$	$g_{-10}{}^7$ or $h_{-10}{}^7$	Absolute term for g, $\frac{1}{2}(a_7+a'_7)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_7+b'_7)$ 1845	1880
(a) 5°	− 4·5147546	− 4·4902775	− 4·4378648	− 4·4076285				
(b) 10	− 6·2985912	− 6·2748390	− 6·2059859	− 6·1766478				
(c) 15	− 7·3176089	− 7·2950421	− 7·1980258	− 7·1701647	− ·00965		·03855	
(d) 20	− 8·0140556	− 7·9931025	− 7·8548734	− 7·8290426	·0110		·0182	
(e) 25	− 8·5259435	− 8·5069886	− 8·3121841	− 8·2889114	·01405	·0139	·0188	− ·0074
(f) 30	− 8·9139868	− 8·8973638	− 8·6265386	− 8·6063454	− ·0068	·0143	·01085	·0020
(g) 35	− 9·2095448	− 9·1955330	− 8·8215075	− 8·8049783	− ·01115	·0095	·0103	·0035
(h) 40	− 9·4298744	− 9·4187018	− 8·8973791	− 8·8854058	− ·01785	·0020	·0080	·0012
(i) 45	− 9·5838576	− 9·5757177	− 8·8141057	− 8·8091170	− ·0209	·0009	·00465	·0043
(k) 50	− 9·6737195	− 9·6688172	− 8·2555775	− 8·2842199	− ·0113	− ·0030	− ·01115	·0060
(l) 55	− 9·6934295	− 9·6921173	+ 8·7265471	+ 8·7032120	·0044	− ·0004	− ·01245	·0059
(m) 60	− 9·6193347	− 9·6226196	+ 9·0900786	+ 9·0789159	− ·0113	·0002	·00295	·0022
(n) 65	− 9·3561532	− 9·3692384	+ 9·2021533	+ 9·1961542	− ·0163	− ·0017	·0153	− ·0040
(o) 70	+ 8·7520763	+ 8·6819804	+ 9·1451111	+ 9·1428858	− ·01655	− ·0056	·00535	− ·0008
(p) 75	+ 9·5880195	+ 9·5798744	+ 8·8148047	+ 8·8170986	− ·0090	− ·0040	− ·0019	·0068
(q) 80	+ 9·8439487	+ 9·8413307	− 8·5646950	− 8·5580977	·0096	·0043	− ·0488	·0087
(r) 85	+ 9·9636575	+ 9·9630790	− 9·0927066	− 9·0919349	− ·0071	·0131	− ·0489	− ·0020
(s) 90	+ 0·0000000	+ 0·0000000	− 9·1983677	− 9·1983677	·0004	·0191	− ·0614	− ·0007

Co-latitude	For Y				Absolute term for g, $\frac{1}{2}(b_7-b'_7)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_7-a'_7)$ 1845	1880
(a) 5°	4·5169430	+ 4·4924116	+ 4·4412072	+ 4·4109037				
(b) 10	6·3074021	+ 6·2834338	+ 6·2195411	+ 6·1899332				
(c) 15	7·3376555	+ 7·3146060	+ 7·2292631	+ 7·2007902	·00905		− ·02035	
(d) 20	8·0502741	+ 8·0284706	+ 7·9124198	+ 7·8854861	·01805		− ·0112	
(e) 25	8·5837802	+ 8·5635110	+ 8·4067117	+ 8·3816733	·0205	·0182	− ·00915	·0130
(f) 30	8·9996625	+ 8·9811684	+ 8·7723850	+ 8·7495393	·0264	·0182	− ·01075	·0047
(g) 35	9·3304580	+ 9·3139248	+ 9·0401025	+ 9·0196792	·01805	·0106	·0016	·0036
(h) 40	9·5952604	+ 9·5808138	+ 9·2260416	+ 9·2081957	·0125	·0045	·00865	·0115
(i) 45	9·8059643	+ 9·7936659	+ 9·3372993	+ 9·3221071	− ·0006	·0039	·0088	·0168
(k) 50	9·9701325	+ 9·9599785	+ 9·3727922	+ 9·3602490	− ·00255	·0040	·0074	·0182
(l) 55	0·0923877	+ 0·0843094	+ 9·3190232	+ 9·3090442	− ·0078	·0150	·0053	·0203
(m) 60	0·1750156	+ 0·1688818	+ 9·1280562	+ 9·1204791	− ·0104	·0209	·00195	·0235
(n) 65	0·2179843	+ 0·2136053	+ 8·4916125	+ 8·4862031	− ·00745	0195	·00285	·0261
(o) 70	0·2182097	+ 0·2153434	− 8·8448213	− 8·8412806	·0040	·0161	·0027	·0235
(p) 75	0·1671688	+ 0·1655283	− 9·1299187	− 9·1278921	− ·00185	·0118	− ·00155	·0212
(q) 80	0·0430127	+ 0·0422745	− 9·1507510	− 9·1498391	− ·0124	·0091	− ·0078	·0146
(r) 85	9·7728140	+ 9·7726281	− 8·9500739	− 8·9498442	− ·0106	·0044	− ·0018	·0057
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000				

Co-latitude	For Z				Absolute term for g, $\frac{1}{2}(a_7-a'_7)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_7-b'_7)$ 1845	1880
(a) 5°	3·5714985	− 3·4910641	+ 3·5825037	− 3·5069770				
(b) 10	5·6612074	− 5·5814606	+ 5·6600835	− 5·5853834				
(c) 15	6·8645836	− 6·7859588	+ 6·8429218	− 6·7695709	− ·0168		·0275	
(d) 20	7·6979821	− 7·6208789	+ 7·6468489	− 7·5753294	− ·0309		− ·00415	
(e) 25	8·3230453	− 8·2478160	+ 8·2326847	− 8·1634232	− ·01895	− ·0097	·0194	− ·0163
(f) 30	8·8115564	− 8·7384951	+ 8·6709694	− 8·6043253	− ·00955	·0007	·0091	− ·0190
(g) 35	9·2015389	− 9·1308726	+ 8·9978503	− 8·9341068	·00155	·0017	·0091	·0004
(h) 40	9·5153553	− 9·4472379	+ 9·2327706	− 9·1721298	− ·0346	− ·0128	·0208	− ·0281
(i) 45	9·7670005	− 9·7015074	+ 9·3849215	− 9·3275053	− ·04985	− ·0365	− ·0054	− ·0145
(k) 50	9·9654619	− 9·9025890	+ 9·4546306	− 9·4004948	− ·0497	− ·0542	− ·00455	− ·0074
(l) 55	0·1163671	− 0·0560305	+ 9·4293666	− 9·3785537	− ·0415	− ·0308	− ·0086	·0082
(m) 60	0·2227295	− 0·1647691	+ 9·2617608	− 9·2145580	− ·00745	− ·0310	·0147	·0259
(n) 65	0·2850535	− 0·2292376	+ 8·6416794	− 8·6032741	·0177	− ·0620	·01385	− ·0067
(o) 70	0·3006528	− 0·2466859	− 9·0156988	+ 8·9681157	·03375	− ·0418	− ·0211	·0065
(p) 75	0·2612975	− 0·2088287	− 9·3115824	+ 9·2676754	·05565	− ·0115	− ·01195	− ·0022
(q) 80	0·1453484	− 0·0939825	− 9·3404015	+ 9·2982701	·0208	− ·0155	− ·0550	·0179
(r) 85	9·8800197	− 9·8293289	− 9·1445115	+ 9·1033578	·00805	·0069	− ·0034	·0144
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000				

Co-latitude	For X: $g_8{}^8$ or $h_8{}^8$	$g_{-8}{}^8$ or $h_{-8}{}^8$	$g_{10}{}^8$ or $h_{10}{}^8$	$g_{-10}{}^8$ or $h_{-10}{}^8$	Absolute term for g, $\frac{1}{2}(a_8-a'_8)$ 1845	Absolute term for g, $\frac{1}{2}(a_8-a'_8)$ 1880	Absolute term for h, $\frac{1}{2}(b_8-b'_8)$ 1845	Absolute term for h, $\frac{1}{2}(b_8-b'_8)$ 1880
(a) 5°	− 3·5180919	− 3·4936069	− 3·4930578	− 3·4628129				
(b) 10	− 5·6077846	− 5·5840045	− 5·5692329	− 5·5398594				
(c) 15	− 6·8111342	− 6·7885026	− 6·7495916	− 6·7216452	·00285		− ·0073	
(d) 20	− 7·6444964	− 7·6234227	− 7·5497265	− 7·5237276	·0158		− ·0021	
(e) 25	− 8·2695151	− 8·2503596	− 8·1300463	− 8·1064698	− ·0066	·0094	− ·00675	− ·0065
(f) 30	− 8·7579747	− 8·7410387	− 8·5603896	− 8·5396611	− ·0037	·0097	− ·0108	− ·0107
(g) 35	− 9·1479001	− 9·1334165	− 8·8756559	− 8·8581591	− ·00795	·0083	·00175	− ·0023
(h) 40	− 9·4616555	− 9·4497815	− 9·0929033	− 9·0790106	− ·00535	·0010	·0056	·0001
(i) 45	− 9·7132381	− 9·7040513	− 9·2161940	− 9·2063661	− ·01205	− ·0030	·0156	− ·0017
(k) 50	− 9·9116369	− 9·9051328	− 9·2324995	− 9·2276369	− ·0018	− ·0034	·01525	− ·0021
(l) 55	− 0·0624814	− 0·0585744	− 9·0794081	− 9·0828638	·0001	− ·0041	·00905	− ·0058
(m) 60	− 0·1687866	− 0·1673128	− 7·9033505	− 8·0658274	− ·0063	·0081	·02235	− ·0013
(n) 65	− 0·2310590	− 0·2317814	+ 9·1521118	+ 9·1371587	− ·01535	·0042	·04105	− ·0035
(o) 70	− 0·2466138	− 0·2492295	+ 9·4502854	+ 9·4440495	·0019	− ·0055	·0296	− ·0026
(p) 75	− 0·2072224	− 0·2113725	+ 9·5530208	+ 9·5505546	·02205	·0014	·02935	− ·0073
(q) 80	− 0·0912467	− 0·0965263	+ 9·5181379	+ 9·5178662	·0381	·0082	·01815	− ·0036
(r) 85	− 9·8259017	− 9·8318727	+ 9·2958361	+ 9·2967766	·0186	− ·0015	·0060	− ·0042
(s) 90	0·0000000	0·0000000	0·0000000	0·0000000				

Co-latitude	For Y				Absolute term for g, $\frac{1}{2}(b_8+b'_8)$ 1845	Absolute term for g, $\frac{1}{2}(b_8+b'_8)$ 1880	Absolute term for h, $-\frac{1}{2}(a_8+a'_8)$ 1845	Absolute term for h, $-\frac{1}{2}(a_8+a'_8)$ 1880
(a) 5°	3·5197951	+ 3·4952637	+ 3·4956569	+ 3·4653534				
(b) 10	5·6146213	+ 5·5906530	+ 5·5797255	+ 5·5501176				
(c) 15	6·8266084	+ 6·8035589	+ 6·7735753	+ 6·7451024	·01375		·0071	
(d) 20	7·6722403	+ 7·6504368	+ 7·5933599	+ 7·5664262	·0001		·00405	
(e) 25	8·3133531	+ 8·2930839	+ 8·2004135	+ 8·1753751	− ·00805	− ·0004	− ·00625	− ·0091
(f) 30	8·8220022	+ 8·8035081	+ 8·6660683	+ 8·6432226	− ·00425	·0028	− ·01405	− ·0047
(g) 35	9·2365852	+ 9·2200520	+ 9·0276984	+ 9·0072751	·0008	·0071	− ·00985	·0028
(h) 40	9·5799742	+ 9·5655276	+ 9·3067203	+ 9·2888744	·0055	·0105	·0079	·0105
(i) 45	9·8668646	+ 9·8545662	+ 9·5157141	+ 9·5005219	·0075	·0099	·00585	·0102
(k) 50	0·1072193	+ 0·0970653	+ 9·6614478	+ 9·6489046	− ·0083	·0050	·01545	·0057
(l) 55	0·3080613	+ 0·2999830	+ 9·7458236	+ 9·7358446	·0024	·0070	·01285	·0043
(m) 60	0·4744766	+ 0·4683428	+ 9·7649333	+ 9·7573562	·0045	·0053	·00785	·0020
(n) 65	0·6102121	+ 0·6058331	+ 9·7042124	+ 9·6988030	− ·00325	·0082	·0036	− ·0007
(o) 70	0·7180442	+ 0·7151779	+ 9·5175137	+ 9·5139730	− ·00735	·0069	− ·0088	·0008
(p) 75	0·8000167	+ 0·7983762	+ 8·9312956	+ 8·9292690	− ·0109	·0026	− ·0054	·0046
(q) 80	0·8575943	+ 0·8568561	− 9·2168923	− 9·2159804	− ·00995	·0010	− ·00705	·0152
(r) 85	0·8917626	+ 0·8915767	− 9·5463056	− 9·5460759	− ·0001	− ·0032	− ·00495	·0153
(s) 90	0·9030900	+ 0·9030900	− 9·6243364	− 9·6243364	·0101	− ·0040	− ·0114	·0142

Co-latitude	For Z				Absolute term for g, $\frac{1}{2}(a_8+a'_8)$ 1845	Absolute term for g, $\frac{1}{2}(a_8+a'_8)$ 1880	Absolute term for h, $\frac{1}{2}(b_8+b'_8)$ 1845	Absolute term for h, $\frac{1}{2}(b_8+b'_8)$ 1880
(a) 5°	2·5166789	− 2·4355597	+ 2·5792243	− 2·5031428				
(b) 10	4·9107550	− 4·8303232	+ 4·9625406	− 4·8872820				
(c) 15	6·2958650	− 6·2165551	+ 6·3295097	− 6·2555942	·00645		·01785	
(d) 20	7·2622770	− 7·1844885	+ 7·2700693	− 7·1979755	− ·00125		·03135	
(e) 25	7·9949473	− 7·9190322	+ 7·9686736	− 7·8988238	− ·01745	− ·0316	·01925	·0082
(f) 30	8·5762254	− 8·5024781	+ 8·5069494	− 8·4396970	− ·0257	− ·0416	·01145	·0026
(g) 35	9·0499957	− 8·9786432	+ 8·9277570	− 8·8633761	− ·02105	− ·0062	·01275	− ·0157
(h) 40	9·4423990	− 9·3735950	+ 9·2557807	− 9·1944593	− ·00455	− ·0047	·0158	− ·0144
(i) 45	9·7702311	− 9·7040511	+ 9·5057007	− 9·4475350	− ·00105	·0037	·00775	− ·0046
(k) 50	0·0448794	− 9·9813192	+ 9·6857081	− 9·6307025	·0161	·0125	− ·01015	− ·0186
(l) 55	0·2743717	− 0·2133475	+ 9·7987060	− 9·7467779	·01885	− ·0007	− ·00905	·0008
(m) 60	0·4645218	− 0·4058735	+ 9·8415063	− 9·7924985	·00915	− ·0016	− ·0045	− ·0147
(n) 65	0·6196129	− 0·5631088	+ 9·8000589	− 9·7537755	·02985	·0088	·01635	− ·0069
(o) 70	0·7428193	− 0·6881638	+ 9·6285234	− 9·5848828	·01815	− ·0055	·00135	− ·0050
(p) 75	0·8364775	− 0·7833199	+ 9·0524467	− 9·0138192	·0081	·0004	− ·02975	·0058
(q) 80	0·9022623	− 0·8502076	− 9·3493657	+ 9·3055216	·04575	·0077	− ·06055	− ·0059
(r) 85	0·9413006	− 0·8899209	− 9·6830778	+ 9·6412384	·01475	·0145	− ·0363	·0039
(s) 90	0·9542425	− 0·9030900	− 9·7626391	+ 9·7212464	− ·0008	·0240	− ·0255	·0073

Co-latitude		For X: $g_9^{\ 8}$ or $h_9^{\ 8}$	$g_{-9}^{\ 8}$ or $h_{-9}^{\ 8}$	Absolute term for g, $\frac{1}{2}(a_8+a'_8)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_8+b'_8)$ 1845	1880
(a)	5°	− 3·5174329	− 3·4900685				
(b)	10	− 5·6007321	− 5·5741552				
(c)	15	− 6·7932427	− 6·7679530	− ·00465		− ·0225	
(d)	20	− 7·6110218	− 7·5874827	·0040		·0095	
(e)	25	− 8·2152453	− 8·1938715	− ·0023	− ·0119	·00885	·0066
(f)	30	− 8·6769920	− 8·6581405	·0033	− ·0066	·0063	− ·0003
(g)	35	− 9·0332258	− 9·0171900	·00095	− ·0032	·00475	·0022
(h)	40	− 9·3046765	− 9·2916871	·00555	− ·0059	·0027	·0053
(i)	45	− 9·5027078	− 9·4929448	·00205	− ·0024	− ·0069	·0020
(k)	50	− 9·6317529	− 9·6253789	·0142	− ·0013	− ·00505	·0023
(l)	55	− 9·6886616	− 9·6859201	·0418	·0003	− ·01325	·0032
(m)	60	− 9·6563151	− 9·6578627	·0368	− ·0043	− ·01535	·0011
(n)	65	− 9·4708712	− 9·4796447	·02755	·0065	·01095	− ·0046
(o)	70	− 8·3407921	− 8·4802142	·0321	− ·0097	− ·0010	− ·0042
(p)	75	+ 9·5084844	+ 9·4981929	·04925	− ·0040	− ·00295	− ·0037
(q)	80	+ 9·8202790	+ 9·8172676	·0089	·0117	·00955	·0075
(r)	85	+ 9·9585824	+ 9·9579324	·0269	·0060	·0177	·0076
(s)	90	+ 0·0000000	0·0000000	·0176	·0117	·0128	·0067
		For Y		Absolute term for g, $\frac{1}{2}(b_8-b'_8)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_8-a'_8)$ 1845	1880
(a)	5°	3·5195602	+ 3·4921428				
(b)	10	5·6092944	+ 5·5825063				
(c)	15	6·8127120	+ 6·7869508	·01515		·0061	
(d)	20	7·6461664	+ 7·6217978	− ·0009		− ·00565	
(e)	25	8·2712988	+ 8·2486450	·00605	·0021	− ·00425	− ·0010
(f)	30	8·7598900	+ 8·7392201	·01175	·0099	− ·00115	− ·0029
(g)	35	9·1499611	+ 9·1314829	·0126	·0105	− ·00415	− ·0052
(h)	40	9·4638716	+ 9·4477253	·0102	·0090	− ·0004	− ·0050
(i)	45	9·7156140	+ 9·7018687	·0043	·0033	− ·00265	− ·0025
(k)	50	9·9141727	+ 9·9028241	− ·0013	·0043	·00995	− ·0038
(l)	55	0·0651720	+ 0·0561434	·0036	·0023	·00175	·0002
(m)	60	0·1716225	+ 0·1647670	− ·0028	·0059	− ·00045	·0061
(n)	65	0·2340262	+ 0·2291320	− ·00165	·0063	− ·0049	·0110
(o)	70	0·2496943	+ 0·2464908	·00495	·0090	− ·0124	·0174
(p)	75	0·2103947	+ 0·2085611	·0062	·0132	− ·0162	·0177
(q)	80	0·0944868	+ 0·0936617	·01275	·0134	− ·01815	·0105
(r)	85	9·8291831	+ 9·8289753	− ·0010	·0097	− ·01195	·0045
(s)	90	0·0000000	0·0000000				
		For Z		Absolute term for g, $\frac{1}{2}(a_8-a'_8)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_8-b'_8)$ 1845	1880
(a)	5°	2·5619451	− 2·4839152				
(b)	10	4·9509283	− 4·8736531				
(c)	15	6·3274676	− 6·2514236	·01285		·00845	
(d)	20	7·2817003	− 7·2073261	·00095		·02315	
(e)	25	7·9983878	− 7·9260700	− ·01335	·0158	·00445	− ·0078
(f)	30	8·5596056	− 8·4896668	− ·0083	·0118	·02165	− ·0229
(g)	35	9·0088611	− 8·9415507	·00825	− ·0018	− ·00035	·0017
(h)	40	9·3717829	− 9·3072695	·03905	·0100	− ·0206	·0040
(i)	45	9·6644639	− 9·6028303	·02635	·0170	− ·00945	·0019
(k)	50	9·8973131	− 9·8385547	·0430	·0079	·00575	·0092
(l)	55	0·0769598	− 0·0209845	·03755	·0107	·01115	·0140
(m)	60	0·2071422	− 0·1537744	·02965	− ·0014	·0248	·0053
(n)	65	0·2888990	− 0·2378843	·03375	− ·0041	·00555	·0065
(o)	70	0·3199391	− 0·2709533	·03295	− ·0150	− ·00245	·0158
(p)	75	0·2923235	− 0·2449816	·0426	− ·0098	− ·00305	·0176
(q)	80	0·1846214	− 0·1384898	− ·01795	− ·0055	·01415	·0024
(r)	85	9·9241870	− 9·8787961	·01165	·0021	·0073	− ·0075
(s)	90	0·0000000	0·0000000				

Co-latitude	For X g_9^9 or h_9^9	g_{-9}^9 or h_{-9}^9	Absolute term for g, $\frac{1}{2}(a_9-a'_9)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_9-b'_9)$ 1845	1880
(a) 5°	− 2·5138700	− 2·4864997				
(b) 10	− 4·9028385	− 4·8762374				
(c) 15	− 6·2793538	− 6·2540078	− ·0022		− ·00245	
(d) 20	− 7·2335540	− 7·2099104	·00955		·00585	
(e) 25	− 7·9502015	− 7·9286541	− ·0004	+ ·0067	·0002	·0037
(f) 30	− 8·5113729	− 8·4922509	− ·0028	− ·0025	·0066	·0050
(g) 35	− 8·9605771	− 8·9441350	·0042	·0042	− ·00405	− ·0008
(h) 40	− 9·3234442	− 9·3098536	·0048	·0010	− ·0121	− ·0013
(i) 45	− 9·6160687	− 9·6054146	·01205	·0009	− ·0099	·0001
(k) 50	− 9·8488615	− 9·8411389	− ·00545	− ·0010	− ·0282	·0011
(l) 55	− 0·0284535	− 0·0235688	·00325	·0005	·01575	·0023
(m) 60	− 0·1585846	− 0·1563585	− ·00105	·0046	·0060	·0011
(n) 65	− 0·2402950	− 0·2404686	− ·0036	− ·0032	− ·0008	− ·0023
(o) 70	− 0·2712950	− 0·2735374	− ·00415	− ·0019	·00095	·0001
(p) 75	− 0·2436469	− 0·2475658	·01735	·0025	− ·00475	·0003
(q) 80	− 0·1359210	− 0·1410740	·0128	− ·0045	− ·02275	− ·0003
(r) 85	− 9·8754717	− 9·8813803	·0203	− ·0048	− ·00795	− ·0021
(s) 90	0·0000000	0·0000000				

Co-latitude	For Y		Absolute term for g, $\frac{1}{2}(b_9+b'_9)$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_9+a'_9)$ 1845	1880
(a) 5°	2·5155729	+ 2·4881555				
(b) 10	4·9096740	+ 4·8228859				
(c) 15	6·2948253	+ 6·2690641	·0141		− ·0083	
(d) 20	7·2612931	+ 7·2369245	·0102		− ·00095	
(e) 25	7·9940322	+ 7·9713784	− ·00305	·0061	·00525	·0166
(f) 30	8·5753902	+ 8·5547203	− ·00315	·0011	·0022	·0033
(g) 35	9·0492487	+ 9·0307705	− ·00015	− ·0020	·00175	·0076
(h) 40	9·4417460	+ 9·4255997	− ·0016	·0002	− ·0014	·0092
(i) 45	9·7696748	+ 9·7559295	·0010	− ·0011	·0003	·0080
(k) 50	0·0444200	+ 0·0330714	·00335	− ·0011	·0115	·0079
(l) 55	0·2740060	+ 0·2649774	·00275	·0031	− ·00645	·0035
(m) 60	0·4642440	+ 0·4573885	·00325	·0046	− ·01825	·0018
(n) 65	0·6194145	+ 0·6145203	·01035	·0065	− ·01805	·0027
(o) 70	0·7426893	+ 0·7394858	·0051	·0128	− ·01505	− ·0011
(p) 75	0·8364031	+ 0·8345695	·0003	·0125	− ·0161	− ·0004
(q) 80	0·9022289	+ 0·9014038	− ·0060	·0134	− ·0155	·0077
(r) 85	0·9412921	+ 0·9410843	− ·00555	·0124	− ·01295	·0169
(s) 90	0·9542425	+ 0·9542425	− ·0043	·0067	− ·0129	·0108

Co-latitude	For Z		Absolute term for g, $\frac{1}{2}(a_9+a'_9)$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_9+b'_9)$ 1845	1880
(a) 5°	1·5070936	− 1·4284514				
(b) 10	4·2004438	− 4·1225561				
(c) 15	5·7587168	− 5·6820603	·02545		− ·0053	
(d) 20	6·8459631	− 6·7709762	·01805		− ·0183	
(e) 25	7·6702576	− 7·5973267	·0129	·0014	− ·0142	− ·0098
(f) 30	8·3242424	− 8·2536903	·00445	− ·0055	·0066	·0000
(g) 35	8·8572857	− 8·7893617	− ·0059	− ·0098	− ·01495	− ·0111
(h) 40	9·2987946	− 9·2336672	− ·0238	− ·0139	− ·03455	− ·0097
(i) 45	9·6676623	− 9·6054145	− ·01565	− ·0056	− ·0153	− ·0028
(k) 50	9·9766983	− 9·9173254	− ·01735	·0083	·0018	− ·0125
(l) 55	0·2349320	− 0·1783419	− ·02635	·0056	·00585	·0042
(m) 60	0·4489022	− 0·3949193	− ·0214	·0042	·01335	·0014
(n) 65	0·6234261	− 0·5717960	− ·0181	− ·0042	− ·0064	·0028
(o) 70	0·7620740	− 0·7124717	− ·0152	·0139	− ·01935	·0083
(p) 75	0·8674711	− 0·8195133	− ·02885	·0139	·00055	− ·0028
(q) 80	0·9415028	− 0·8947553	− ·03875	·0139	− ·01435	− ·0042
(r) 85	0·9854354	− 0·9394285	− ·04405	·0168	·0190	·0056
(s) 90	1·0000000	− 0·9542425	− ·0618	·0083	·0274	·0111

Co-latitude	For X: g_{10}^{9} or h_{10}^{9}	g_{-10}^{9} or h_{-10}^{9}	Absolute term for g, $\frac{1}{2}(a_9+a'_9)$ 1845	Absolute term for g, $\frac{1}{2}(a_9+a'_9)$ 1880	Absolute term for h, $\frac{1}{2}(b_9+b'_9)$ 1845	Absolute term for h, $\frac{1}{2}(b_9+b'_9)$ 1880
(a) 5°	− 2·5132580	− 2·4830066				
(b) 10	− 4·8959782	− 4·8665777				
(c) 15	− 6·2619080	− 6·2338978	·0057		·00725	
(d) 20	− 7·2009081	− 7·1747878	·00425		− ·00115	
(e) 25	− 7·8973064	− 7·8735216	− ·0069	− ·0016	− ·0066	− ·0035
(f) 30	− 8·4325283	− 8·4114603	− ·0070	− ·0092	− ·0074	− ·0057
(g) 35	− 8·8491113	− 8·8310700	·0006	− ·0012	− ·01185	− ·0079
(h) 40	− 9·1712031	− 9·1564253	·0035	·0057	− ·0011	− ·0058
(i) 45	− 9·4125418	− 9·4011998	·00155	·0092	− ·0100	− ·0020
(k) 50	− 9·5795338	− 9·5717613	− ·00525	·0076	− ·0073	− ·0022
(l) 55	− 9·6712972	− 9·6672644	·00355	·0036	− ·02105	− ·0073
(m) 60	− 9·6750788	− 9·6752302	− ·01355	·0054	− ·0019	− ·0054
(n) 65	− 9·5438122	− 9·5500252	− ·0178	·0116	− ·0174	− ·0012
(o) 70	− 8·9602331	− 8·9969364	− ·01685	− ·0010	− ·02405	− ·0073
(p) 75	+ 9·4175680	+ 9·4043668	− ·02275	− ·0018	− ·03965	− ·0005
(q) 80	+ 9·7959040	+ 9·7924733	− ·0139	·0094	− ·03425	·0026
(r) 85	+ 9·9534798	+ 9·9527574	− ·0396	·0063	− ·05875	·0052
(s) 90	0·0000000	0·0000000	− ·0367	·0091	− ·0528	− ·0041

Co-latitude	For Y		Absolute term for g, $\frac{1}{2}(b_9-b'_9)$ 1845	Absolute term for g, $\frac{1}{2}(b_9-b'_9)$ 1880	Absolute term for h, $-\frac{1}{2}(a_9-a'_9)$ 1845	Absolute term for h, $-\frac{1}{2}(a_9-a'_9)$ 1880
(a) 5°	2·5153379	+ 2·4850344				
(b) 10	4·9043471	+ 4·8747392				
(c) 15	6·2809288	+ 6·2524559	− ·0046		− ·0091	
(d) 20	7·2352193	+ 7·2082856	·0022		− ·00185	
(e) 25	7·9519780	+ 7·9269396	− ·00005	− ·0054	− ·00345	− ·0019
(f) 30	8·5132780	+ 8·4904323	− ·00195	− ·0023	− ·0024	− ·0054
(g) 35	8·9626247	+ 8·9422014	− ·00205	− ·0029	·00115	·0037
(h) 40	9·3256434	+ 9·3077975	− ·0059	− ·0006	− ·0028	·0039
(i) 45	9·6184243	+ 9·6032321	− ·0048	− ·0015	·0020	·0028
(k) 50	9·8513734	+ 9·8388302	− ·00345	− ·0062	·0010	− ·0001
(l) 55	0·0311167	+ 0·0211377	− ·00385	− ·0070	− ·00435	− ·0038
(m) 60	0·1613900	+ 0·1538129	− ·00815	− ·0055	·00195	− ·0046
(n) 65	0·2432286	+ 0·2378192	− ·01155	− ·0013	·00145	− ·0110
(o) 70	0·2743394	+ 0·2707987	− ·0061	− ·0026	− ·00765	− ·0063
(p) 75	0·2467811	+ 0·2447545	− ·0059	·0056	− ·0098	− ·0017
(q) 80	0·1391213	+ 0·1382094	− ·0073	·0065	− ·0154	·0003
(r) 85	9·8787126	+ 9·8784829	·00195	·0042	− ·01245	·0010
(s) 90	0·0000000	0·0000000				

Co-latitude	For Z		Absolute term for g, $\frac{1}{2}(a_9-a'_9)$ 1845	Absolute term for g, $\frac{1}{2}(a_9-a'_9)$ 1880	Absolute term for h, $\frac{1}{2}(b_9-b'_9)$ 1845	Absolute term for h, $\frac{1}{2}(b_9-b'_9)$ 1880
(a) 5°	1·5480152	− 1·4713796				
(b) 10	4·2362729	− 4·1604586				
(c) 15	5·7859803	− 5·7115013	·03735		− ·0080	
(d) 20	6·8610427	− 6·7883866	·02395		− ·0259	
(e) 25	7·6693549	− 7·5989371	·0160	·0014	− ·0064	·0070
(f) 30	8·3032797	− 8·2354515	− ·00325	·0084	·0073	·0083
(g) 35	8·8118092	− 8·7468419	− ·0069	·0042	·00805	·0028
(h) 40	9·2238367	− 9·1619143	− ·0063	·0056	·02385	·0042
(i) 45	9·5575538	− 9·4987663	− ·01275	− ·0056	·0232	·0056
(k) 50	9·8247913	− 9·7691334	·00355	·0000	·0046	·0125
(l) 55	0·0331797	− 9·9805515	− ·00325	− ·0056	·00045	·0236
(m) 60	0·1871829	− 0·1373928	·0078	− ·0070	− ·01685	·0097
(n) 65	0·2883728	− 0·2411441	·0032	− ·0042	− ·0246	− ·0028
(o) 70	0·3348540	− 0·2898338	·0213	− ·0028	− ·01995	·0111
(p) 75	0·3189783	− 0·2757476	·02425	− ·0083	− ·02935	·0167
(q) 80	0·2195235	− 0·1776101	·01565	− ·0111	− ·00905	·0097
(r) 85	9·9639834	− 9·9228765	·00565	− ·0001	·0013	·0001
(s) 90	0·0000000	0·0000000				

Co-latitude	For X g_{10}^{10} or h_{10}^{10}	g_{-10}^{10} or h_{-10}^{10}	Absolute term for g, $\frac{1}{2}(a_{10} - a'_{10})$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_{10} - b'_{10})$ 1845	1880
(a) 5°	– 1·5042529	– 1·4739962				
(b) 10	– 4·1924972	– 4·1630753				
(c) 15	– 5·7421779	– 5·7141180	·0043		·00655	
(d) 20	– 6·8172157	– 6·7910032	– ·00605		·00045	
(e) 25	– 7·6254915	– 7·6015537	·0015	– ·0024	·0072	– ·0065
(f) 30	– 8·2593740	– 8·2380681	·0051	– ·0032	·0028	·0060
(g) 35	– 8·7678564	– 8·7494585	·0022	– ·0016	·0040	·0011
(h) 40	– 9·1798345	– 9·1645308	·0073	·0020	·01565	– ·0006
(i) 45	– 9·5135004	– 9·5013830	·0081	·0021	·01585	– ·0020
(k) 50	– 9·7806864	– 9·7717500	·0077	.0018	·01165	– ·0018
(l) 55	– 9·9890252	– 9·9831682	·02865	·0007	·0105	·0037
(m) 60	– 0·1429816	- 0·1400094	·00725	– ·0006	·0043	– ·0017
(n) 65	– 0·2441292	– 0·2437608	·0129	·0007	·0092	– ·0014
(o) 70	– 0·2905739	– 0·2924503	·0201	·0018	·0290	– ·0034
(p) 75	– 0·2746688	– 0·2783642	·01515	– ·0056	·0272	·0012
(q) 80	– 0·1751921	– 0·1802267	·00365	·0012	·02245	·0022
(r) 85	– 9·9196387	– 9·9254930	– ·00765	– ·0044	·00465	– ·0017
(s) 90	0·0000000	0·0000000				

Co-latitude	For Y		Absolute term for g, $\frac{1}{2}(b_{10} + b'_{10})$ 1845	1880	Absolute term for h, $-\frac{1}{2}(a_{10} + a'_{10})$ 1845	1880
(a) 5°	1·5059555	+ 1·4756520				
(b) 10	4·1993317	+ 4·1697238				
(c) 15	5·7576472	+ 5·7291743	– ·02195		·0054	
(d) 20	6·8449510	+ 6·8180173	– ·00445		– ·00825	
(e) 25	7·6693164	+ 7·6442780	– ·00065	– ·0004	·00525	·0006
(f) 30	8·3233832	+ 8·3005375	– ·0018	·0032	·0053	·0006
(g) 35	8·8565173	+ 8·8360940	– ·00575	·0012	·00685	– ·0040
(h) 40	9·2981228	+ 9·2802769	– ·0006	·0032	– ·00145	– ·0078
(i) 45	9·6670901	+ 9·6518979	– ·00575	·0041	– ·0013	– ·0052
(k) 50	9·9762257	+ 9·9636825	·00595	·0033	·0013	– ·0031
(l) 55	0·2345558	+ 0·2245768	– ·00075	·0021	– ·00135	– ·0019
(m) 60	0·4486165	+ 0·4410394	·00255	·0019	·00135	– ·0033
(n) 65	0·6232219	+ 0·6178125	·0040	– ·0027	·00365	– ·0049
(o) 70	0·7619394	+ 0·7583987	– ·0006	·0039	– ·0023	– ·0033
(p) 75	0·8673945	+ 0·8653679	·00035	·0034	– ·00275	– ·0040
(q) 80	0·9414684	+ 0·9405565	– ·00155	·0054	– ·0059	– ·0021
(r) 85	0·9854267	+ 0·9851970	·0074	– ·0010	– ·00805	– ·0029
(s) 90	1·0000000	+ 1·0000000	·0087	– ·0043	– ·0085	– ·0040

Co-latitude	For Z		Absolute term for g, $\frac{1}{2}(a_{10} + a'_{10})$ 1845	1880	Absolute term for h, $\frac{1}{2}(b_{10} + b'_{10})$ 1845	1880
(a) 5°	0·4931373	– 0·4159480				
(b) 10	3·4857621	– 3·4093940				
(c) 15	5·2171984	– 5·1421705	– ·0210		– ·0189	
(d) 20	6·4252793	– 6·3520690	– ·0160		– ·0091	
(e) 25	7·3411984	– 7·2702262	– ·00205	·0096	– ·0038	·0042
(f) 30	8·0678903	– 7·9995075	·02125	·0068	·00265	·0157
(g) 35	8·6602070	– 8·5946852	·00725	·0245	·01735	·0189
(h) 40	9·1508219	– 9·0[illegible]83444	·0219	·0219	·0210	·0023
(i) 45	9·5607258	– 9·5013829	·01925	·0120	·01195	·0026
(k) 50	9·9041500	– 9·8479365	·0143	·0002	·0104	– ·0033
(l) 55	0·1911256	– 0·1379413	·0260	– ·0035	·0171	·0024
(m) 60	0·4289164	– 0·3785701	·02255	·0080	·0267	– ·0006
(n) 65	0·6228733	– 0·5750882	·03825	·0062	·0134	– ·0152
(o) 70	0·7769614	– 0·7313846	·03255	·0057	·0026	·0016
(p) 75	0·8940994	– 0·8503117	·01225	+ ·0178	– ·0099	– ·0041
(q) 80	0·9763783	– 0·9339080	·01255	·0083	– ·0268	– ·0004
(r) 85	1·0252054	– 0·9835413	·0316	·0157	– ·02455	·0028
(s) 90	1·0413927	– 1·0000000	·0275	·0089	– ·0240	– ·0016

TABLE OF EQUATIONS OF CONDITION MULTIPLIED BY THE

To form the following table of Equations each equation of the previous table for a given latitude is multiplied by the square root of the weight (w) of the observations for that belt of latitude, and the order of the equations is reversed so as to start from the Equator, to which the equation (s) corresponds. The signs and the logarithms of the coefficients of g_n^m or h_n^m are given in the tables. Also the signs and the logarithms of the products

$m = 0$. n ODD.

Co-latitude	For X g_1^0 or h_1^0	g_{-1}^0 or h_{-1}^0	g_3^0 or h_3^0	g_{-3}^0 or h_{-3}^0	g_5^0 or h_5^0	g_{-5}^0 or h_{-5}^0
(s) 90°	9·8480308	+ 9·8480308	− 9·6261821	− 9·6261821	+ 9·2247815	+ 9·2247815
(r) 85	9·9961826	+ 9·9960842	− 9·7578121	− 9·7575784	+ 9·3254981	+ 9·3251195
(q) 80	9·9890513	+ 9·9886605	− 9·6975235	− 9·6965409	+ 9·1460773	+ 9·1442273
(p) 75	9·9770239	+ 9·9761556	− 9·5815797	− 9·5791089	+ 8·5781217	+ 8·5779798
(o) 70	9·9598804	+ 9·9583637	− 9·3655723	− 9·3598813	− 8·8621623	− 8·8659824
(n) 65	9·9372940	+ 9·9349774	− 8·7944658	− 8·7709545	− 9·2255272	− 9·2248182
(m) 60	9·9088108	+ 9·9055671	+ 9·0546871	+ 9·0646740	− 9·3565590	− 9·3532785
(l) 55	9·8738178	+ 9·8695473	+ 9·4477433	+ 9·4492357	− 9·3754704	− 9·3696551
(k) 50	9·8314919	+ 9·8261260	+ 9·6280652	+ 9·6264054	− 9·2923044	− 9·2834875
(i) 45	9·7807217	+ 9·7742251	+ 9·7279859	+ 9·7240848	− 9·0489269	− 9·0351832
(h) 40	9·7199798	+ 9·7123513	+ 9·7791723	+ 9·7733932	− 7·4978518	− 7·1875715
(g) 35	9·6471089	+ 9·6383820	+ 9·7929763	+ 9·7855101	+ 9·0294262	+ 9·0238232
(f) 30	9·5573500	+ 9·5491803	+ 9·7730641	+ 9·7640850	+ 9·2906553	+ 9·2801788
(e) 25	9·4505475	+ 9·4398556	+ 9·7184303	+ 9·7081193	+ 9·3870560	+ 9·3739686
(d) 20	9·3136885	+ 9·3021905	+ 9·6232460	+ 9·6118035	+ 9·3881693	+ 9·3732123
(c) 15	9·1329055	+ 9·1207531	+ 9·4736098	+ 9·4612585	+ 9·3026988	+ 9·2863480
(b) 10	8·8735063	+ 8·8608715	+ 9·2358394	+ 9·2228227	+ 9·1064776	+ 9·0891426
(a) 5	8·4248063	+ 8·4118760	+ 8·7998842	+ 8·7864615	+ 8·6940295	+ 8·6761050
	For Z					
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·2347618	− 8·9380360	− 9·3092487	+ 9·1858796	+ 9·0743200	− 8·9960044
(q) 80	9·5319003	− 9·2349792	− 9·5898548	+ 9·4660906	+ 9·3239610	− 9·2450245
(p) 75	9·7014482	− 9·4042082	− 9 7309378	+ 9·6065184	+ 9·4080128	− 9·3280032
(o) 70	9·8171029	− 9·5194295	− 9·8046272	+ 9·6792944	+ 9·3864211	− 9·3047670
(n) 65	9·9018580	− 9 6036500	− 9·8312052	+ 9·7046950	+ 9·2457366	− 9·1613749
(m) 60	9·9658346	− 9·6670064	− 9·8160681	+ 9·6880866	+ 8·8362427	− 8·7438208
(l) 55	0·0142895	− 9·7147742	− 9·7556157	+ 9·6257703	− 8·7405584	+ 8·6689063
(k) 50	0·0501883	− 9·7499396	− 9·6336009	+ 9·5011601	− 9·2220882	+ 9·1400120
(i) 45	0·0752314	− 9·7742252	− 9·3958845	+ 9·2585189	− 9·3818831	+ 9·2962171
(h) 40	0·0903029	− 9·7885378	− 8·6298930	+ 8·4490832	− 9·4150341	+ 9·3265328
(g) 35	0·0956572	− 9·7931553	+ 9·2269176	− 9·1067577	− 9·3404464	+ 9·2490846
(f) 30	0·0909353	− 9·7877409	+ 9·5611957	− 9·4334013	− 9·1025090	+ 9·0070516
(e) 25	0·0750042	− 9·7711831	+ 9·7188471	− 9·5881502	− 7·5968651	+ 7·3182432
(d) 20	0·0454874	− 9·7411247	+ 9·7983118	− 9·6658492	+ 9·0796370	− 8·9905182
(c) 15	9·9975031	− 9·6927006	+ 9·8223793	− 9·6887084	+ 9·3325092	− 9·2399061
(b) 10	9·9196796	− 9·6145528	+ 9·7909935	− 9·6565083	+ 9·4055390	− 9·3113985
(a) 5	9·7752498	− 9·4699242	+ 9·6727639	− 9·5378029	+ 9·3380047	− 9·2430764

SQUARE ROOT OF THE WEIGHTS OF THE OBSERVATIONS.

$w^{\frac{1}{2}}x'_m$, $w^{\frac{1}{2}}y'_m$ and $w^{\frac{1}{2}}z'_m$, which are derived from the *absolute terms* for 1845 and 1880, are given in the tables.

The following is the type of these equations:

$$X'^m_n w^{\frac{1}{2}} g^m_n + X'^m_{n_1} w^{\frac{1}{2}} g^m_{n_1} + \&c. = w^{\frac{1}{2}} x'_m,$$

with similar equations for Y and Z.

g_7^0 or h_7^0	g_{-7}^0 or h_{-7}^0	g_9^0 or h_9^0	g_{-9}^0 or h_{-9}^0	Log $(w^{\frac{1}{2}}x'_0)$ 1845	Log $(w^{\frac{1}{2}}x'_0)$ 1880	
− 8·7596415	− 8·7596415	+ 8·2615864	+ 8·2615864	0·7152507	0·70865	(s)
− 8·8128096	− 8·8122642	+ 8·2463867	+ 8·2456379	0·8638796	0·85642	(r)
− 8·3525487	− 8·3484346	− 7·2470443	− 7·2604370	0·8549486	0·84662	(q)
+ 8·4640367	+ 8·4663273	− 8·3031472	− 8·3024661	0·8374469	0·82975	(p)
+ 8·8400830	+ 8·8386689	− 8·4101240	− 8·4066231	0·8137929	0·80532	(o)
+ 8·9114657	+ 8·9073370	− 8·1749797	− 8·1668569	0·7833724	0·77357	(n)
+ 8·7856961	+ 8·7779251	+ 7·7331870	+ 7·7325291	0·7461696	0·73649	(m)
+ 8·1968996	+ 8·1754615	+ 8·3515188	+ 8·3440116	0·7014411	0·69204	(l)
− 8·5593511	− 8·5560113	+ 8·4025010	+ 8·3901930	0·6511081	0·64008	(k)
− 8·8689360	− 8·8593493	+ 8·0750786	+ 8·0553963	0·5942336	0·57946	(i)
− 8·9140099	− 8·9005304	− 7·9650013	− 7·9562729	0·5256056	0·51175	(h)
− 8·7567930	− 8·7390482	− 8·3910181	− 8·3734562	0·4453318	0·43343	(g)
− 7·9603884	− 7·9205183	− 8·3871257	− 8·3650942	0·3478724	0·34143	(f)
+ 8·6318087	+ 8·6173050	− 7·9376865	− 7·9082885	0·2319849	0·23048	(e)
+ 8·8951186	+ 8·8763413	+ 8·1016159	+ 8·0804358	0·0923595		(d)
+ 8·9307785	+ 8·9098422	+ 8·4212249	+ 8·3956119	9·9156873		(c)
+ 8·8017019	+ 8·7793913	+ 8·3979044	+ 8·3703951			(b)
+ 8·4245084	+ 8·4014043	+ 8·0694545	+ 8·0409292			(a)
				Log $(w^{\frac{1}{2}}z'_0)$ 1845	Log $(w^{\frac{1}{2}}z'_0)$ 1880	
0·0000000	0·0000000	0·0000000	0·0000000			(s)
− 8·7198184	+ 8·6623296	+ 8·2995571	− 8·2540614	0·1082287	0·13998	(r)
− 8·9217977	+ 8·8634396	+ 8·4329829	− 8·3863310	0·4082565	0·41981	(q)
− 8·9054873	+ 8·8454828	+ 8·2271493	− 8·1777287	0·5750112	0·58172	(p)
− 8·6468229	+ 8·5831688	− 7·6294167	+ 7·5921714	0·6874334	0·69437	(o)
+ 8·0065966	− 7·9634591	− 8·3569485	+ 8·3078314	0·7691362	0·77383	(n)
+ 8·7835247	− 8·7228120	− 8·4254586	+ 8·3427932	0·8282037	0·83104	(m)
+ 8·9370585	− 8·8724612	− 8·1229647	+ 8·0632647	0 8741764	0·87069	(l)
+ 8·8838346	− 8·8154180	+ 7·9454921	− 7·8969455	0·9072220	0·89802	(k)
+ 8·5391423	− 8·4635778	+ 8·4036329	− 8·3455581	0·9261796	0·91615	(i)
− 8·3439103	+ 8·2844247	+ 8·4045491	− 8·3415345	0·9344018	0·92394	(h)
− 8·8442269	+ 8·7740280	+ 7·9425201	− 7·8700868	0·9340227	0·92405	(g)
− 8·9466883	+ 8·8726490	− 8·1386342	+ 8·0768930	0·9242089	0·91524	(f)
− 8·8424076	+ 8·7650494	− 8·4363393	+ 8·3686690	0·9025448	0·89666	(e)
− 8·3179836	+ 8·2330974	− 8·3567655	+ 8·2855343	0·8675103		(d)
+ 8·5700374	− 8·4952762	− 7·4574616	+ 7·3715422	0·8145359		(c)
+ 8·8949339	− 8·8166539	+ 8·2781321	− 8·2069701			(b)
+ 8·9109646	− 8·8315040	+ 8·4308079	− 8·3578653			(a)

On multiplying an equation in X given in this table by the coefficient of g_n^m in that equation, *i.e.* by $X'^m_n w^{\frac{1}{2}}$, we get an equation for g_n^m of the type

$$(X'^m_n)^2 wg_n^m + X'^m_n X'^m_{n_1} wg^m_{n_1} + \&c. = X'^m_n wx'_m.$$

Adding all such equations in X for the different belts of latitude

$m = 0$. n EVEN.

Co-latitude	g_2^0 or h_2^0	g_{-2}^0 or h_{-2}^0	g_4^0 or h_4^0	g_{-4}^0 or h_{-4}^0	g_6^0 or h_6^0	g_{-6}^0 or h_{-6}^0
	For X					
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·2331609	+ 9·2354845	− 9·1593128	− 9·1604518	+ 8·8719621	+ 8·8726198
(q) 80	9·5254695	+ 9·5274677	− 9·4281206	− 9·4287355	+ 9·1019415	+ 9·1018459
(p) 75	9·6868757	+ 9·6883423	− 9·5481442	− 9·5478769	+ 9·1467917	+ 9·1453559
(o) 70	9·7909334	+ 9·7916783	− 9·5887566	− 9·5872390	+ 9·0470935	+ 9·0433989
(n) 65	9·8604239	+ 9·8602780	− 9·5644188	− 9·5612026	+ 8·6875469	+ 8·6780346
(m) 60	9·9051718	+ 9·9039930	− 9·4664646	− 9·4608588	− 8 4651562	− 8·4720852
(l) 55	9·9300321	+ 9·9277093	− 9·2459940	− 9·2361336	− 8·9940451	− 8·9905264
(k) 50	9·9374302	+ 9·9338864	− 8·5283908	− 8·4888574	− 9·1414680	− 9·1342284
(i) 45	9·9283337	+ 9·9235294	+ 9·0581069	+ 9·0605771	− 9·1351758	− 9·1246193
(h) 40	9·9026175	+ 9·8965508	+ 9·3928568	+ 9·3883295	− 8·9688665	− 8·9542098
(g) 35	9·8591068	+ 9·8518146	+ 9·5392698	+ 9·5316256	− 8·3270177	− 8·2956004
(f) 30	9·7953376	+ 9·7868940	+ 9·6004526	+ 9·5905849	+ 8·7594589	+ 8·7506738
(e) 25	9·7069038	+ 9·6974184	+ 9·6007250	+ 9·5890788	+ 9·0783525	+ 9·0640286
(d) 20	9·5859387	+ 9·5755531	+ 9·5441684	+ 9·5310798	+ 9·1707789	+ 9·1538861
(c) 15	9·4172631	+ 9·4061465	+ 9·4216117	+ 9·4073928	+ 9·1358613	+ 9·1172493
(b) 10	9·1663817	+ 9·1547264	+ 9·2018773	+ 9·1868421	+ 8·9694526	+ 8·9496721
(a) 5	8·7227420	+ 8·7107566	+ 8·7762846	+ 8·7607547	+ 8·5730750	+ 8·5526052
	For Z					
(s) 90°	− 9·8480308	+ 9·6719395	+ 9·4800540	− 9·3831440	− 9·0284868	+ 8·9615401
(r) 85	− 9·9879504	+ 9·8116935	+ 9·5963900	− 9·4991742	− 9·1058556	+ 9·0384480
(q) 80	− 9·9550459	+ 9·7782749	+ 9·4809342	− 9·3826432	− 8·8100928	+ 8·7404716
(p) 75	− 9·8959691	+ 9·7182709	+ 9·2158933	− 9·1145922	+ 8·2705227	− 8·2139747
(o) 70	− 9·8020065	+ 9·6227891	+ 6·8974504	+ 6·9366555	+ 8·9837304	− 8·9169184
(n) 65	− 9·6526755	+ 9·4707822	− 9·2134830	+ 9·1198236	+ 9·1591456	− 9·0896563
(m) 60	− 9·3844633	+ 9·1960068	− 9·4820182	+ 9·3845735	+ 9·1681590	− 9·0959737
(l) 55	− 8·3529949	+ 8·0216630	− 9·5991437	+ 9·4993071	+ 9·0174044	− 8·9414592
(k) 50	9·2994331	− 9·1356193	− 9·6338183	+ 9·5317745	+ 8·4446670	− 8·3542362
(i) 45	9·6148097	− 9·4425086	− 9·5984671	+ 9·4940914	− 8·7610803	+ 8·6921111
(h) 40	9·7812720	− 9·6056314	− 9·4783759	+ 9·3711510	− 9·0983518	+ 9·0226454
(g) 35	9·8841879	− 9·7064396	− 9·1962640	+ 9·0837570	− 9·1861001	+ 9·1069108
(f) 30	9·9479528	− 9·7686265	+ 8·0899661	− 8·0666595	− 9·1219350	+ 9·0396496
(e) 25	9·9819212	− 9·8013403	+ 9·2563827	− 9·1542509	− 8·8347073	+ 8·7479945
(d) 20	9·9890380	− 9·8074539	+ 9·5043969	− 9·3982439	+ 8·2640409	− 8·1964959
(c) 15	9·9674057	− 9·7850453	+ 9·6065443	− 9·4984993	+ 8·9964713	− 8·9140683
(b) 10	9·9074102	− 9·7244943	+ 9·6175662	− 9·5083967	+ 9·1634124	− 9·0788266
(a) 5	9·7733157	− 9·5900654	+ 9·5212559	− 9·4114609	+ 9·1327157	− 9·0471606

together we get the final equation in X for g_n^m of the type

$$\Sigma\left[(X'^m_n)^2 w\right] g_n^m + \Sigma\left[X'^m_n X'^m_{n_1} w\right] g^m_{n_1} + \&\text{c.} = \Sigma\left[X'^m_n w x'_m\right].$$

The coefficients in the final equations for g_n^m and h_n^m are the same.

In the same way the final equations in Y and Z for g_n^m or h_n^m are formed.

$g_8{}^0$ or $h_8{}^0$	$g_{-8}{}^0$ or $h_{-8}{}^0$	$g_{10}{}^0$ or $h_{10}{}^0$	$g_{-10}{}^0$ or $h_{-10}{}^0$	Log $(w^{\frac{1}{2}}x'_0)$ 1845	Log $(w^{\frac{1}{2}}x'_0)$ 1880	
0·0000000	0·0000000	0·0000000	0·0000000			(s)
− 8·4895634	− 8·4899330	+ 8·0513343	+ 8·0514957	8·8942662	9·18243	(r)
− 8·6623150	− 8·6616706	+ 8·1427286	+ 8·1415623	9·0941257	9·43716	(q)
− 8·5738015	− 8·5710866	+ 7·7681583	+ 7·7628691	9·2927844	9·54558	(p)
− 7·9871488	− 7·9754625	− 7·8267217	− 7·8266552	9·3446441	9·59348	(o)
+ 8·3654347	+ 8·3648227	− 8·1517098	− 8·1466423	9·2932522	9·60270	(n)
+ 8·6467574	+ 8·6413467	− 8·0316560	− 8·0217449	9·2200394	9·55598	(m)
+ 8·6306641	+ 8·6212524	+ 7·0058837	+ 7·0330329	9·0487280	9·45929	(l)
+ 8·2791421	+ 8·2623640	+ 8·0825528	+ 8·0717983	8·4218790	9·24398	(k)
− 8·1675493	− 8·1637821	+ 8·1427703	+ 8·1264752	− 8·7275138	8·68966	(i)
− 8·6137775	− 8·6007470	+ 7·6924812	+ 7·6663824	− 9·1400648	− 8·81277	(h)
− 8·6672250	− 8·6498731	− 7·8999534	− 7·8835731	− 9·2920834	− 9·20531	(g)
− 8·4471258	− 8·4249285	− 8·1694543	− 8·1466346	− 9·4025134	− 9·34700	(f)
+ 7·6913985	+ 7·6915933	− 8·0114368	− 7·9838493	− 9·4329485	− 9·39563	(e)
+ 8·5516436	+ 8·5312258	+ 7·3284906	+ 7·3139055	− 9·4271989		(d)
+ 8·6921616	+ 8·6688793	+ 8·1156223	+ 8·0877435	− 9·3379813		(c)
+ 8·6097951	+ 8·5849018	+ 8·1687408	+ 8·1385945			(b)
+ 8·2551400	+ 8·2293446	+ 7·8705633	+ 7·8392807			(a)
				Log $(w^{\frac{1}{2}}z'_0)$ 1845	Log $(w^{\frac{1}{2}}z'_0)$ 1880	
+ 8·5377928	− 8·4866403	− 8·0242255	+ 7·9828328	− 9·1692148	− 9·51576	(s)
+ 8·5577199	− 8·5059237	− 7·9624540	+ 7·9201823	− 9·2551786	− 9·63593	(r)
+ 7·6680609	− 7·6056453	+ 7·5700341	− 7·5316034	− 9·1536715	− 9·52268	(q)
− 8·4669588	+ 8·4161291	+ 8·1386238	− 8·0958503	− 8·8007716	− 9·36283	(p)
− 8·6822473	+ 8·6286240	+ 8·1193872	− 8·0734506	8·8104793	− 8·96675	(o)
− 8·6217921	+ 8·5649436	+ 7·3658886	− 7·3045859	9·2301906	8·65804	(n)
− 8·1347930	+ 8·0687264	− 8·0170416	+ 7·9707863	9·4189780	9·20078	(m)
+ 8·3409869	− 8·2873618	− 8·1784484	+ 8·1268750	9·5176420	9·36016	(l)
+ 8·6644568	− 8·6043909	− 7·9016319	+ 7·8431031	9·4983555	9·47284	(k)
+ 8·6644792	− 8·6000326	+ 7·7376981	− 7·6887954	9·4487465	9·49235	(i)
+ 8·3358783	− 8·2644564	+ 8·1674245	− 8·1091829	9·3346815	9·49966	(h)
− 8·1632669	+ 8·1047516	+ 8·0926398	− 8·0289674	9·1579727	9·41076	(g)
− 8·6359592	+ 8·5680367	+ 6·3068217	− 5·9871999	8·5496144	9·22587	(f)
− 8·6887630	+ 8·6170965	− 8·0869711	+ 8·0224998	− 8·8235707	8·77104	(e)
− 8·4472041	+ 8·3717035	− 8·1738096	+ 8·1051979	− 9·2062854		(d)
+ 7·9157470	− 7·8507053	− 7·7478096	+ 7·6743545	− 9·3688120		(c)
+ 8·6006931	− 8·5266524	+ 7·9198267	− 7·8506794			(b)
+ 8·6762302	− 8·6007501	+ 8·1762536	− 8·1048202			(a)

For X

Co-latitude	g_1^1 or h_1^1	g_{-1}^1 or h_{-1}^1	g_3^1 or h_3^1	g_{-3}^1 or h_{-3}^1	g_5^1 or h_5^1	g_{-5}^1 or h_{-5}^1	g_7^1 or h_7^1
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	− 8·9293511	− 8·9380360	+ 9·2721395	+ 9·2738556	− 9·0605599	− 9·0614571	+ 8·7130456
(q) 80	− 9·2265894	− 9·2349792	+ 9·5558885	+ 9·5571831	− 9·3136922	− 9·3139541	+ 8·9188480
(p) 75	− 9·3963001	− 9·4042082	+ 9·7026093	+ 9·7032091	− 9·4045969	− 9·4037704	+ 8·9114841
(o) 70	− 9·5121758	− 9·5194295	+ 9·7851527	+ 9·7847947	− 9·3958334	− 9·3933738	+ 8·6807237
(n) 65	− 9·5972033	− 9·6036500	+ 9·8251402	+ 9·8235719	− 9·2828244	− 9·2778369	− 7·6746874
(m) 60	− 9·6614953	− 9·6670064	+ 9·8305207	+ 9·8274949	− 8·9768565	− 8·9662002	− 8·7461195
(l) 55	− 9·7102992	− 9·7147742	+ 9·8030296	+ 9·7982867	+ 8·3399882	− 8·3661587	− 8·9292398
(k) 50	− 9·7464891	− 9·7499396	+ 9·7395586	+ 9·7328777	+ 9·1272503	+ 9·1252475	− 8·9079185
(i) 45	− 9·7719962	− 9·7742252	+ 9·6301063	+ 9·6207639	+ 9·3399996	+ 9·3336636	− 8·6546684
(h) 40	− 9·7874509	− 9·7885378	+ 9·4482104	+ 9·4350364	+ 9·4141252	+ 9·4048052	+ 7·9125412
(g) 35	− 9·7931766	− 9·7931553	+ 9·0981223	+ 9·0752239	+ 9·3972895	+ 9·3852912	+ 8·7656858
(f) 30	− 9·7888032	− 9·7877409	− 8·3806504	− 8·4401040	+ 9·2879274	+ 9·2731536	+ 8·9358276
(e) 25	− 9·7731869	− 9·7711831	− 9·1836231	− 9·1840886	+ 9·0332445	+ 9·0145465	+ 8·9173676
(d) 20	− 9·7439418	− 9·7411247	− 9·3933421	− 9·3880491	+ 8·1224978	+ 8·0686572	+ 8·7114774
(c) 15	− 9·6961780	− 9·6927006	− 9·4739356	− 9·4661353	− 8·8220335	− 8·8119358	+ 7·9023955
(b) 10	− 9·6185167	− 9·6145528	− 9·4713805	− 9·4621847	− 9·0443049	− 9·0300578	− 8·4408547
(a) 5	− 9·4741864	− 9·4699242	− 9·3673949	− 9·3574530	− 9·0244968	− 9·0088832	− 8·5846877

For Y

Co-latitude	g_1^1 or h_1^1	g_{-1}^1 or h_{-1}^1	g_3^1 or h_3^1	g_{-3}^1 or h_{-3}^1	g_5^1 or h_5^1	g_{-5}^1 or h_{-5}^1	g_7^1 or h_7^1
(s) 90°	9·8480308	+ 9·8480308	− 9·1490608	− 9·1490608	+ 8·5258115	+ 8·5258115	− 7·9145435
(r) 85	9·9977728	+ 9·9977400	− 9·2822349	− 9·2821583	+ 8·6280207	+ 8·6279004	− 7·9691421
(q) 80	9·9954393	+ 9·9953090	− 9·2265762	− 9·2262722	+ 8·4529210	+ 8·4524433	− 7·5121691
(p) 75	9·9915014	+ 9·9912119	− 9·1182966	− 9·1176211	+ 7·8890855	+ 7·8880240	+ 7·6357995
(o) 70	9·9858837	+ 9·9853779	− 8·9125562	− 8·9113760	− 8·1929738	− 8·1911192	+ 8·0222570
(n) 65	9·9784745	+ 9·9777017	− 8·3472821	− 8·3454790	− 8·5699264	− 8·5670929	+ 8·1084355
(m) 60	9·9691188	+ 9·9680364	+ 8·6465076	+ 8·6439819	− 8·7197123	− 8·7157434	+ 8·0010894
(l) 55	9·9576084	+ 9·9561828	+ 9·0590672	+ 9·0557409	− 8·7618647	− 8·7566376	+ 7·4299133
(k) 50	9·9436640	+ 9·9418721	+ 9·2669379	+ 9·2627568	− 8·7066619	− 8·7000917	− 7·8318108
(i) 45	9·9269104	+ 9·9247401	+ 9·4006042	+ 9·3955401	− 8·4958786	− 8·4879208	− 8·1735812
(h) 40	9·9068333	+ 9·9042839	+ 9·4923808	+ 9·4864322	− 6·8508486	− 6·8415008	− 8·2587543
(g) 35	9·8827084	+ 9·8797908	+ 9·5549426	+ 9·5481348	+ 8·5727064	+ 8·5620085	− 8·1494780
(f) 30	9·8534740	+ 9·8502103	+ 9·5940182	+ 9·5864030	+ 8·8913530	+ 8·8793862	− 7·4017428
(e) 25	9·8174843	+ 9·8139074	+ 9·6118544	+ 9·6035083	+ 9·0597683	+ 9·0466530	+ 8·1601842
(d) 20	9·7719865	+ 9·7681388	+ 9·6081013	+ 9·5991234	+ 9·1521197	+ 9·1380116	+ 8·5136498
(c) 15	9·7118245	+ 9·7077569	+ 9·5791485	+ 9·5696575	+ 9·1872380	+ 9·1723236	+ 8·6696535
(b) 10	9·6254310	+ 9·6212013	+ 9·5144324	+ 9·5045631	+ 9·1640225	+ 9·1485136	+ 8·7134945
(a) 5	9·4759091	+ 9·4715800	+ 9·3776860	+ 9·3675848	+ 9·0507599	+ 9·0348866	+ 8·6354510

For Z

Co-latitude	g_1^1 or h_1^1	g_{-1}^1 or h_{-1}^1	g_3^1 or h_3^1	g_{-3}^1 or h_{-3}^1	g_5^1 or h_5^1	g_{-5}^1 or h_{-5}^1	g_7^1 or h_7^1
(s) 90°	0·1490608	− 9·8480308	− 9·7511208	+ 9·6261821	+ 9·3039628	− 9·2247815	− 8·8176335
(r) 85	0·2971798	− 9·9960842	− 9·8827229	+ 9·7575632	+ 9·4046521	− 9·3250995	− 8·8707735
(q) 80	0·2899510	− 9·9886605	− 9·8223436	+ 9·6964728	+ 9·2251194	− 9·1441198	− 8·4102999
(p) 75	0·2777646	− 9·9761556	− 9·7062142	+ 9·5789161	+ 8·6564535	− 8·5671839	+ 8·5222547
(o) 70	0·2604052	− 9·9583637	− 9·4897856	+ 9·3593461	− 8·9418060	+ 8·8665616	+ 8·8981072
(n) 65	0·2375527	− 9·9349774	− 8·9162430	+ 8·7679395	− 9·3048160	+ 9·2251135	+ 8·9693757
(m) 60	0·2087615	− 9·9055671	+ 9·1812530	− 9·0666687	− 9·4356825	+ 9·3534850	+ 8·8434413
(l) 55	0·1734277	− 9·8695473	+ 9·5731351	− 9·4502340	− 9·4544401	+ 9·3697915	+ 8·2538352
(k) 50	0·1307386	− 9·8261260	+ 9·7530659	− 9·6271605	− 9·3710835	+ 9·2835218	− 8·6175700
(i) 45	0·0795939	− 9·7742251	+ 9·8527309	− 9·7247443	− 9·1273460	+ 9·0349442	− 8·9268316
(h) 40	0·0184776	− 9·7123513	+ 9·9037109	− 9·7739823	− 7·5566188	+ 7·1467409	− 8·9717411
(g) 35	9·9452439	− 9·6383820	+ 9·9173358	− 9·7860630	+ 9·1087478	− 9·0246689	− 8·8143334
(f) 30	9·8567336	− 9·5491803	+ 9·8972660	− 9·7646140	+ 9·3696098	− 9·2807476	− 8·0165848
(e) 25	9·7480346	− 9·4398556	+ 9·8424951	− 9·7086318	+ 9·4658433	− 9·3744465	+ 8·6897285
(d) 20	9·6109102	− 9·3021905	+ 9·7471952	− 9·6123046	+ 9·4668474	− 9·3736469	+ 8·9528214
(c) 15	9·4299118	− 9·1207531	+ 9·5974666	− 9·4617520	+ 9·3812996	− 9·2867584	+ 8·9883927
(b) 10	9·1703541	− 8·8608715	+ 9·3596288	− 9·2233110	+ 9·1850254	− 9·0895393	+ 8·8592655
(a) 5	8·7215570	− 8·4118760	+ 8·9236326	− 8·7869469	+ 8·7725460	− 8·6764945	+ 8·4820442

$g_{-7}{}^1$ or $h_{-7}{}^1$	$g_9{}^1$ or $h_9{}^1$	$g_{-9}{}^1$ or $h_{-9}{}^1$	Log $(w^{\frac{1}{2}}x'_1)$ for g		Log $(w^{\frac{1}{2}}x'_1)$ for h		
			1845	1880	1845	1880	
0·0000000	0·0000000	0·0000000					(*s*)
+ 8·7135594	− 8·2958140	− 8·2960798	9·2982268	9·26327	8·8212142	8·80051	(*r*)
+ 8·9184837	− 8·4335587	− 8·4326643	9·5689927	9·52870	9·3400941	9·14484	(*q*)
+ 8·9094792	− 8·2423034	− 8·2387032	9·6380566	9·64654	9·5434541	9·37807	(*p*)
+ 8·6752975	+ 7·5079205	+ 7·5201478	9·6385904	9·69064	9·6878084	9·56114	(*o*)
− 7·7100691	+ 8·3419786	+ 8·3386728	9·5661975	9·66443	9·7940093	9·70451	(*n*)
− 8·7436400	+ 8·4311550	+ 8·4237595	9·4363099	9·55955	9·8601032	9·80405	(*m*)
− 8·9226439	+ 8·1766830	+ 8·1633150	9·0515265	9·35342	9·9181257	9·88524	(*l*)
− 8·8975728	− 7·8112015	− 7·8114727	− 8·9818807	8·82211	9·9725932	9·94217	(*k*)
− 8·6386748	− 8·3806175	− 8·3683779	− 9·4113123	− 8·98279	9·9945746	9·97508	(*i*)
+ 7·9295339	− 8·4250652	− 8·4078593	− 9·5797873	− 9·36322	9·9997671	9·99623	(*h*)
+ 8·7539494	− 8·1021373	− 8·0779993	− 9·6651919	− 9·52600	9·9979564	0·00410	(*g*)
+ 8·9197871	+ 7·9589804	+ 7·9453782	− 9·7005195	− 9·60540	9·9695652	9·99928	(*f*)
+ 8·8982366	+ 8·4026901	+ 8·3808368	− 9·7023677	− 9·63856	9·9226753	9·96662	(*e*)
+ 8·6891614	+ 8·4246928	+ 8·3993792	− 9·6783362		9·8446154		(*d*)
+ 7·8650606	+ 8·1379483	+ 8·1092209	− 9·6420533		9·7244142		(*c*)
− 8·4221499	− 7·5232327	− 7·5042596					(*b*)
− 8·5634170	− 8·0856940	− 8·0587908					(*a*)
			Log $(w^{\frac{1}{2}}y'_1)$ for g		Log $(w^{\frac{1}{2}}y'_1)$ for h		
			1845	1880	1845	1880	
− 7·9145435	+ 7·3073439	+ 7·3073439	9·4720033	9·25406	− 9·9617068	− 9·95768	(*s*)
− 7·9689781	+ 7·2934934	+ 7·2932856	9·6330226	9·43517	− 0·1128510	− 0·11091	(*r*)
− 7·5115177	− 6·3069045	− 6·3060794	9·6679609	9·49085	− 0·1130773	− 0·11032	(*q*)
+ 7·6343520	− 7·3643749	− 7·3625413	9·7039878	9·56152	− 0·1140010	− 0·10884	(*p*)
+ 8·0197279	− 7·4824020	− 7·4791985	9·7430775	9·63335	− 0·1065765	− 0·10295	(*o*)
+ 8·1045717	− 7·2612501	− 7·2563559	9·7815001	9·69968	− 0·0959664	− 0·09504	(*n*)
+ 7·9956772	+ 6·8468349	+ 6·8399794	9·8105208	9·75415	− 0·0820633	− 0·08258	(*m*)
+ 7·4227854	+ 7·4837628	+ 7·4747342	9·8416006	9·79295	− 0·0608236	− 0·06593	(*l*)
− 7·8228514	+ 7·5623053	+ 7·5509567	9·8539878	9·81629	− 0·0373901	− 0·04100	(*k*)
− 8·1627296	+ 7·2667896	+ 7·2530443	9·8571378	9·82994	− 0·0117580	− 0·01088	(*i*)
− 8·2460073	− 7·2049117	− 7·1887654	9·8432037	9·82419	− 9·9799038	− 9·97476	(*h*)
− 8·1348899	− 7·6767218	− 7·6582436	9·8169576	9·80992	− 9·9333303	− 9·92932	(*g*)
− 7·3854245	− 7·7310213	− 7·7103514	9·7868246	9·79053	− 9·8848388	− 9·87499	(*f*)
+ 8·1422996	− 7·3515746	− 7·3289208	9·7456957	9·74765	− 9·8247160	− 9·81525	(*e*)
+ 8·4944114	+ 7·6124377	+ 7·5880691	9·6875569		− 9·7512010		(*d*)
+ 8·6493157	+ 8·0513525	+ 8·0255913	9·6219413		− 9·6655796		(*c*)
+ 8·6923460	+ 8·2007925	+ 8·1740044					(*b*)
+ 8·6138057	+ 8·1714472	+ 8·1440298					(*a*)
			Log $(w^{\frac{1}{2}}z'_1)$ for g		Log $(w^{\frac{1}{2}}z'_1)$ for h		
			1845	1880	1845	1880	
+ 8·7596415	+ 8·3073439	− 8·2615864	9·4983383	8·83748	− 0·2592969	− 0·26138	(*s*)
+ 8·8122418	+ 8·2921136	− 8·2456117	9·6833474	9·17005	− 0·4061665	− 0·41433	(*r*)
+ 8·3482154	− 7·2935527	+ 7·2611823	9·8052725	9·47474	− 0·4075259	− 0·41333	(*q*)
− 8·4666027	− 8·3489461	+ 8·3025482	9·8894158	9·67354	− 0·3961995	− 0·41006	(*p*)
− 8·8387913	− 8·4558321	+ 8·4066451	9·9747832	9·86015	− 0·3787513	− 0·40540	(*o*)
− 8·9073929	− 8·2205167	+ 8·1667439	0·0535953	0·00190	− 0·3504389	− 0·38253	(*n*)
− 8·7778708	+ 7·7794896	− 7·7386104	0·1098310	0·09146	− 0·3192624	− 0·35105	(*m*)
− 8·1746286	+ 8·3972614	− 8·3441967	0·1373973	0·14090	− 0·2734206	− 0·30798	(*l*)
+ 8·5565527	+ 8·4480889	− 8·3902752	0·1386146	0·15063	− 0·2107191	− 0·25328	(*k*)
+ 8·8596396	+ 8·1203821	− 8·0552433	0·1131764	0·14078	− 0·1356336	− 0·18340	(*i*)
+ 8·9007240	− 8·0109814	+ 7·9568467	0·0625408	0·10973	− 0·0341833	− 0·10057	(*h*)
+ 8·7391153	− 8·4366295	+ 8·3737197	9·9917755	0·06346	− 9·8903241	− 9·98984	(*g*)
+ 7·9192210	− 8·4325948	+ 8·3652639	9·9039032	0·00969	− 9·7116850	− 9·86423	(*f*)
− 8·6178677	− 7·9828555	+ 7·9081970	9·7962239	9·93841	− 9·5162518	− 9·70573	(*e*)
− 8·8767372	+ 8·1472851	− 8·0808860	9·6657109		− 9·2903221		(*d*)
− 8·9101885	+ 8·4667197	− 8·3959199	9·5140386		− 8·9558528		(*c*)
− 8·7797156	+ 8·4433430	− 8·3706696					(*b*)
− 8·4017183	+ 8·1148663	− 8·0411906					(*a*)

For X

Co-latitude	g_2^1 or h_2^1	g_{-2}^1 or h_{-2}^1	g_4^1 or h_4^1	g_{-4}^1 or h_{-4}^1	g_6^1 or h_6^1	g_{-6}^1 or h_{-6}^1	g_8^1 or h_8^1
(s) 90°	9·8480308	+ 9·8480308	− 9·4800541	− 9·4800541	+ 9·0284869	+ 9·0284869	− 8·5377928
(r) 85	9·9912896	+ 9·9911247	− 9·5998534	− 9·5995492	+ 9·1095400	+ 9·1090818	− 8·5617678
(q) 80	9·9690965	+ 9·9684316	− 9·4975810	− 9·4962372	+ 8·8344012	+ 8·8318420	− 7·7535787
(p) 75	9·9305845	+ 9·9290639	− 9·2743214	− 9·2703754	− 8·0929233	− 8·1090929	+ 8·4479831
(o) 70	9·8730248	+ 9·8702438	− 8·5570377	− 8·5323459	− 8·9522092	− 8·9525187	+ 8·6782943
(n) 65	9·7913960	+ 9·7868330	+ 9·0821657	+ 9·0873655	− 9·1477016	− 9·1451806	+ 8·6356449
(m) 60	9·6759855	+ 9·6688160	+ 9·4172180	+ 9·4171181	− 9·1764679	− 9·1712397	+ 8·2262356
(l) 55	9·5050429	+ 9·4934910	+ 9·5643885	+ 9·5615745	− 9·0630047	− 9·0542884	− 8·2601182
(k) 50	9·2114254	+ 9·1889846	+ 9·6269363	+ 9·6218065	− 8·6736544	− 8·6566817	− 8·6454874
(i) 45	7·9965194	+ 7·4488279	+ 9·6289191	+ 9·6215181	+ 8·5446035	+ 8·5472854	− 8·6815474
(h) 40	− 9·1179096	− 9·1348988	+ 9·5734426	+ 9·5636533	+ 9·0297134	+ 9·0213445	− 8·4437168
(g) 35	− 9·4043497	− 9·4090927	+ 9·4476916	+ 9·4351160	+ 9·1719612	+ 9·1597319	+ 7·8002126
(f) 30	− 9·5457962	− 9·5459003	+ 9·2024463	+ 9·1855006	+ 9·1732826	+ 9·1580777	+ 8·5710500
(e) 25	− 9·6220466	− 9·6194487	+ 8·5039550	+ 8·4606842	+ 9·0452487	+ 9·0271267	+ 8·7005463
(d) 20	− 9·6547509	− 9·6503221	− 8·9078062	− 8·9052784	+ 8·6907860	+ 8·6679390	+ 8·6040439
(c) 15	− 9·6492281	− 9·6435098	− 9·2041060	− 9·1945953	− 8·1695775	− 8·1663125	+ 8·1636571
(b) 10	− 9·5991876	− 9·5925954	− 9·2810075	− 9·2692502	− 8·7657463	− 8·7491390	− 8·0502809
(a) 5	− 9·4705463	− 9·4634437	− 9·2123351	− 9·1995560	− 8·8134727	− 8·7950281	− 8·3414199

For Y

Co-latitude	g_2^1 or h_2^1	g_{-2}^1 or h_{-2}^1	g_4^1 or h_4^1	g_{-4}^1 or h_{-4}^1	g_6^1 or h_6^1	g_{-6}^1 or h_{-6}^1	g_8^1 or h_8^1
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	·00000000
(r) 85	8·9351933	+ 8·9351386	− 8·5595746	− 8·5594762	+ 8·0958756	+ 8·0957334	− 7·5884015
(q) 80	9·2323317	+ 9·2321146	− 8·8331813	− 8·8327905	+ 8·3306045	+ 8·3300400	− 7·7658364
(p) 75	9·4018794	+ 9·4013969	− 8·9612800	− 8·9604115	+ 8·3834060	+ 8·3821515	− 7·6850062
(o) 70	9·5175338	+ 9·5166908	− 9·0133735	− 9·0118561	+ 8·2948107	+ 8·2926188	− 7·1060887
(n) 65	9·6022886	+ 9·6010007	− 9·0040693	− 9·0017510	+ 7·9482967	+ 7·9449480	+ 7·5065522
(m) 60	9·6662648	+ 9·6644607	− 8·9248436	− 8·9215963	− 7·7550479	− 7·7503573	+ 7·8057981
(l) 55	9·7147191	+ 9·7123431	− 8·7265259	− 8·7222492	− 8·3030126	− 8·2968351	+ 7·8125538
(k) 50	9·7506174	+ 9·7476309	− 8·0222383	− 8·0168627	− 8·4781511	− 8·4703863	+ 7·4871093
(i) 45	9·7756599	+ 9·7720427	+ 8·6101274	+ 8·6036165	− 8·5054728	− 8·4960681	− 7·4180648
(h) 40	9·7907307	+ 9·7864817	+ 8·9827663	+ 8·9751181	− 8·3790084	− 8·3679610	− 7·9016707
(g) 35	9·7960844	+ 9·7912217	+ 9·1772898	+ 9·1685370	− 7·7783218	− 7·7656788	− 8·0031655
(f) 30	9·7913618	+ 9·7859224	+ 9·2972042	+ 9·2874132	+ 8·2830548	+ 8·2689122	− 7·8409334
(e) 25	9·7754300	+ 9·7694685	+ 9·3698237	+ 9·3590930	+ 8·6725020	+ 8·6570020	+ 7·1709295
(d) 20	9·7459126	+ 9·7394998	+ 9·4046336	+ 9·3930906	+ 8·8558926	+ 8·8392193	+ 8·1126710
(c) 15	9·6979281	+ 9·6911487	+ 9·4027249	+ 9·3905222	+ 8·9414715	+ 8·9238454	+ 8·3732405
(b) 10	9·6201041	+ 9·6130546	+ 9·3560254	+ 9·3433363	+ 8·9480302	+ 8·9297015	+ 8·4637256
(a) 5	9·4756741	+ 9·4684590	+ 9·2296319	+ 9·2166447	+ 8·8508215	+ 8·8320622	+ 8·4081993

For Z

Co-latitude	g_2^1 or h_2^1	g_{-2}^1 or h_{-2}^1	g_4^1 or h_4^1	g_{-4}^1 or h_{-4}^1	g_6^1 or h_6^1	g_{-6}^1 or h_{-6}^1	g_8^1 or h_8^1
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·4097215	− 9·2359695	− 9·2563561	+ 9·1605945	+ 8·9389672	− 8·8726849	− 8·5407462
(q) 80	9·7019757	− 9·5279527	− 9·5251251	+ 9·4288688	+ 9·1689119	− 9·1018964	− 8·7134634
(p) 75	9·8632931	− 9·6888273	− 9·6450794	+ 9·5479922	+ 9·2136970	− 9·1453740	− 8·6248695
(o) 70	9·9672302	− 9·7921632	− 9·6855992	+ 9·5873233	+ 9·1138769	− 9·0433394	− 8·0377503
(n) 65	0·0365719	− 9·8607630	− 9·6611262	+ 9·5612316	+ 8·7539495	− 8·6776432	+ 8·4167523
(m) 60	0·0811476	− 9·9044780	− 9·5629693	+ 9·4607767	− 8·5329035	+ 8·4730847	+ 8·6978723
(l) 55	0·1058174	− 9·9281942	− 9·3420984	+ 9·2357477	− 9·0610610	+ 8·9908879	+ 8·6816329
(k) 50	0·1130123	− 9·9343714	− 8·6214049	+ 8·4852025	− 9·2082847	+ 9·1344674	+ 8·3297760
(i) 45	0·1037065	− 9·9240143	+ 9·1557324	− 9·0619205	− 9·2018279	+ 9·1247717	− 8·2190463
(h) 40	0·0777810	− 9·8970358	+ 9·4897742	− 9·3890970	− 9·0352953	+ 8·9542148	− 8·6648247
(g) 35	0·0340673	− 9·8522995	+ 9·6359144	− 9·5322329	− 8·3922769	+ 8·2944530	− 8·7181139
(f) 30	9·9701075	− 9·7873789	+ 9·6969198	− 9·5911187	+ 8·8265082	− 8·7514424	− 8·4978211
(e) 25	9·8815018	− 9·6979035	+ 9·6970579	− 9·5895718	+ 9·1450496	− 9·0645146	+ 7·7435117
(d) 20	9·7603881	− 9·5760381	+ 9·6403961	− 9·5315479	+ 9·2373433	− 9·1542954	+ 8·6025931
(c) 15	9·5915919	− 9·4066315	+ 9·5177583	− 9·4078453	+ 9·2023462	− 9·1176243	+ 8·7430001
(b) 10	9·3406218	− 9·1552114	+ 9·2979660	− 9·1872848	+ 9·0358868	− 8·9500292	+ 8·6605810
(a) 5	8·8969277	− 8·7112416	+ 8·8723387	− 8·7611923	+ 8·6394800	− 8·5529534	+ 8·3058990

g_{-8}^1 or h_{-8}^1	g_{10}^1 or h_{10}^1	g_{-10}^1 or h_{-10}^1	Log $(w^{\frac{1}{2}}x'_1)$ for g 1845	Log $(w^{\frac{1}{2}}x'_1)$ for g 1880	Log $(w^{\frac{1}{2}}x'_1)$ for h 1845	Log $(w^{\frac{1}{2}}x'_1)$ for h 1880	
− 8.5377928	+ 8.0242255	+ 8.0242255	− 9.7686237	− 9.76810	9.0835592	9.32283	(s)
− 8.5611285	+ 7.9670880	+ 7.9662159	− 9.9270312	− 9.91818	9.1463427	9.45954	(r)
− 7.7442156	− 7.5405337	− 7.5437667	− 9.9369695	− 9.92335	8.9461605	9.43669	(q)
+ 8.4483934	− 8.1336660	− 8.1322925	− 9.9330462	− 9.92307	8.9188389	9.41593	(p)
+ 8.6758270	− 8.1273774	− 8.1228517	− 9.9225948	− 9.92525	8.8965355	9.34312	(o)
+ 8.6299908	− 7.4746360	− 7.4579801	− 9.8972505	− 9.89829	8.8829577	9.15869	(n)
+ 8.2130023	+ 7.9904158	+ 7.9857241	− 9.8481923	− 9.84078	8.5039631	8.88712	(m)
− 8.2587841	+ 8.1805844	+ 8.1704047	− 9.7786451	− 9.75984	− 8.2700500	8.31602	(l)
− 8.6366289	+ 7.9536207	+ 7.9370051	− 9.6899438	− 9.65055	− 8.8530297	− 8.53737	(k)
− 8.6682919	− 7.6109527	− 7.6060166	− 9.5275594	− 9.47582	− 8.8103840	− 9.04859	(i)
− 8.4247344	− 8.1499723	− 8.1331642	− 9.2381340	− 9.20553	− 8.7652205	− 9.21794	(h)
+ 7.8045512	− 8.1307303	− 8.1086067	8.0068957	− 8.06163	− 8.6145907	− 9.26504	(g)
+ 8.5545574	− 7.4391341	− 7.4018125	9.2856565	9.13980	− 8.6846310	− 9.26535	(f)
+ 8.6800389	+ 8.0015107	+ 7.9788569	9.5541121	9.45933	− 8.6748440	− 9.25513	(e)
+ 8.5804688	+ 8.1906688	+ 8.1634606	9.6746893		− 8.5459280		(d)
+ 8.1353447	+ 8.0186432	+ 7.9882503	9.7367346		− 8.2592069		(c)
− 8.0301365	− 4.9846147	− 5.6256209					(b)
− 8.3173289	− 7.8187169	− 7.7890118					(a)

			Log $(w^{\frac{1}{2}}y'_1)$ for g 1845	Log $(w^{\frac{1}{2}}y'_1)$ for g 1880	Log $(w^{\frac{1}{2}}y'_1)$ for h 1845	Log $(w^{\frac{1}{2}}y'_1)$ for h 1880	
0.0000000	0.0000000	0.0000000					(s)
− 7.5882156	+ 7.0531731	+ 7.0529434	− 8.7936200	− 8.85144	− 7.7459280	8.58094	(r)
− 7.7650982	+ 7.1491571	+ 7.1482452	− 9.1367588	− 9.13519	− 7.6080929	8.85684	(q)
− 7.6833657	+ 6.7813156	+ 6.7792890	− 9.3064473	− 9.31673	7.7355049	9.03221	(p)
− 7.1032224	− 6.8552220	− 6.8516813	− 9.4204255	− 9.44748	8.4036792	9.11086	(o)
+ 7.5021732	− 7.1940980	− 7.1886886	− 9.5258453	− 9.55703	8.4634231	9.12846	(n)
+ 7.7996643	− 7.0921827	− 7.0846056	− 9.6156150	− 9.65298	8.2871425	9.14931	(m)
+ 7.8044755	+ 6.1114121	+ 6.1014331	− 9.6967823	− 9.72858	7.8931989	9.15291	(l)
+ 7.4769553	+ 7.1980422	+ 7.1854990	− 9.7602300	− 9.78417	− 8.1265635	9.14843	(k)
− 7.4057664	+ 7.2912610	+ 7.2760688	− 9.8019098	− 9.83483	− 8.2520990	9.11451	(i)
− 7.8872241	+ 6.8784109	+ 6.8605650	− 9.8426535	− 9.86594	− 8.4293287	9.05288	(h)
− 7.9866323	− 7.1415396	− 7.1211163	− 9.8715042	− 9.88540	− 8.4348853	8.95570	(g)
− 7.8224393	− 7.4683221	− 7.4454764	− 9.8841565	− 9.89275	− 8.5157913	8.79321	(f)
+ 7.1506603	− 7.3817705	− 7.3567321	− 9.8715925	− 9.89059	− 8.5231774	8.36045	(e)
+ 8.0908675	+ 6.7983859	+ 6.7714522	− 9.8479891		− 8.5886686		(d)
+ 8.3501910	+ 7.7002257	+ 7.6717528	− 9.8192882		− 8.4826387		(c)
+ 8.4397573	+ 7.9259688	+ 7.8963609					(b)
+ 8.3836679	+ 7.9268687	+ 7.8965652					(a)

			Log $(w^{\frac{1}{2}}z'_1)$ for g 1845	Log $(w^{\frac{1}{2}}z'_1)$ for g 1880	Log $(w^{\frac{1}{2}}z'_1)$ for h 1845	Log $(w^{\frac{1}{2}}z'_1)$ for h 1880	
0.0000000	0.0000000	0.0000000					(s)
+ 8.4899683	+ 8.0927436	− 8.0515166	− 9.1427809	− 9.23528	8.4193439	9.23854	(r)
+ 8.6616870	+ 8.1841013	− 8.1415589	− 9.5011370	− 9.57131	8.5722260	9.39860	(q)
+ 8.5710475	+ 7.8093788	− 7.7627326	− 9.6744841	− 9.73574	− 7.4383699	9.52307	(p)
+ 7.9749807	− 7.8682498	+ 7.8268535	− 9.8273627	− 9.86773	8.3978384	9.59476	(o)
− 8.3651073	− 8.1930719	+ 8.1467100	− 9.9296945	− 9.94600	8.6771058	9.63351	(n)
− 8.6414793	− 8.0728723	+ 8.0217058	− 0.0041461	− 0.00686	8.9287450	9.64446	(m)
− 8.6212927	+ 7.0490280	− 7.0348152	− 0.0610112	− 0.06959	9.0311820	9.63351	(l)
− 8.2621215	+ 8.1239222	− 8.0720025	− 0.1103254	− 0.11447	8.9063674	9.59893	(k)
+ 8.1644574	+ 8.1839792	− 8.1265672	− 0.1375800	− 0.14808	8.6809957	9.49889	(i)
+ 8.6010418	+ 7.7333315	− 7.6661573	− 0.1585056	− 0.17239	8.5342029	9.43010	(h)
+ 8.6500677	− 7.9413623	+ 7.8839578	− 0.1564710	− 0.18288	− 7.7460782	9.36052	(g)
+ 8.4249821	− 8.2106508	+ 8.1468528	− 0.1374072	− 0.18283	− 8.1243681	9.31485	(f)
− 7.6930575	− 8.0524928	+ 7.9839659	− 0.0861436	− 0.17021	− 7.3190574	9.29002	(e)
− 8.5316263	+ 7.3702436	− 7.3147368	− 0.0056825		8.2524386		(d)
− 8.6692035	+ 8.1567973	− 8.0880418	− 9.8703661		8.5871395		(c)
− 8.5851989	+ 8.2098524	− 8.1388499					(b)
− 8.2296301	+ 7.9116482	− 7.8395218					(a)

Co-latitude		g_2^2 or h_2^2	g_{-2}^2 or h_{-2}^2	g_4^2 or h_4^2	g_{-4}^2 or h_{-4}^2	g_6^2 or h_6^2	g_{-6}^2 or h_{-6}^2	g_8^2 or h_8^2
		For X						
(s)	90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r)	85	− 9·2302233	− 9·2374211	+ 9·2859994	+ 9·2874648	− 8·9766835	− 8·9774360	+ 8·5760641
(q)	80	− 9·5225331	− 9·5294044	+ 9·5609074	+ 9·5618466	− 9·2136774	− 9·2136786	+ 8·7557527
(p)	75	− 9·6839411	− 9·6902791	+ 9 6922787	+ 9·6923406	− 9·2732588	− 9·2719397	+ 8·6892203
(o)	70	− 9·7880013	− 9·7936149	+ 9·7518615	+ 9·7506911	− 9·2057574	− 9·2023085	+ 8·2309244
(n)	65	− 9·8574948	− 9·8622148	+ 9·7592651	+ 9·7564846	− 8 9556967	− 8·9479436	− 8·3635141
(m)	60	− 9·9022462	− 9·9059297	+ 9·7186360	+ 9·7137957	+ 7·5792273	+ 7·6895357	− 8·7171569
(l)	55	− 9·9271105	− 9·9296461	+ 9·6237259	+ 9·6161544	+ 8·9829019	+ 8·9813771	− 8·7504083
(k)	50	− 9·9345125	− 9·9358232	+ 9·4510181	+ 9·4392581	+ 9·2109150	+ 9·2046562	− 8·5293782
(i)	45	− 9·9254203	− 9·9254661	+ 9·1124386	+ 9·0903726	+ 9·2718121	+ 9·2621444	+ 7·6256275
(h)	40	− 9·8997082	− 9·8984874	− 8·2299282	− 8·3228271	+ 9·2205906	+ 9·2075399	+ 8·6010709
(g)	35	− 9·8562019	− 9·8537514	− 9·1390818	− 9·1421420	+ 9·0408824	+ 9 0235606	+ 8·7582868
(f)	30	− 9·7924364	− 9·7888307	− 9·3392247	− 9·3351035	+ 8·5726010	+ 8·5436940	+ 8·7021095
(e)	25	− 9·7040061	− 9·6993551	− 9·4008540	− 9·3933416	− 8·4598915	− 8·4595468	+ 8·4017717
(d)	20	− 9·5830440	− 9·5774899	− 9·3777772	− 9·3680746	− 8·8478851	− 8·8352078	− 7·5675487
(c)	15	− 9·4143688	− 9·4080833	− 9·2750663	− 9·2638321	− 8·8982372	− 8·8822694	− 8·3411692
(b)	10	− 9·1634912	− 9·1566632	− 9·0671527	− 9·0548774	− 8·7696554	− 8·7519639	− 8·3593051
(a)	5	− 8·7198527	− 8·7126934	− 8·6478976	− 8·6350118	− 8·3908146	− 8·3722038	− 8·0405778
		For Y						
(s)	90°	0·1490608	+ 0·1490608	− 9·3039628	− 9·3039628	+ 8·6305469	+ 8·6305469	− 7·9937248
(r)	85	0·2971798	+ 0·2971251	− 9·4286962	− 9·4285978	+ 8·7166936	+ 8·7165514	− 8·0230755
(q)	80	0·2899513	+ 0·2897342	− 9·3434723	− 9·3430815	+ 8·4648795	+ 8·4643150	− 7·0831628
(p)	75	0·2777653	+ 0·2772828	− 9·1627656	− 9·1618971	− 7·4258276	− 7·4245731	+ 7·9001954
(o)	70	0·2604063	+ 0·2595633	− 8·6962008	− 8·6946834	− 8·5486249	− 8·5464330	+ 8·1571442
(n)	65	0·2375544	+ 0·2362665	+ 8·7668911	+ 8·7645728	− 8·7807401	− 8·7773914	+ 8·1464844
(m)	60	0·2087638	+ 0·2069597	+ 9·2291831	+ 9·2259358	− 8·8481028	− 8·8434122	+ 7·8159318
(l)	55	0·1734307	+ 0·1710547	+ 9·4372660	+ 9·4329893	− 8·7905964	− 8·7844189	− 7·7072878
(k)	50	0·1307422	+ 0·1277557	+ 9·5585704	+ 9·5531948	− 8·5354254	− 8·5276606	− 8·1969380
(i)	45	0·0795982	+ 0·0759810	+ 9·6297962	+ 9·6232853	+ 7·8653227	+ 7·8559180	− 8·3024323
(h)	40	0·0184825	+ 0·0142335	+ 9·6643407	+ 9·6566925	+ 8·7424371	+ 8·7313897	− 8·1740215
(g)	35	9·9452496	+ 9·9403869	+ 9·6675106	+ 9·6587578	+ 8·9991334	+ 8·9864904	− 6·9399112
(f)	30	9·8567395	+ 9·8513001	+ 9·6404000	+ 9·6306090	+ 9·1195212	+ 9·1053786	+ 8·2508898
(e)	25	9·7480409	+ 9·7420794	+ 9·5807339	+ 9·5700032	+ 9·1590030	+ 9·1435030	+ 8·5394824
(d)	20	9·6109168	+ 9·6045040	+ 9·4819938	+ 9·4704508	+ 9·1301794	+ 9·1135061	+ 8·6375443
(c)	15	9·4299189	+ 9·4231396	+ 9·3298802	+ 9·3176775	+ 9·0272841	+ 9·0096580	+ 8·6117751
(b)	10	9·1703612	+ 9·1633117	+ 9·0904738	+ 9·0777847	+ 8·8207823	+ 8·8024536	+ 8·4530170
(a)	5	8·7215643	+ 8·7143492	+ 8·6535856	+ 8·6405984	+ 8·4028498	+ 8·3840905	+ 8·0614879
		For Z						
(s)	90°	0·3251521	− 0·1490608	− 9·7019028	+ 9·6049928	+ 9·1746149	− 9·1076682	− 8·6469373
(r)	85	0·4716518	− 0·2954694	− 9·8250752	+ 9·7279028	+ 9·2592591	− 9·1918853	− 8·6748548
(q)	80	0·4595388	− 0·2830857	− 9·7351832	+ 9·6371337	+ 9·0031921	− 8·9339319	− 7·9331223
(p)	75	0·4391220	− 0·2622265	− 9·5468581	+ 9·4467910	− 7·9396802	+ 7·9040820	+ 8·5375190
(o)	70	0·4100452	− 0·2325491	− 9·0719504	+ 8·9633622	− 9·0647299	+ 8·9983730	+ 8·7834205
(n)	65	0·3717797	− 0·1935421	+ 9·1177832	− 9·0293674	− 9·2820780	+ 9·2129108	+ 8·7579391
(m)	60	0·3235876	− 0·1444904	+ 9·5635861	− 9·4674444	− 9·3304491	+ 9·2587556	+ 8·4099583
(l)	55	0·2644692	− 0·0844192	+ 9·7485024	− 9·6496041	− 9·2497227	+ 9·1749863	− 8·2707199
(k)	50	0·1930764	− 0·0120096	+ 9·8413814	− 9·7403940	− 8·9674611	+ 8·8872729	− 8·7347126
(i)	45	0·1075832	− 9·9254660	+ 9·8784238	− 9·7755524	+ 8·2451333	− 8·2010456	− 8·8066774
(h)	40	0·0054703	− 9·8223009	+ 9·8720852	− 9·7674429	+ 9·0943653	− 9·0204926	− 8·6383230
(g)	35	9·8831690	− 9·6989781	+ 9·8262724	− 9·7199669	+ 9·3029946	− 9·2250135	− 7·3870768
(f)	30	9·7354209	− 9·5502700	+ 9·7399911	− 9·6321524	+ 9·3545351	− 9·2838789	+ 8·6036723
(e)	25	9·5540476	− 9·3680276	+ 9·6077036	− 9·4984935	+ 9·3315569	− 9·2487787	+ 8·8206054
(d)	20	9·3253266	− 9·1385557	+ 9·4174089	− 9·3070223	+ 9·2112723	− 9·1267676	+ 8·8274271
(c)	15	9·0235165	− 8·8361358	+ 9·1445153	− 9·0331781	+ 8·9876552	− 8·9017918	+ 8·6810303
(b)	10	8·5908122	− 8·4029819	+ 8·7319852	− 8·6199492	+ 8·6080654	− 8·5212175	+ 8·3492310
(a)	5	7·8427509	− 7·6546452	+ 7·9958463	− 7·8833828	+ 7·8909017	− 7·8034563	+ 7·6584936

g_{-8}^2 or h_{-8}^2	g_{10}^2 or h_{10}^2	g_{-10}^2 or h_{-10}^2	Log $(w^{\frac{1}{2}}x'_2)$ for g 1845	Log $(w^{\frac{1}{2}}x'_2)$ for g 1880	Log $(w^{\frac{1}{2}}x'_2)$ for h 1845	Log $(w^{\frac{1}{2}}x'_2)$ for h 1880	
0·0000000	0·0000000	0·0000000					(s)
+ 8·5764743	− 8·1246457	− 8·1248284	− 9·0003380	− 8·64414	8·9589237	8·88026	(r)
+ 8·7551543	− 8·2251999	− 8·2240644	− 9·2465821	− 9·00006	9·1630693	9·18057	(q)
+ 8·6866107	− 7·8938989	− 7·8889557	− 9·4130625	− 9·17391	9·2981728	9·34857	(p)
+ 8·2217238	+ 7·8421800	+ 7·8428848	− 9·4494201	− 9·28401	9·3775470	9·43747	(o)
− 8·3643056	+ 8·2212460	+ 8·2163973	− 9·3857720	− 9·24134	9·4356593	9·46342	(n)
− 8·7121853	+ 8·1484847	+ 8·1389858	− 9·2815873	− 9·14326	9·4496940	9·44502	(m)
− 8·7414667	+ 7·0565795	+ 7·0017982	− 9·0355450	− 8·85326	9·4349657	9·41376	(l)
− 8·5148558	− 8·1024397	− 8·0925142	− 8·1121338	7·79920	9·4159426	9·39273	(k)
+ 7·6531148	− 8·2372423	− 8·2215082	8·6335857	8·91106	9·4156112	9·36217	(i)
+ 8·5895544	− 7·9766187	− 7·9539601	8·8303690	9·13165	9·3435376	9·30274	(h)
+ 8·7418553	+ 7·6995652	+ 7·6883623	8·8864708	9·25197	9·1746978	9·23562	(g)
+ 8·6817047	+ 8·1960180	+ 8·1741540	8·9608000	9·31021	9·0024986	9·10669	(f)
+ 8·3759299	+ 8·2051797	+ 8·1788591	8·9380855	9·30765	8·8412570	8·68897	(e)
− 7·5768056	+ 7·8579191	+ 7·8260756	8·9632078		8·7074084		(d)
− 8·3211194	− 7·4874665	− 7·4673613	8·9834137		8·5925523		(c)
− 8·3362555	− 7·8642717	− 7·8359786					(b)
− 8·0162446	− 7·6324295	− 7·6023775					(a)

			Log $(w^{\frac{1}{2}}y'_2)$ for g 1845	Log $(w^{\frac{1}{2}}y'_2)$ for g 1880	Log $(w^{\frac{1}{2}}y'_2)$ for h 1845	Log $(w^{\frac{1}{2}}y'_2)$ for h 1880	
− 7·9937248	+ 7·3710130	+ 7·3710130	8·9589570	− 8·22278	− 9·5947428	− 9·61273	(s)
− 8·0228896	+ 7·3197369	+ 7·3195072	9·0867615	− 8·57752	− 9·7466642	− 9·76184	(r)
− 7·0824246	− 6·8646773	− 6·8637654	8·9237049	− 8·87954	− 9·7313468	− 9·74233	(q)
+ 7·8985549	− 7·4906716	− 7·4886450	8·6190896	− 9·05940	− 9·7025973	− 9·71818	(p)
+ 8·1542779	− 7·5081552	− 7·5046145	6·2864079	− 9·20313	− 9·6622543	− 9·68077	(o)
+ 8·1421054	− 6·9453078	− 6·9398984	− 8·6511834	− 9·30116	− 9·6058351	− 9·63284	(n)
+ 7·8097980	+ 7·3723589	+ 7·3647818	− 8·9462169	− 9·36789	− 9·5407919	− 9·57699	(m)
− 7·6992095	+ 7·6150620	+ 7·6050830	− 9·0544803	− 9·41753	− 9·4620108	− 9·51990	(l)
− 8·1867840	+ 7·4593087	+ 7·4467655	− 9·1113996	− 9·46038	− 9·4003589	− 9·46327	(k)
− 8·2901339	− 6·9360932	− 6·9209010	− 9·1709924	− 9·48165	− 9·3255370	− 9·40875	(i)
− 8·1595749	− 7·6694338	− 7·6515879	− 9·1484373	− 9·46895	− 9·2425411	− 9·36095	(h)
− 6·9233780	− 7·7511071	− 7·7306838	− 9·0777603	− 9·43355	− 9·1670326	− 9·32054	(g)
+ 8·2323957	− 7·3652396	− 7·3423939	− 9·0008133	− 9·35793	− 9·0892594	− 9·25930	(f)
+ 8·5192132	+ 7·6151712	+ 7·5901328	− 8·9161695	− 9·22687	− 8·9872394	− 9·18108	(e)
+ 8·6157408	+ 8·0169194	+ 7·9899857	− 8·8392841		− 8·8673013		(d)
+ 8·5887256	+ 8·1145181	+ 8·0860452	− 8·9101090		− 8·6502610		(c)
+ 8·4290487	+ 8·0220476	+ 7·9924397					(b)
+ 8·0369565	+ 7·6649740	+ 7·6346705					(a)

			Log $(w^{\frac{1}{2}}z'_2)$ for g 1845	Log $(w^{\frac{1}{2}}z'_2)$ for g 1880	Log $(w^{\frac{1}{2}}z'_2)$ for h 1845	Log $(w^{\frac{1}{2}}z'_2)$ for h 1880	
+ 8·5957848	+ 8·1113757	− 8·0699830	9·2444045	− 8·32660	− 9·8297135	− 9·85171	(s)
+ 8·6230897	+ 8·0587588	− 8·0165207	9·3652823	− 8·85447	− 9·9757835	− 9·98200	(r)
+ 7·8741703	− 7·5963747	+ 7·5584760	9·1952012	− 9·15428	− 9·9475685	− 9·95311	(q)
− 8·4869027	− 8·2158270	+ 8·1731301	8·6805208	− 9·33185	− 9·9027424	− 9·90798	(p)
− 8·7299524	− 8·2222317	+ 8·1764773	− 8·6039489	− 9·42927	− 9·8593376	− 9·86813	(o)
− 8·7014480	− 7·6476485	+ 7·5920063	− 9·1057777	− 9·53328	− 9·7708982	− 9·80784	(n)
− 8·3472102	+ 8·0494009	− 8·0037278	− 9·2475893	− 9·62947	− 9·6007467	− 9·71280	(m)
+ 8·2201459	+ 8·2696603	− 8·2184550	− 9·3678025	− 9·71099	− 9·4981363	− 9·62345	(l)
+ 8·6752876	+ 8·0866115	− 8·0295373	− 9·3905011	− 9·73377	− 9·3158868	− 9·57189	(k)
+ 8·7430124	− 7·5198725	+ 7·4789221	− 9·3624907	− 9·71156	− 9·1590042	− 9·51869	(i)
+ 8·5699101	− 8·2187486	+ 8·1612199	− 9·3096303	− 9·64826	− 8·9246452	− 9·42411	(h)
+ 7·2450250	− 8·2523937	+ 8·1897634	− 9·2624382	− 9·52186	− 8·8819570	− 9·29576	(g)
− 8·5373675	− 7·8098321	+ 7·7390918	− 9·1794047	− 9·36622	− 8·7428613	− 9·08951	(f)
− 8·7500810	+ 7·9820073	− 7·9194719	− 9·0595434	− 9·15830	− 8·2246846	− 8·73871	(e)
− 8·7543574	+ 8·2935574	− 8·2259479	− 8·7276572		7·9776538		(d)
− 8·6061165	+ 8·2707101	− 8·2005895	7·5466060		8·2561463		(c)
− 8·2730225	+ 8·0052617	− 7·9335019					(b)
− 7·5815098	+ 7·3490057	− 7·2762862					(a)

Co-latitude	g_3^2 or h_3^2	g_{-3}^2 or h_{-3}^2	g_5^2 or h_5^2	g_{-5}^2 or h_{-5}^2	g_7^2 or h_7^2	g_{-7}^2 or h_{-7}^2
	For X					
(s) 90°	9·8480308	+ 9·8480308	− 9·3709096	− 9·3709096	+ 8·8687860	+ 8·8687860
(r) 85	9·9863720	+ 9·9861399	− 9·4786687	− 9·4782921	+ 8·9295499	+ 8·9290091
(q) 80	9·9487128	+ 9·9477613	− 9·3296111	− 9·3278471	+ 8·5285027	+ 8·5248152
(p) 75	9·8816519	+ 9·8794031	− 8·9373089	− 8·9304052	− 8·4783552	− 8·4818566
(o) 70	9·7764092	+ 9·7720359	+ 8·7477462	+ 8·7573750	− 8·9177814	− 8·9166963
(n) 65	9·6125179	+ 9·6044081	+ 9·2723970	+ 9·2729279	− 9·0201900	− 9·0163376
(m) 60	9·3272316	+ 9·3099911	+ 9·4525567	+ 9·4500502	− 8·9454901	− 8·9382586
(l) 55	8·3164902	+ 8·1115896	+ 9·5165375	+ 9·5114272	− 8·5975586	− 8·5831744
(k) 50	− 9·1792226	− 9·1965643	+ 9·5007426	+ 9·4929002	+ 8·3601654	+ 8·3642941
(i) 45	− 9·4624249	− 9·4677915	+ 9·4056363	+ 9·3945693	+ 8·8844720	+ 8·8762393
(h) 40	− 9·5899972	− 9·5906424	+ 9·1980083	+ 9·1821437	+ 9·0138313	+ 9·0012708
(g) 35	− 9·6444103	− 9·6420969	+ 8·6901074	+ 8·6573749	+ 8·9803126	+ 8·9640401
(f) 30	− 9·6493033	− 9·6447774	− 8·6857852	− 8·6920740	+ 8·7779371	+ 8·7571735
(e) 25	− 9·6108577	− 9·6045736	− 9·0633882	− 9·0558372	+ 8·1116563	+ 8·0726248
(d) 20	− 9·5265698	− 9·5188805	− 9·1531394	− 9·1416661	− 8·4198889	− 8·4078874
(c) 15	− 9·3842658	− 9·3754893	− 9·1096932	− 9·0960512	− 8·6434723	− 8·6253225
(b) 10	− 9·1512203	− 9·1416659	− 8·9357505	− 8·9207619	− 8·5760855	− 8·5557052
(a) 5	− 8·7189186	− 8·7078956	− 8·5345233	− 8·5187747	− 8·2245069	− 8·2030346
	For Y					
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·2346004	+ 9·2345238	− 8·7476220	− 8·7475017	+ 8·2310911	+ 8·2309271
(q) 80	9·5268437	+ 9·5265397	− 9·0091883	− 9·0087106	+ 8·4454953	+ 8·4448439
(p) 75	9·6881433	+ 9·6874678	− 9·1151374	− 9·1140759	+ 8·4548136	+ 8·4533661
(o) 70	9·7920564	+ 9·7908762	− 9·1303157	− 9·1284611	+ 8·2601100	+ 8·2575809
(n) 65	9·8613684	+ 9·8595653	− 9·0569073	− 9·0540738	+ 6·1969281	+ 6·1930643
(m) 60	9·9059097	+ 9·9033840	− 8·8403026	− 8·8363337	− 8·2917530	− 8·2863408
(l) 55	9·9305414	+ 9·9272151	− 7·7944086	− 7·7891815	− 8·5301667	− 8·5230388
(k) 50	9·9376956	+ 9·9335145	+ 8·8231129	+ 8·8165427	− 8·5673784	− 8·5584190
(i) 45	9·9283476	+ 9·9232835	+ 9·1428291	+ 9·1348713	− 8·4152819	− 8·4044303
(h) 40	9·9023799	+ 9·8964313	+ 9·3024355	+ 9·2930877	− 7·4435780	− 7·2308310
(g) 35	9·8586255	+ 9·8518177	+ 9·3852434	+ 9·3745455	+ 8·4342169	+ 8·4196288
(f) 30	9·7946272	+ 9·7870120	+ 9·4139574	+ 9·4019906	+ 8·7465941	+ 8·7302758
(e) 25	9·7059866	+ 9·6976405	+ 9·3950853	+ 9·3819700	+ 8·8751730	+ 8·8572884
(d) 20	9·5848430	+ 9·5758651	+ 9·3264249	+ 9·3123168	+ 8·8995848	+ 8·8803464
(c) 15	9·4160225	+ 9·4065315	+ 9·1960432	+ 9·1811288	+ 8·8310121	+ 8·8106743
(b) 10	9·1650343	+ 9·1551650	+ 8·9713924	+ 8·9558835	+ 8·6463454	+ 8·6251969
(a) 5	8·7213293	+ 8·7112281	+ 8·5430803	+ 8·5272070	+ 8·2406511	+ 8·2190058
	For Z					
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·5332828	− 9·4099310	− 9·2226559	+ 9·1443513	+ 8·8312018	− 8·7737214
(q) 80	9·8206472	− 9·6969541	− 9·4793973	+ 9·4005091	+ 9·0408418	− 8·9825238
(p) 75	9·9737252	− 9·8494744	− 9·5772306	+ 9·4973505	+ 9·0422096	− 8·9823431
(o) 70	0·0659330	− 9·9409249	− 9·5808989	+ 9·4995573	+ 8·8366204	− 8·7736800
(n) 65	0·1198467	− 9·9939039	− 9·4924925	+ 9·4089957	+ 6·9239275	− 6·3609825
(m) 60	0·1450044	− 0·0179775	− 9·2576536	+ 9·1701410	− 8·8304334	+ 8·7705187
(l) 55	0·1458703	− 0·0176425	− 8·2141353	+ 8·0648599	− 9·0460602	+ 8·9819713
(k) 50	0·1243413	− 9·9948313	+ 9·1827208	− 9·1034589	− 9·0551806	+ 8·9875997
(i) 45	0·0806648	− 9·9498326	+ 9·4696339	− 9·3852363	− 8·8697855	+ 8·7976962
(h) 40	0·0137222	− 9·8815616	+ 9·5887463	− 9·5014869	− 7·8755713	+ 7·7604106
(g) 35	9·9209205	− 9·7874719	+ 9·6227455	− 9·5332081	+ 8·7947251	− 8·7266073
(f) 30	9·7977039	− 9·6630449	+ 9·5923846	− 9·5008842	+ 9·0491285	− 8·9762556
(e) 25	9·6364062	− 9·5006516	+ 9·5009502	− 9·4077496	+ 9·1054555	− 9·0298155
(d) 20	9·4236811	− 9·2869796	+ 9·3407738	− 9·2461395	+ 9·0384892	− 8·9607710
(c) 15	9·1340608	− 8·9965906	+ 9·0896378	− 8·9938562	+ 8·8492354	− 8·7699291
(b) 10	8·7099353	− 8·5718981	+ 8·6918800	− 8·5952597	+ 8·4915026	− 8·4110599
(a) 5	7·9669716	− 7·8285869	+ 7·9643264	− 7·8671948	+ 7·7865882	− 7·7054599

g_9^2 or h_9^2	g_{-9}^2 or h_{-9}^2	Log $(w^{\frac{1}{2}}x'_2)$ for g 1845	Log $(w^{\frac{1}{2}}x'_2)$ for g 1880	Log $(w^{\frac{1}{2}}x'_2)$ for h 1845	Log $(w^{\frac{1}{2}}x'_2)$ for h 1880	
− 8·3487365	− 8·3487365	− 9·4165845	− 9·46660	− 9·1020953	− 8·83075	(s)
− 8·3420552	− 8·3413140	− 9·5998000	− 9·61306	− 9·2499861	− 8·93173	(r)
+ 6·9856038	+ 7·0166510	− 9·6157053	− 9·62339	− 9·1412819	− 8·75271	(q)
+ 8·3685074	+ 8·3680005	− 9·6207744	− 9·62920	− 9·0886428	− 8·58671	(p)
+ 8·5000167	+ 8·4966557	− 9·6167202	− 9·61804	− 8·9158175	− 8·65932	(o)
+ 8·3168924	+ 8·3092851	− 9·5953310	− 9·57198	− 8·5803042	− 8·71250	(n)
− 7·4299548	− 7·4523989	− 9·5073007	− 9·46745	− 8·5001535	− 8·48918	(m)
− 8·3899161	− 8·3830853	− 9·3714066	− 9·27984	7·9361862	− 7·91522	(l)
− 8·5016171	− 8·4898153	− 9·1202734	− 8·94013	8·3568454	− 6·78697	(k)
− 8·3047692	− 8·2871091	8·2967310	8·03868	8·1108485	7·70289	(i)
+ 7·4822555	+ 7·4962089	9·1232131	9·01285	7·9888602	− 7·24670	(h)
+ 8·3839648	+ 8·3676885	9·3401625	9·27037	7·7936048	7·05588	(g)
+ 8·5012981	+ 8·4802008	9·4443818	9·37487	8·4379213	8·01158	(f)
+ 8·3611111	+ 8·3357632	9·4188890	9·37739	8·4700056	8·42563	(e)
+ 7·7261103	+ 7·6888375	9·3448256		8·3002559		(d)
− 7·9747095	− 7·9534329	9·2293526		8·1159969		(c)
− 8·1225096	− 8·0960172					(b)
− 7·8424166	− 7·8152235					(a)

		Log $(w^{\frac{1}{2}}y'_2)$ for g 1845	Log $(w^{\frac{1}{2}}y'_2)$ for g 1880	Log $(w^{\frac{1}{2}}y'_2)$ for h 1845	Log $(w^{\frac{1}{2}}y'_2)$ for h 1880	
0·0000000	0·0000000					(s)
− 7·6918832	− 7·6916754	− 8·8292899	− 8·75967	7·7420330	7·27649	(r)
− 7·8385833	− 7·8377582	− 9·0766563	− 9·05714	6·3932490	7·93979	(q)
− 7·6687576	− 7·6669240	− 9·2835787	− 9·26467	− 8·3296684	8·16440	(p)
+ 6·7766652	+ 6·7734617	− 9·3998498	− 9·43424	− 8·5476708	8·10595	(o)
+ 7·7675669	+ 7·7626727	− 9·5172169	− 9·56169	− 8·7354772	− 6·93194	(n)
+ 7·8961527	+ 7·8892972	− 9·6084184	− 9·64810	− 8·8601310	− 8·16394	(m)
+ 7·7058535	+ 7·6968249	− 9·6829915	− 9·71152	− 8·8891636	− 8·41858	(l)
− 7·1388144	− 7·1274658	− 9·7432415	− 9·75022	− 8·7243447	− 8·46691	(k)
− 7·9042522	− 7·8905069	− 9·7809250	− 9·77089	− 8·6944866	− 8·35610	(i)
− 8·0322429	− 8·0160966	− 9·7910844	− 9·76944	− 8·6669625	− 7·65247	(h)
− 7·8535102	− 7·8350320	− 9·7870679	− 9·74861	− 8·6197586	8·13021	(g)
+ 7·3522563	+ 7·3315864	− 9·7485789	− 9·70875	− 8·5474396	8·37525	(f)
+ 8·1363213	+ 8·1136675	− 9·6745747	− 9·64616	− 8·5467031	8·55506	(e)
+ 8·3442632	+ 8·3198946	− 9·5440404		− 8·3771993		(d)
+ 8·3723953	+ 8·3466341	− 9·3076945		− 7·9910581		(c)
+ 8·2441366	+ 8·2173485					(b)
+ 7·8687812	+ 7·8413638					(a)

		Log $(w^{\frac{1}{2}}z'_2)$ for g 1845	Log $(w^{\frac{1}{2}}z'_2)$ for g 1880	Log $(w^{\frac{1}{2}}z'_2)$ for h 1845	Log $(w^{\frac{1}{2}}z'_2)$ for h 1880	
0·0000000	0·0000000					(s)
− 8·3889936	+ 8·3435046	− 8·9605827	− 9·18357	− 8·1908646	− 8·31789	(r)
− 8·5310063	+ 8·4843927	− 9·3441005	− 9·45015	− 8·5762340	− 8·33970	(q)
− 8·3536165	+ 8·3044361	− 9·5188418	− 9·60590	− 8·8964679	− 8·33951	(p)
+ 7·4383451	− 7·4114932	− 9·6561338	− 9·72748	− 8·9163269	− 8·45667	(o)
+ 8·4234618	− 8·3746363	− 9·7752462	− 9·81218	− 8·9726784	8·28090	(n)
+ 8·5334777	− 8·4806479	− 9·8776726	− 9·88095	− 8·7826169	8·78693	(m)
+ 8·3207463	− 8·2625588	− 9·9510276	− 9·92251	− 8·8312441	8·93709	(l)
− 7·7145595	+ 7·6755651	− 9·9670755	− 9·94196	− 9·0449914	8·95555	(k)
− 8·4531268	+ 8·3957899	− 9·9649071	− 9·93968	− 9·1114140	8·94344	(i)
− 8·5411098	+ 8·4789320	− 9·9386717	− 9·93056	− 9·1339657	8·85416	(h)
− 8·3146287	+ 8·2469825	− 9·8894320	− 9·90222	− 9·0375498	8·62486	(g)
+ 7·7452413	− 7·6937328	− 9·8122954	− 9·83698	− 9·0554147	8·37135	(f)
+ 8·4627445	− 8·3961884	− 9·6719932	− 9·75928	− 9·0406362	− 8·39937	(e)
+ 8·5797208	− 8·5098675	− 9·4381990		− 8·9491244		(d)
+ 8·4872985	− 8·4153040	− 9·0464126		− 8·7787178		(c)
+ 8·1860276	− 8·1125716					(b)
+ 7·5114765	− 7·4371537					(a)

Co-latitude		g_3^3 or h_3^3	g_{-3}^3 or h_{-3}^3	g_5^3 or h_5^3	g_{-5}^3 or h_{-5}^3	g_7^3 or h_7^3	g_{-7}^3 or h_{-7}^3
		For X					
(s)	90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r)	85	− 9·4051791	− 9·4118677	+ 9·2921179	+ 9·2934355	− 8·9042670	− 8·9049161
(q)	80	− 9·6925825	− 9·6988908	+ 9·5580602	+ 9·5587421	− 9·1247071	− 9·1244846
(p)	75	− 9·8457243	− 9·8514113	+ 9·6736613	+ 9·6732724	− 9·1505140	− 9·1487136
(o)	70	− 9·9380187	− 9·9428617	+ 9·7090841	+ 9·7071570	− 9·0106835	− 9·0060559
(n)	65	− 9·9920390	− 9·9958407	+ 9·6806883	+ 9·6766619	− 8·4805580	− 8·4649922
(m)	60	− 0·0173203	− 0·0199142	+ 9·5852377	+ 9·5782685	+ 8·7060792	+ 8·7078609
(l)	55	− 0·0183232	− 0·0195793	+ 9·3953985	+ 9·3835774	+ 9·0584023	+ 9·0533667
(k)	50	− 9·9969398	− 9·9967681	+ 8·9868444	+ 8·9604145	+ 9·1512494	+ 9·1423007
(i)	45	− 9·9534145	− 9·9517682	− 8·6323743	− 8·6660394	+ 9·1055361	+ 9·0926765
(h)	40	− 9·8866214	− 9·8834984	− 9·1737142	− 9·1751411	+ 8·9102533	+ 8·8921275
(g)	35	− 9·7939654	− 9·7894086	− 9·3198509	− 9·3150242	+ 8·3499326	+ 8·3132314
(f)	30	− 9·6708856	− 9·6649816	− 9·3423087	− 9·3339987	− 8·4520847	− 8·4499489
(e)	25	− 9·5097115	− 9·5025884	− 9·2808513	− 9·2700516	− 8·7494247	− 8·7363151
(d)	20	− 9·2970930	− 9·2889165	− 9·1392106	− 9·1265054	− 8·7617711	− 8·7448610
(c)	15	− 9·0075594	− 8·9985273	− 8·8998891	− 8·8857380	− 8·6106106	− 8·5914138
(b)	10	− 8·5834976	− 8·5738350	− 8·5094856	− 8·4943094	− 8·2732122	− 8·2525341
(a)	5	− 7·8405728	− 7·8305237	− 7·7859773	− 7·7701863	− 7·5785800	− 7·5570479
		For Y					
(s)	90°	0·3251521	+ 0·3251521	− 9·3709096	− 9·3709096	+ 8·6469373	+ 8·6469373
(r)	85	0·4716483	+ 0·4715717	− 9·4870941	− 9·4869738	+ 8·7166021	+ 8·7164381
(q)	80	0·4595246	+ 0·4592206	− 9·3699509	− 9·3694732	+ 8·3696606	+ 8·3690092
(p)	75	0·4390905	+ 0·4384150	− 9·0920722	− 9·0910107	− 8·1535966	− 8·1521491
(o)	70	0·4099903	+ 0·4088101	+ 8·0626029	+ 8·0607483	− 8·6896857	− 8·6871566
(n)	65	0·3716955	+ 0·3698924	+ 9·1887079	+ 9·1858744	− 8·8402822	− 8·8364184
(m)	60	0·3234699	+ 0·3209442	+ 9·4589613	+ 9·4549924	− 8·8300034	− 8·8245912
(l)	55	0·2643142	+ 0·2609879	+ 9·5975593	+ 9·5923322	− 8·6316823	− 8·6245544
(k)	50	0·1928817	+ 0·1887006	+ 9·6694877	+ 9·6629175	− 6·4407835	− 6·4318241
(i)	45	0·1073472	+ 0·1022831	+ 9·6948642	+ 9·6869064	+ 8·6947768	+ 8·6839252
(h)	40	0·0051931	+ 9·9992445	+ 9·6812594	+ 9·6719116	+ 8·9732997	+ 8·9605527
(g)	35	9·8828519	+ 9·8760441	+ 9·6306019	+ 9·6199040	+ 9·0879643	+ 9·0733762
(f)	30	9·7350662	+ 9·7274510	+ 9·5409512	+ 9·5289844	+ 9·1099588	+ 9·0936405
(e)	25	9·5536588	+ 9·5453127	+ 9·4062664	+ 9·3931511	+ 9·0556821	+ 9·0377975
(d)	20	9·3249085	+ 9·3159306	+ 9·2142578	+ 9·2001497	+ 8·9226001	+ 8·9033617
(c)	15	9·0230746	+ 9·0135836	+ 8·9401610	+ 8·9252466	+ 8·5909619	+ 8·6706241
(b)	10	8·5903528	+ 8·5804835	+ 8·5268326	+ 8·5113237	+ 8·3064288	+ 8·2852803
(a)	5	7·8422807	+ 7·8321795	+ 7·7902369	+ 7·7743636	+ 7·5865614	+ 7·5649161
		For Z					
(s)	90°	0·4500908	− 0·3251521	− 9·6719396	+ 9·5927583	+ 9·0729060	− 9·0149140
(r)	85	0·5949695	− 0·4699159	− 9·7865709	+ 9·7070916	+ 9·1410830	− 9·0826139
(q)	80	0·5779667	− 0·4525721	− 9·6648076	+ 9·5842935	+ 8·7901407	− 8·7291947
(p)	75	0·5493107	− 0·4233586	− 9·3796692	+ 9·2962736	− 8·5619008	+ 8·5084421
(o)	70	0·5085047	− 0·3817958	+ 8·3144198	− 8·2815050	− 9·0882002	+ 9·0293430
(n)	65	0·4548111	− 0·3271681	+ 9·4456260	− 9·3677864	− 9·2238439	+ 9·1624120
(m)	60	0·3872013	− 0·2584748	+ 9·6972769	− 9·6162506	− 9·1946364	+ 9·1303489
(l)	55	0·3042791	− 0·1743523	+ 9·8123863	− 9·7291165	− 8·9736325	+ 8·9049336
(k)	50	0·2041627	− 0·0729545	+ 9·8557773	− 9·7704410	− 7·0137040	+ 5·4333448
(i)	45	0·0842998	− 9·9517681	+ 9·8469196	− 9·7595746	+ 8·9696986	− 8·9050844
(h)	40	9·9411693	− 9·8073118	+ 9·7924061	− 9·7031075	+ 9·2083195	− 9·1391134
(g)	35	9·7697800	− 9·6346353	+ 9·6927525	− 9·6015865	+ 9·2743108	− 9·2020703
(f)	30	9·5627754	− 9·4264210	+ 9·5439244	− 9·4510201	+ 9·2372839	− 9·1625222
(e)	25	9·3087103	− 9·1712608	+ 9·3366151	− 9·2421467	+ 9·1104730	− 9·0335473
(d)	20	8·9883779	− 8·8499822	+ 9·0530509	− 8·9572357	+ 8·8858923	− 8·8071462
(c)	15	8·5657438	− 8·4265798	+ 8·6581741	− 8·5612678	+ 8·5335114	− 8·4533088
(b)	10	7·9598844	− 7·8201537	+ 8·0717224	− 7·9740129	+ 7·9758786	− 7·8946115
(a)	5	6·9125534	− 6·7724755	+ 7·0358763	− 6·9376751	+ 6·9567741	− 6·8748579

g_9^3 or h_9^3	g_{-9}^3 or h_{-9}^3	Log $(w^{\frac{1}{2}}x'_3)$ for g		Log $(w^{\frac{1}{2}}x'_3)$ for h		
		1845	1880	1845	1880	
0.0000000	0·0000000					(s)
+ 8·4556348	+ 8·4559624	7·4370727	6·77589	8·6712217	8·37065	(r)
+ 8·6100030	+ 8·6091840	8·4997799	8·26013	8·9705110	8·92011	(q)
+ 8·4716859	+ 8·4683753	8·5992028	8·38741	9·1312483	9·08147	(p)
+ 7·2182685	+ 7·1633930	8·1629144	8·69295	9·2538389	9·13767	(o)
− 8·4345877	− 8·4322523	− 5·9777017	8·79593	9·2993001	9·16324	(n)
− 8·6095550	− 8·6027939	− 8·0632058	8·83374	9·2777730	9·10855	(m)
− 8·4995520	− 8·4879702	− 7·7950319	8·90752	9·2103682	9·02696	(l)
− 7·7368227	− 7·7016122	8·4656186	8·94230	9·0699481	8·96840	(k)
+ 8·3630246	+ 8·3536595	8·4816454	8·88947	9·0353298	8·91017	(i)
+ 8·5922023	+ 8·5765607	7·7104639	8·88474	8·9840077	8·79860	(h)
+ 8·5558849	+ 8·5354987	8·1808209	8·79148	8·9549728	8·68188	(g)
+ 8·2592075	+ 8·2321484	8·1278195	8·65842	8·9055886	8·51485	(f)
− 7·4078296	− 7·4250861	8·1718422	8·40050	8·6907024	8·46422	(e)
− 8·1799489	− 8·1599259	− 7·7295599		8·4145425		(d)
− 8·1963258	− 8·1722330	− 8·0216241		7·6497650		(c)
− 7·9385690	− 7·9124079					(b)
− 7·2833888	− 7·2561164					(a)
		Log $(w^{\frac{1}{2}}y'_3)$ for g		Log $(w^{\frac{1}{2}}y'_3)$ for h		
		1845	1880	1845	1880	
− 7·9807598	− 7·9807598	− 8·9087286	− 9·23649	− 9·5033614	− 9·43372	(s)
− 7·9837709	− 7·9835631	− 9·0490855	− 9·37723	− 9·6597419	− 9·57374	(r)
− 5·9022650	− 5·9014399	− 9·0041220	− 9·35344	− 9·6482319	− 9·54798	(q)
+ 7·9932370	+ 7·9914034	− 9·0152869	− 9·31857	− 9·6086316	− 9·51169	(p)
+ 8·1614177	+ 8·1582142	− 9·0069806	− 9·28205	− 9·5678390	− 9·45476	(o)
+ 8·0385406	+ 8·0336464	− 8·9669290	− 9·24086	− 9·5122355	− 9·38848	(n)
+ 6·7455904	+ 6·7387349	− 8·9416260	− 9·19541	− 9·4604475	− 9·33744	(m)
− 8·0543139	− 8·0452853	− 8·9237308	− 9·14623	− 9·4034959	− 9·28983	(l)
− 8·2598551	− 8·2485065	− 8·8487456	− 9·09355	− 9·3649541	− 9·24160	(k)
− 8·1915417	− 8·1777964	− 8·7436255	− 9·03667	− 9·3078370	− 9·18314	(i)
− 7·5279100	− 7·5117637	− 8·7080823	− 8·96874	− 9·2307222	− 9·12203	(h)
+ 8·1372439	+ 8·1187657	− 8·5682107	− 8·91039	− 9·1524446	− 9·05122	(g)
+ 8·4578190	+ 8·4371491	− 8·4975933	− 8·89904	− 8·9802222	− 8·95230	(f)
+ 8·5512518	+ 8·5285980	− 8·4371567	− 8·78750	− 8·7897985	− 8·74537	(e)
+ 8·5101302	+ 8·4857616	− 8·4462017		− 8·5726190		(d)
+ 8·3392714	+ 8·3135102	− 8·3862753		− 8·4287426		(c)
+ 7·9939883	+ 7·9672002					(b)
+ 7·2963100	+ 7·2688926					(a)
		Log $(w^{\frac{1}{2}}z'_3)$ for g		Log $(w^{\frac{1}{2}}z'_3)$ for h		
		1845	1880	1845	1880	
− 8·5036385	+ 8·4578810	− 9·1587241	− 9·23453	− 9·5991559	− 9·53216	(s)
− 8·5052404	+ 8·4588014	− 9·2680694	− 9·33759	− 9·7390886	− 9·63812	(r)
− 6·4895046	+ 6·2766573	− 9·2538266	− 9·26248	− 9·7341292	− 9·59563	(q)
+ 8·5011848	− 8·4544120	− 9·1671584	− 9·21935	− 9·7067138	− 9·56455	(p)
+ 8·6575875	− 8·6086939	− 9·2328602	− 9·22040	− 9·6985323	− 9·48917	(o)
+ 8·5201268	− 8·4675377	− 9·2461627	− 9·15897	− 9·6392778	− 9·18679	(n)
+ 7·2374026	− 7·1172657	− 9·2528054	− 9·12246	− 9·5523111	− 9·30888	(m)
− 8·4901191	+ 8·4382588	− 9·0187648	− 8·98880	− 9·4684001	− 9·28943	(l)
− 8·6680194	+ 8·6112573	− 8·8419660	− 8·76469	− 9·2710665	− 9·22608	(k)
− 8·5662688	+ 8·5048289	− 8·8955517	− 8·77170	− 9·1939027	− 9·03801	(i)
− 7·8682170	+ 7·7899781	− 8·8507362	− 8·63506	− 9·1280394	− 8·95272	(h)
+ 8·4187914	− 8·3584888	− 8·2829114	− 8·46525	− 9·0041321	− 8·90673	(g)
+ 8·6813702	− 8·6159413	− 7·9436320	− 8·43227	− 8·9308368	− 8·85367	(f)
+ 8·7025229	− 8·6339861	8·0892188	− 8·57883	− 8·9118585	− 8·86080	(e)
+ 8·5700096	− 8·4990727	6·4671088		− 8·8314722		(d)
+ 8·2784632	− 8·2056659	− 8·5453453		− 8·6547002		(c)
+ 7·7601107	− 7·6859724					(b)
+ 6·7632110	− 6·6882602					(a)

Co-latitude	g_4^3 or h_4^3	g_{-4}^3 or h_{-4}^3	g_6^3 or h_6^3	g_{-6}^3 or h_{-6}^3	g_8^3 or h_8^3	g_{-8}^3 or h_{-8}^3
	For X					
(*s*) 90°	9·8480308	+ 9·8480308	− 9·2837594	− 9·2837594	+ 8·7340874	+ 8·7340874
(*r*) 85	9·9814287	+ 9·9811292	− 9·3792582	− 9·3788072	+ 8·7738883	+ 8·7732606
(*q*) 80	9·9278717	+ 9·9266203	− 9·1773499	− 9·1750828	+ 8·1894498	+ 8·1835239
(*p*) 75	9·8298003	+ 9·8267270	− 8·4239385	− 8·4056399	− 8·5482583	− 8·5505619
(*o*) 70	9·6662551	+ 9·6597502	+ 9·0098338	+ 9·0127185	− 8·8467462	− 8·8446976
(*n*) 65	9·3694571	+ 9·3541557	+ 9·3137189	+ 9·3123860	− 8·8693026	− 8·8641846
(*m*) 60	8·0577755	+ 7·5436536	+ 9·4179855	+ 9·4137730	− 8·6578131	− 8·6478277
(*l*) 55	− 9·2601213	− 9·2748257	+ 9·4157874	+ 9·4085623	+ 7·3360205	+ 7·4223857
(*k*) 50	− 9·5168149	− 9·5213555	+ 9·3169507	+ 9·3060162	+ 8·6816720	+ 8·6751934
(*i*) 45	− 9·6213907	− 9·6215180	+ 9·0821448	+ 9·0651603	+ 8·8658516	+ 8·8540973
(*h*) 40	− 9·6515150	− 9·6486344	+ 8·4045856	+ 8·3536013	+ 8·8456126	+ 8·8295033
(*g*) 35	− 9·6280551	− 9·6227712	− 8·7378852	− 8·7404845	+ 8·6353113	+ 8·6136666
(*f*) 30	− 9·5564877	− 9·5491763	− 9·0117308	− 9·0035547	+ 7·8758656	+ 7·8249633
(*e*) 25	− 9·4342778	− 9·4252443	− 9·0557599	− 9·0434760	− 8·3149814	− 8·3032654
(*d*) 20	− 9·2509932	− 9·2405287	− 8·9744157	− 8·9595449	− 8·5002381	− 8·4815216
(*c*) 15	− 8·9829255	− 8·9713276	− 8·7729983	− 8·7563119	− 8·4175675	− 8·3958962
(*b*) 10	− 8·5735539	− 8·5611333	− 8·4063464	− 8·3884174	− 8·1159588	− 8·0925362
(*a*) 5	− 7·8392058	− 7·8262857	− 7·6960362	− 7·6773746	− 7·4396387	− 7·4152363
	For Y					
(*s*) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(*r*) 85	9·4090688	+ 9·4089704	− 8·8327135	− 8·8325713	+ 8·2663374	+ 8·2661515
(*q*) 80	9·6964170	+ 9·6960262	− 9·0820441	− 9·0814796	+ 8·4597634	+ 8·4590252
(*p*) 75	9·8494685	+ 9·8486000	− 9·1647329	− 9·1634784	+ 8·4203488	+ 8·4187083
(*o*) 70	9·9416404	+ 9·9401230	− 9·1382252	− 9·1360333	+ 8·0635147	+ 8·0606484
(*n*) 65	9·9955096	+ 9·9931913	− 8·9786465	− 8·9752978	− 7·9805350	− 7·9761560
(*m*) 60	0·0206158	+ 0·0173685	− 8·4232583	− 8·4185677	− 8·4505241	− 8·4443903
(*l*) 55	0·0214248	+ 0·0171481	+ 8·7492072	+ 8·7430297	− 8·5565220	− 8·5484437
(*k*) 50	9·9998351	+ 9·9944595	+ 9·1383994	+ 9·1306346	− 8·4543282	− 8·4441742
(*i*) 45	9·9560966	+ 9·9495857	+ 9·3076446	+ 9·2982399	− 7·8349288	− 7·8226304
(*h*) 40	9·8890905	+ 9·8814423	+ 9·3833350	+ 9·3722876	+ 8·3462939	+ 8·3318473
(*g*) 35	9·7962278	+ 9·7874750	+ 9·3951208	+ 9·3824778	+ 8·6913485	+ 8·6748153
(*f*) 30	9·6729540	+ 9·6631630	+ 9·3515974	+ 9·3374548	+ 8·8150698	+ 8·7965757
(*e*) 25	9·5116045	+ 9·5008738	+ 9·2517328	+ 9·2362328	+ 8·8219245	+ 8·8016553
(*d*) 20	9·2988346	+ 9·2872916	+ 9·0859489	+ 9·0692756	+ 8·7295692	+ 8·7077657
(*c*) 15	9·0091781	+ 8·9969754	+ 8·8310674	+ 8·8134413	+ 8·5257066	+ 8·5026571
(*b*) 10	8·5850259	+ 8·5723368	+ 8·4309070	+ 8·4125783	+ 8·1594058	+ 8·1354375
(*a*) 5	7·8420457	+ 7·8290585	+ 7·7020078	+ 7·6832485	+ 7·4499341	+ 7·4254027
	For Z					
(*s*) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(*r*) 85	9·6287171	− 9·5329822	− 9·1986752	+ 9·1324117	+ 8·7415538	− 8·6907907
(*q*) 80	9·9111897	− 9·8150453	− 9·4431909	+ 9·3762605	+ 8·9302334	− 8·8785328
(*p*) 75	0·0560248	− 9·9592114	− 9·5177893	+ 9·4497118	+ 8·8829436	− 8·8294337
(*o*) 70	0·1364985	− 0·0387764	− 9·4798430	+ 9·4100149	+ 8·5159624	− 8·4576752
(*n*) 65	0·1749781	− 0·0761346	− 9·3055512	+ 9·2327884	− 8·4122776	+ 8·3634538
(*m*) 60	0·1807110	− 0·0805668	− 8·7355844	+ 8·6506268	− 8·8651219	+ 8·8094823
(*l*) 55	0·1577656	− 0·0561802	+ 9·0281839	− 8·9622104	− 8·9479957	+ 8·8886850
(*k*) 50	0·1075044	− 0·0043809	+ 0·3906232	− 9·3185107	− 8·8179497	+ 8·7544051
(*i*) 45	0·0294508	− 9·9247384	+ 9·5260351	− 9·4508054	− 8·1702618	+ 8·0911924
(*h*) 40	9·9214813	− 9·8151774	+ 9·5609837	− 9·4831957	+ 8·6308437	− 8·5701989
(*g*) 35	9·7795833	− 9·6717339	+ 9·5238622	− 9·4437775	+ 8·9283141	− 8·8620974
(*f*) 30	9·5971025	− 9·4878006	+ 9·4212153	− 9·3390520	+ 8·9932442	− 8·9237944
(*e*) 25	9·3631061	− 9·2524897	+ 9·2487618	− 9·1647550	+ 8·9276540	− 8·8556471
(*d*) 20	9·0587636	− 8·9470109	+ 8·9914464	− 8·9058653	+ 8·7438444	− 8·6697461
(*c*) 15	8·6483144	− 8·5356392	+ 8·6158018	− 8·5289513	+ 8·4192639	− 8·3435130
(*b*) 10	8·0510301	− 7·9376746	+ 8·0425290	− 7·9547466	+ 7·8798777	− 7·8029259
(*a*) 5	7·0087944	− 6·8950220	+ 7·0143856	− 6·9260338	+ 6·8711764	− 6·7934944

g_{10}^3 or h_{10}^3	g_{-10}^3 or h_{-10}^3	Log ($w^{\frac{1}{2}}x'_3$) for g		Log ($w^{\frac{1}{2}}x'_3$) for h		
		1845	1880	1845	1880	
− 8·1839263	− 8·1839263	8·1844905	8·70657	7·6261821	8·68054	(*s*)
− 8·1449281	− 8·1440757	8·4034278	8·82961	8·4770273	8·81794	(*r*)
+ 7·5501895	+ 7·5554364	8·5337571	8·62268	8·4111167	8·90806	(*q*)
+ 8·2686541	+ 8·2675083	8·5207706	8·53528	8·4956828	8·94977	(*p*)
+ 8·3120597	+ 8·3078154	8·4333108	8·37632	8·2742975	8·91018	(*o*)
+ 7·9077224	+ 7·8963964	8·5340042	8·26996	− 8·1951856	8·81655	(*n*)
− 7·9981640	− 7·9956009	8·7158365	8·30450	− 8·2953953	8·84020	(*m*)
− 8·3239187	− 8·3145229	8·3859351	8·16301	− 8·5652433	8·72777	(*l*)
− 8·2508871	− 8·2360488	− 8·2783318	7·70530	− 8·7698872	8·23190	(*k*)
− 7·4402748	− 7·3996055	− 8·5991419	− 8·31744	− 8·7922076	− 7·59684	(*i*)
+ 8·1388764	+ 8·1245648	− 8·8024604	− 8·54277	− 8·8178329	− 8·13217	(*h*)
+ 8·3285337	+ 8·3080495	− 8·8071613	− 8·68597	− 8·8232854	− 8·17426	(*g*)
+ 8·2169423	+ 8·1913313	− 8·6018742	− 8·71768	− 8·7229491	− 8·27509	(*f*)
+ 7·6489598	+ 7·6107379	− 7·7115345	− 8·71428	− 8·5728193	− 8·40830	(*e*)
− 7·7689928	− 7·7490909	7·7101469		− 8·4213513		(*d*)
− 7·9473046	− 7·9208722	− 8·2909557		− 7·9520340		(*c*)
− 7·7438166	− 7·7149242					(*b*)
− 7·1128454	− 70·827033					(*a*)

		Log ($w^{\frac{1}{2}}y'_3$) for g		Log ($w^{\frac{1}{2}}y'_3$) for h		
		1845	1880	1845	1880	
0·0000000	0·0000000					(*s*)
− 7·6945973	− 7·6943676	7·7936200	8·38335	7·8489983	8·12484	(*r*)
− 7·8089312	− 7·8080193	7·7841841	8·68284	8·2257579	8·45621	(*q*)
− 7·5234477	− 7·5214211	8·3050791	8·74632	8 6366342	8·75163	(*p*)
+ 7·3710701	+ 7·3675294	8·5392609	8·83725	8·8276750	8·90289	(*o*)
+ 7·8280082	+ 7·8225988	8·7848984	8·97246	8·8999080	8·94758	(*n*)
+ 7·8180852	+ 7·8105081	8·9462169	9·08165	8·9554792	8·97236	(*m*)
+ 7·1955239	+ 7·1855449	9·0829634	9·17498	8·9579165	8·98718	(*l*)
− 7·7242800	− 7·7117368	9·1656276	9·23766	9·0250162	9·00295	(*k*)
− 7·9751031	− 7·9599109	9·1687697	9·27109	8·9820257	9·02477	(*i*)
− 7·8778602	− 7·8600143	9·1444586	9·27479	8·9139472	8·99348	(*h*)
+ 6·3716753	+ 6·3512520	9·1094727	9·25344	8·7362177	8·92511	(*g*)
+ 8·0101841	+ 7·9873384	9·0303362	9·18001	8·4796199	8·74948	(*f*)
+ 8·2411811	+ 8·2161427	8·9371055	9·06749	7·9985988	8·30527	(*e*)
+ 8·2660663	+ 8·2391326	8·8822492		8·0173372		(*d*)
+ 8·1344139	+ 8·1059410	8·7338814		− 6·9118769		(*c*)
+ 7·8131615	+ 7·7835536					(*b*)
+ 7·1287132	+ 7·0984097					(*a*)

		Log ($w^{\frac{1}{2}}z'_3$) for g		Log ($w^{\frac{1}{2}}z'_3$) for h		
		1845	1880	1845	1880	
0·0000000	0·0000000					(*s*)
− 8·2570406	+ 8·2158260	− 8·3041650	6·59980	8·0823163	8·99206	(*r*)
− 8·3667171	+ 8·3242524	− 8·4774677	− 8·17429	8·5024900	9·04992	(*q*)
− 8·0739936	+ 8·0281817	8·2976369	− 8·31136	8·4460568	8·89701	(*p*)
+ 7·9062998	− 7·8663176	8·7296709	8·06093	8·9055012	8·83234	(*o*)
+ 8·3496717	− 8·3037066	8·6333203	8·93291	8·9704761	8·98673	(*n*)
+ 8·3212104	− 8·2709763	8·6630801	9·17943	9·0897603	9·08697	(*m*)
+ 7·6801662	− 7·6159779	9·0138489	9·30273	9·1588073	9·16516	(*l*)
− 8·1722928	+ 8·1219341	9·0482330	9·39273	9·1341607	9·15432	(*k*)
− 8·3901480	+ 8·3338956	9·1419608	9·40890	9·2258787	9·17198	(*i*)
− 8·2530066	+ 8·1914382	9·1882632	9·40204	9·2061816	9·12360	(*h*)
+ 6·6334990	− 6·6947981	9·1415294	9·40328	9·1256735	9·01237	(*g*)
+ 8·2745015	− 8·2125695	9·1828507	9·36236	9·0233966	8·83021	(*f*)
+ 8·4336049	− 8·3675527	9·0986764	9·32001	8·8934504	8·44025	(*e*)
+ 8·3671930	− 8·2983757	8·9019969		8·5715959		(*d*)
+ 8·1148904	− 8·0439799	8·1963076		7·8316085		(*c*)
+ 7·6205866	− 7·5481911					(*b*)
+ 6·6369254	− 6·5636342					(*a*)

For X

Co-latitude	$g_4{}^4$ or $h_4{}^4$	$g_{-4}{}^4$ or $h_{-4}{}^4$	$g_6{}^4$ or $h_6{}^4$	$g_{-6}{}^4$ or $h_{-6}{}^4$	$g_8{}^4$ or $h_8{}^4$	$g_{-8}{}^4$ or $h_{-8}{}^4$
(*s*) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(*r*) 85	− 9·5287383	− 9·5351617	+ 9·2948430	+ 9·2960554	− 8·8405862	− 8·8411529
(*q*) 80	− 9·8112411	− 9·8172247	+ 9·5517273	+ 9·5521926	− 9·0440593	− 9·0435771
(*p*) 75	− 9·9561258	− 9·9613909	+ 9·6511763	+ 9·6503675	− 9·0332229	− 9·0309267
(*o*) 70	− 0·0366666	− 0·0409558	+ 9·6611698	+ 9·6584823	− 8·7993636	− 8·7931435
(*n*) 65	− 0·0752292	− 0·0783142	+ 9·5929881	+ 9·5875881	+ 7·7368415	+ 7·7833683
(*m*) 60	− 0·0810578	− 0·0827462	+ 9·4290124	+ 9·4192452	+ 8·8475311	+ 8·8449838
(*l*) 55	− 0·0582186	− 0·0583598	+ 9·0774425	+ 9·0565589	+ 9·0337927	+ 9·0264916
(*k*) 50	− 0·0080705	− 0·0065605	− 8·3495127	− 8·4166400	+ 9·0321061	+ 9·0205924
(*i*) 45	− 9·9301335	− 9·9269179	− 9·1143281	− 9·1174680	+ 8·8714396	+ 8·8545532
(*h*) 40	− 9·8222806	− 9·8173568	− 9·2758178	− 9·2715881	+ 8·4068152	+ 8·3754516
(*g*) 35	− 9·6804957	− 9·6739133	− 9·2952842	− 9·2870423	− 8·2666439	− 8·2705344
(*f*) 30	− 9·4981211	− 9·4899801	− 9·2241538	− 9·2129622	− 8·6363698	− 8·6241421
(*e*) 25	− 9·2642205	− 9·2546691	− 9·0711841	− 9·0576286	− 8·6609979	− 8·6439645
(*d*) 20	− 8·9599607	− 8·9491904	− 8·8265424	− 8·8110755	− 8·5200580	− 8·5000418
(*c*) 15	− 8·5495789	− 8·5378187	− 8·4592379	− 8·4422779	− 8·2192079	− 8·1970844
(*b*) 10	− 7·9523439	− 7·9398541	− 7·8912629	− 7·8732272	− 7·6934341	− 7·6698585
(*a*) 5	− 6·9101385	− 6·8972015	− 6·8660691	− 6·8473827	− 6·6918828	− 6·6674472

For Y

Co-latitude	$g_4{}^4$ or $h_4{}^4$	$g_{-4}{}^4$ or $h_{-4}{}^4$	$g_6{}^4$ or $h_6{}^4$	$g_{-6}{}^4$ or $h_{-6}{}^4$	$g_8{}^4$ or $h_8{}^4$	$g_{-8}{}^4$ or $h_{-8}{}^4$
(*s*) 90°	0·4500908	+ 0·4500908	− 9·4086981	− 9·4086981	+ 8·6371774	+ 8·6371774
(*r*) 85	0·5949641	+ 0·5948657	− 9·5162213	− 9·5160791	+ 8·6899155	+ 8·6897296
(*q*) 80	0·5779453	+ 0·5775545	− 9·3643903	− 9·3638258	+ 8·2180933	+ 8·2173551
(*p*) 75	0·5492631	+ 0·5483946	− 8·9431033	− 8·9418488	− 8·3757758	− 8·3741353
(*o*) 70	0·5084216	+ 0·5069042	+ 8·9013204	+ 8·8991285	− 8·7519404	− 8·7490741
(*n*) 65	0·4546842	+ 0·4523659	+ 9·3883859	+ 9·3850372	− 8·8371943	− 8·8328153
(*m*) 60	0·3870235	+ 0·3837762	+ 9·5825036	+ 9·5778130	− 8·7379269	− 8·7317931
(*l*) 55	0·3040451	+ 0·2997684	+ 9·6763052	+ 9·6701277	− 8·2469253	− 8·2388470
(*k*) 50	0·2038686	+ 0·1984930	+ 9·7088854	+ 9·7011206	+ 8·4953157	+ 8·4851617
(*i*) 45	0·0839437	+ 0·0774328	+ 9·6936412	+ 9·6842365	+ 8·8888370	+ 8·8765386
(*h*) 40	9·9407511	+ 9·9331029	+ 9·6350013	+ 9·6239539	+ 9·0323318	+ 9·0178852
(*g*) 35	9·7693016	+ 9·7605488	+ 9·5325260	+ 9·5198830	+ 9·0620046	+ 9·0454714
(*f*) 30	9·5622405	+ 9·5524495	+ 9·3816981	+ 9·3675555	+ 9·0061135	+ 8·9876194
(*e*) 25	9·3081241	+ 9·2973934	+ 9·1729450	+ 9·1574450	+ 8·8680678	+ 8·8477986
(*d*) 20	8·9877475	+ 8·9762045	+ 8·8883368	+ 8·8716635	+ 8·6363367	+ 8·6145332
(*c*) 15	8·5650777	+ 8·5528750	+ 8·4927204	+ 8·4750943	+ 8·2793116	+ 8·2562621
(*b*) 10	7·9591917	+ 7·9465026	+ 7·9057746	+ 7·8874459	+ 7·7187521	+ 7·6947838
(*a*) 5	6·9118445	+ 6·8988573	+ 6·8696448	+ 6·8508855	+ 6·6980256	+ 6·6734942

For Z

Co-latitude	$g_4{}^4$ or $h_4{}^4$	$g_{-4}{}^4$ or $h_{-4}{}^4$	$g_6{}^4$ or $h_6{}^4$	$g_{-6}{}^4$ or $h_{-6}{}^4$	$g_8{}^4$ or $h_8{}^4$	$g_{-8}{}^4$ or $h_{-8}{}^4$
(*s*) 90°	0·5470008	− 0·4500908	− 9·6517361	+ 9·5847894	+ 8·9893599	− 8·9382074
(*r*) 85	0·6902577	− 0·5932099	− 9·7577122	+ 9·6904352	+ 9·0406230	− 8·9889467
(*q*) 80	0·6683631	− 0·5709060	− 9·6013097	+ 9·5328378	+ 8·5652106	− 8·3091423
(*p*) 75	0·6314642	− 0·5333383	− 9·1734628	+ 9·1005140	− 8·7115785	+ 8·6620020
(*o*) 70	0·5789242	− 0·4798900	+ 9·1144124	− 9·0523255	− 9·0769562	+ 9·0239775
(*n*) 65	0·5097968	− 0·4096416	+ 9·5880950	− 9·5202223	− 9·1472138	+ 9·0915844
(*m*) 60	0·4227622	− 0·3213069	+ 9·7632155	− 9·6928304	− 9·0291936	+ 8·9701376
(*l*) 55	0·3160287	− 0·2131329	+ 9·8334232	− 9·7608079	− 8·5179270	+ 8·4490680
(*k*) 50	0·1871804	− 0·0827468	+ 9·8374233	− 9·7626006	+ 8·7300919	− 8·6748296
(*i*) 45	0·0329396	− 9·9269178	+ 9·7879244	− 9·7108938	+ 9·0910802	− 9·0294784
(*h*) 40	9·8487831	− 9·7411703	+ 9·6883658	− 9·6091588	+ 9·1940523	− 9·1289398
(*g*) 35	9·6282977	− 9·5191400	+ 9·5368898	− 9·4555879	+ 9·1748968	− 9·1068426
(*f*) 30	9·3620291	− 9·2514195	+ 9·3268826	− 9·2436224	+ 9·0599214	− 8·9892551
(*e*) 25	9·0352653	− 8·9233416	+ 9·0455048	− 8·9604781	+ 8·8493104	− 8·7763421
(*d*) 20	8·6233156	− 8·5102562	+ 8·6693416	− 8·5827913	+ 8·5260633	− 8·4511345
(*c*) 15	8·0798528	− 7·9658712	+ 8·1529463	− 8·0651605	+ 8·0482854	− 7·9717779
(*b*) 10	7·3008344	− 7·1861728	+ 7·3928780	− 7·3041820	+ 7·3146204	− 7·2369546
(*a*) 5	5·9542316	− 5·8391533	+ 6·0574984	− 5·9682450	+ 5·9946538	− 5·9162802

g_{10}^4 or h_{10}^4	g_{-10}^4 or h_{-10}^4	Log $(w^{\frac{1}{2}}x'_4)$ for g		Log $(w^{\frac{1}{2}}x'_4)$ for h		
		1845	1880	1845	1880	
0·0000000	0·0000000					(s)
+ 8·3480521	+ 8·3483093	8·4185206	8·55765	7·6789812	7·14387	(r)
+ 8·4750152	+ 8·4739805	8·1445281	8·76690	− 8·1240313	7·50046	(q)
+ 8·2502564	+ 8·2460577	8·6326860	8·57441	8·2452764	− 7·98685	(p)
− 7·7666604	− 7·7731956	8·5512257	8·56744	8·1407139	− 8·42471	(o)
− 8·4078864	− 8·4038901	8·7092905	8·57867	− 7·9841677	− 8·53641	(n)
− 8·4604375	− 8·4519782	8·5754914	8·63174	− 7·7604281	− 8·63829	(m)
− 8·1503970	− 8·1345275	8·3963044	8·46133	− 7·6679900	− 8·60746	(l)
+ 7·9449754	+ 7·9426261	8·5053532	8·31478	− 7·7869701	− 8·58729	(k)
+ 8·4136293	+ 8·4000043	8·6224053	8·10658	8·0770284	− 8·64820	(i)
+ 8·4506518	+ 8·4315765	8·6036880	8·15470	7·9515588	− 8·55749	(h)
+ 8·2279073	+ 8·2022317	8·4840170	8·16759	− 7·1698255	− 8·43246	(g)
+ 6·9811443	+ 6·8447778	7·9032887	8·19460	− 7·9821496	− 8·05906	(f)
− 7·9971385	− 7·9788438	− 7·9410122	7·55427	− 7·3190574	8·04436	(e)
− 8·0721721	− 8·0479445	− 6·8288366		7·6489524		(d)
− 7·8744311	− 7·8472060	8·1235646		− 7·6472762		(c)
− 7·4067293	− 7·3776227					(b)
− 6·4359486	− 6·4057644					(a)
		Log $(w^{\frac{1}{2}}y'_4)$ for g		Log $(w^{\frac{1}{2}}y'_4)$ for h		
		1845	1880	1845	1880	
− 7·9408883	− 7·9408883	− 8·3134137	− 8·88266	8·7511208	8·87984	(s)
− 7·9163381	− 7·9161084	− 8·3479880	− 9·03477	8·8956419	9·00463	(r)
+ 7·1553529	+ 7·1544410	− 8·0464615	− 8·96612	8·8735435	8·94665	(q)
+ 8·0203996	+ 8·0183730	8·1980378	− 8·82244	8·8703078	8·93668	(p)
+ 8·1119381	+ 8·1083974	8·4985955	− 8·57867	8·8933264	8·93477	(o)
+ 7·8348820	+ 7·8294726	8·5033942	− 8·33563	8·8439890	8·95634	(n)
− 7·6477539	− 7·6401768	8·5391627	− 7·57010	8·7554969	9·00026	(m)
− 8·1480304	− 8·1380514	8·4025646	7·89068	8·6629006	8·98067	(l)
− 8·1970979	− 8·1845547	8·3974782	8·22517	8·7743810	8·92278	(k)
− 7·8758370	− 7·8606448	8·2710931	8·18956	8·8440797	8·87315	(i)
+ 7·8535675	+ 7·8357216	7·9941890	8·07160	8·8019110	8·78395	(h)
+ 8·3288306	+ 8·3084073	7·9553379	7·95897	8·7198970	8·67775	(g)
+ 8·4585149	+ 8·4356692	7·8935726	7·58260	8·4677348	8·48670	(f)
+ 8·4331205	+ 8·4080821	− 7·7608507	− 7·92785	8·0822513	8·06188	(e)
+ 8·2773032	+ 8·2503695	− 7·7412667		− 7·2229837		(d)
+ 7·9724858	+ 7·9440129	8·0442166		− 6·8380907		(c)
+ 7·4463871	+ 7·4167792					(b)
+ 6·4453767	+ 6·4150732					(a)
		Log $(w^{\frac{1}{2}}z'_4)$ for g		Log $(w^{\frac{1}{2}}z'_4)$ for h		
		1845	1880	1845	1880	
− 8·3802210	+ 8·3388283	− 8·0548567	− 9·10858	8·7395683	8·28736	(s)
− 8·3542852	+ 8·3121413	− 6·9754636	− 9·26562	8·8937147	8·65766	(r)
+ 7·5842864	− 7·5500017	7·6627620	− 9·20348	8·5078600	8·55040	(q)
+ 8·4448229	− 8·4019577	8·6256892	− 9·14074	8·8697337	8·63070	(p)
+ 8·5247586	− 8·4794792	8·3127368	− 8·97970	9·0067740	8·80029	(o)
+ 8·2336718	− 8·1833633	8·3515328	− 8·81085	9·0825301	8·81655	(n)
− 8·0219050	+ 7·9801152	7·9658595	− 8·46910	9·1758054	8·74691	(m)
− 8·5010740	+ 8·4509697	8·4760108	7·88560	9·1276167	8·82600	(l)
− 8·5222628	+ 8·4675208	8·8878328	7·92414	8·6992681	8·55465	(k)
− 8·1686700	+ 8·1066737	8·9049708	7·80555	8·5510805	8·14222	(i)
+ 8·0994672	− 8·0469219	8·9094644	8·30911	7·8207378	− 7·66771	(h)
+ 8·5280190	− 8·4676311	8·8296686	8·41635	7·6469468	− 8·12530	(g)
+ 8·5989667	− 8·5347845	8·7969081	8·39179	− 6·6631237	− 7·80445	(f)
+ 8·5011388	− 8·4339666	8·0594201	8·35424	8·0193824	7·97826	(e)
+ 8·2538719	− 8·1842441	− 8·1954626		8·0626050		(d)
+ 7·8283392	− 7·7567649	6·9118769		8·3231809		(c)
+ 7·1291568	− 7·0561652					(b)
+ 5·8289178	− 5·7550633					(a)

Co-latitude	g_5^4 or h_5^4	g_{-5}^4 or h_{-5}^4	g_7^4 or h_7^4	g_{-7}^4 or h_{-7}^4	g_9^4 or h_9^4
For X					
(s) 90°	9·8480308	+ 9·8480308	− 9·2112087	− 9·2112087	+ 8·6175819
(r) 85	9·9764605	+ 9·9760921	− 9·2942177	− 9·2936902	+ 8·6356867
(q) 80	9·9069536	+ 9·9049757	− 9·0315680	− 9·0286755	+ 7·7138301
(p) 75	9·7744835	+ 9·7704610	+ 8·0562778	+ 8·0881538	− 8·5495161
(o) 70	9·5360524	+ 9·5264333	+ 9·0978783	+ 9·0986538	− 8·7597350
(n) 65	8·9413990	+ 8·8987631	+ 9·3069401	+ 9·3042858	− 8·6996105
(m) 60	− 9·1470309	− 9·1682667	+ 9·3526928	+ 9·3469111	− 8·2490365
(l) 55	− 9·4958023	− 9·5017568	+ 9·2838303	+ 9·2742931	+ 8·3524060
(k) 50	− 9·6194104	− 9·6202228	+ 9·0778704	+ 9·0625169	+ 8·7079213
(i) 45	− 9·6508706	− 9·6483080	+ 8·5221409	+ 8·4838799	+ 8·7478645
(h) 40	− 9·6201410	− 9·6148464	− 8·6107711	− 8·6176698	+ 8·5851900
(g) 35	− 9·5358163	− 9·5281475	− 8·9295381	− 8·9222422	+ 8·0228546
(f) 30	− 9·3974023	− 9·3876383	− 8·9770050	− 8·9647845	− 8·1188943
(e) 25	− 9·1973779	− 9·1857880	− 8·8922014	− 8·8768002	− 8·3705309
(d) 20	− 8·9189556	− 8·9058247	− 8·6918430	− 8·6740755	− 8·3135783
(c) 15	− 8·5276547	− 8·5132971	− 8·3542790	− 8·3347315	− 8·0583092
(b) 10	− 7·9435591	− 7·9282957	− 7·8056415	− 7·7848349	− 7·5593334
(a) 5	− 6·9090549	− 6·8932430	− 6·7914133	− 6·7698522	− 6·5722376
For Y					
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·5323847	+ 9·5322644	− 8·8812400	− 8·8810760	+ 8·2685926
(q) 80	9·8148378	+ 9·8143601	− 9·1181309	− 9·1174795	+ 8·4403297
(p) 75	9·9596411	+ 9·9585796	− 9·1763004	− 9·1748529	+ 8·3453615
(o) 70	0·0400717	+ 0·0382171	− 9·1017401	− 9·0992110	+ 7·6625610
(n) 65	0·0784983	+ 0·0756648	− 8·8123677	− 8·8085039	− 8·2220056
(m) 60	0·0841694	+ 0·0802005	+ 8·3083284	+ 8·3029162	− 8·4922960
(l) 55	0·0611558	+ 0·0559287	+ 9·0410664	+ 9·0339385	− 8·4889584
(k) 50	0·0108220	+ 0·0042518	+ 9·2652647	+ 9·2563053	− 8·1675968
(i) 45	9·9326932	+ 9·9247354	+ 9·3588294	+ 9·3479778	+ 8·0606888
(h) 40	9·8246485	+ 9·8153007	+ 9·3738814	+ 9·3611344	+ 8·5770310
(g) 35	9·6826776	+ 9·6719797	+ 9·3253794	+ 9·3107913	+ 8·7363328
(f) 30	9·5001283	+ 9·4881615	+ 9·2155517	+ 9·1992336	+ 8·7545482
(e) 25	9·2660698	+ 9·2529545	+ 9·0383125	+ 9·0204279	+ 8·6657484
(d) 20	8·9616737	+ 8·9475656	+ 8·7777279	+ 8·7584895	+ 8·4685354
(c) 15	8·5511813	+ 8·5362669	+ 8·3998597	+ 8·3795219	+ 8·1357363
(b) 10	7·9538648	+ 7·9383559	+ 7·8251427	+ 7·8039942	+ 7·5913488
(a) 5	6·9116096	+ 6·8957363	+ 6·7961842	+ 6·7745389	+ 6·5799318
For Z					
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	9·7063736	− 9·6281019	− 9·1803092	+ 9·1228534	+ 8·6646635
(q) 80	9·9839532	− 9·9052048	− 9·4123935	+ 9·3541908	+ 8·8316708
(p) 75	0·1205437	− 0·0410167	− 9·4624969	+ 9·4029931	+ 8·7289194
(o) 70	0·1892809	− 0·1086961	− 9·3765861	+ 9·3150079	+ 8·0387579
(n) 65	0·2123238	− 0·1304337	− 9·0731001	+ 9·0071378	− 8·5762652
(m) 60	0·1986282	− 0·1152244	+ 8·5395133	− 8·4932797	− 8·8281607
(l) 55	0·1518677	− 0·0667865	+ 9·2549593	− 9·1929450	− 8·8017030
(k) 50	0·0728705	− 9·9859989	+ 9·4511424	− 9·3854775	− 8·4535685
(i) 45	9·9604347	− 9·8717136	+ 9·5106469	− 9·4421128	+ 8·3055577
(h) 40	9·8114350	− 9·7208614	+ 9·4848747	− 9·4137151	+ 8·7838368
(g) 35	9·6204367	− 9·5280640	+ 9·3874269	− 9·3138150	+ 8·8946297
(f) 30	9·3786881	− 9·2846247	+ 9·2184553	− 9·1425839	+ 8·8538700
(e) 25	9·0719895	− 8·9763960	+ 8·9686155	− 8·8907219	+ 8·6925563
(d) 20	8·6760265	− 8·5791105	+ 8·6164928	− 8·5368630	+ 8·4038561
(c) 15	8·1447462	− 8·0467563	+ 8·1178574	− 8·0368235	+ 7·9503215
(b) 10	7·3743011	− 7·2755193	+ 7·3700258	− 7·2879591	+ 7·2328388
(a) 5	6·0327925	− 5·9335255	+ 6·0418220	− 5·9591237	+ 5·9221873

g_{-9}^{4} or h_{-9}^{4}	Log $(w^{\frac{1}{2}}x'_4)$ for g		Log $(w^{\frac{1}{2}}x'_4)$ for h		
	1845	1880	1845	1880	
+ 8·6175819	8·2793946	8·30287	− 8·4608147	8·63265	(*s*)
+ 8·6349671	8·6927837	8·47486	− 8·6538382	8·67989	(*r*)
+ 7·7004680	8·9490687	8·40355	− 8·4160896	8·51118	(*q*)
− 8·5497573	8·9793247	8·21393	− 8·3443584	7·91029	(*p*)
− 8·7568644	8·9561895	8·51686	− 8·2754125	− 8·36559	(*o*)
− 8·6931182	8·8283479	8·69370	− 8·1749823	− 8·37564	(*n*)
− 8·2330145	8·4839102	8·67133	− 7·9270778	− 8·15556	(*m*)
+ 8·3506059	7·9875913	8·44614	− 8·2876101	− 8·05998	(*l*)
+ 8·6980917	8·0137541	7·93310	− 8·1645886	− 7·54393	(*k*)
+ 8·7331109	− 7·6918960	− 6·70289	− 8·6440714	6·82783	(*i*)
+ 8·5647117	− 8·2516139	− 7·80737	− 8·7613147	7·05041	(*h*)
+ 7·9845761	− 8·4138170	− 8·17646	− 8·7434111	− 7·56103	(*g*)
− 8·1110460	− 8·3992136	− 7·96080	− 8·6540087	− 7·54918	(*f*)
− 8·3523013	− 7·9722699	7·15633	− 8·0994647	− 7·82251	(*e*)
− 8·2913923	− 8·1135125		7·4841421		(*d*)
− 8·0336249	− 8·0120320		6·7491496		(*c*)
− 7·5329910					(*b*)
− 6·5449277					(*a*)

	Log $(w^{\frac{1}{2}}y'_4)$ for g		Log $(w^{\frac{1}{2}}y'_4)$ for h		
	1845	1880	1845	1880	
0·0000000					(*s*)
+ 8·2683848	− 8·0981105	7·35947	7·7178993	7·46014	(*r*)
+ 8·4395046	− 8·1367588	7·87037	8·0060329	8·30706	(*q*)
+ 8·3435279	− 8·2699655	7·93073	8·2086958	8·61755	(*p*)
+ 7·6593575	− 8·0925879	7·99398	8·4918829	8·78678	(*o*)
− 8·2171114	− 7·7219947	8·28733	8·5620329	8·86700	(*n*)
− 8·4854405	7·4663470	8·47455	8·6326784	8·85958	(*m*)
− 8·4799298	7·4544934	8·66714	8·7020380	8·84041	(*l*)
− 8·1562482	8·1861492	8·72291	8·5702610	8·80579	(*k*)
+ 8·0469435	8·4118785	8·70289	8·6009763	8·74164	(*i*)
+ 8·5608847	8·5170678	8·63828	8·5736008	8·65402	(*h*)
+ 8·7178546	8·6356658	8·46185	8·5726378	8·54349	(*g*)
+ 8·7338783	8·6072265	8·26351	8·4891985	8·43904	(*f*)
+ 8·6430946	8·5632572	8·24848	8·3815218	8·20661	(*e*)
+ 8·4441668	8·3386817		8·0934492		(*d*)
+ 8·1099751	− 6·4859082		7·8396962		(*c*)
+ 7·5645607					(*b*)
+ 6·5525144					(*a*)

	Log $(w^{\frac{1}{2}}z'_4)$ for g		Log $(w^{\frac{1}{2}}z'_4)$ for h		
	1845	1880	1845	1880	
0·0000000					(*s*)
− 8·6191946	− 7·8459291	− 7·77589	− 7·5540425	− 8·52922	(*r*)
− 8·7851668	− 6·8703703	− 8·04063	7·8732560	− 8·71547	(*q*)
− 8·6803121	7·8662732	− 8·64346	− 8·5450949	− 8·78917	(*p*)
− 7·9793145	8·7884935	− 8·54886	− 8·4816154	− 8·31172	(*o*)
+ 8·5287138	8·7882702	− 8·46057	− 7·8469334	7·27873	(*n*)
+ 8·7762391	8·8762533	− 8·16670	7·6963902	8·16394	(*m*)
+ 8·7458796	8·8783891	7·56896	8·7770408	8·60062	(*l*)
+ 8·3909910	8·1922921	7·11796	8·9209650	8·66778	(*k*)
− 8·2544628	8·0566794	− 8·39750	8·8196098	8·68516	(*i*)
− 8·7238337	7·9611888	− 8·35299	8·5422737	8·63587	(*h*)
− 8·8307727	8·4061302	− 8·21825	8·0212407	8·39566	(*g*)
− 8·7869443	− 7·8651506	− 8·25333	8·0277468	8·38424	(*f*)
− 8·6230148	− 8·3726160	− 8·66822	7·5659558	8·44535	(*e*)
− 8·3321157	− 8·5236325		− 7·1105615		(*d*)
− 7·8768217	− 7·9172719		8·2598164		(*c*)
− 7·1580525					(*b*)
− 5·8466159					(*a*)

Co-latitude	For X g_5^5 or h_5^5	g_{-5}^5 or h_{-5}^5	g_7^5 or h_7^5	g_{-7}^5 or h_{-7}^5	g_9^5 or h_9^5
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r) 85	− 9·6241715	− 9·6304270	+ 9·2957778	+ 9·2969072	− 8·7837019
(q) 80	− 9·9017758	− 9·9075298	+ 9·5435343	+ 9·5438030	− 8·9697778
(p) 75	− 0·0384068	− 0·0433417	+ 9·6264360	+ 9·6252174	− 8·9188508
(o) 70	− 0·1071991	− 0·1110212	+ 9·6095705	+ 9·6060918	− 8·5515785
(n) 65	− 0·1303099	− 0·1327589	+ 9·4963762	+ 9·4893863	+ 8·4194668
(m) 60	− 0·1166930	− 0·1175495	+ 9·2399847	+ 9·2259768	+ 8·8762365
(l) 55	− 0·0700194	− 0·0691116	+ 8·4210149	+ 8·3424620	+ 8·9645704
(k) 50	− 9·9911151	− 9·9883242	− 8·9715292	− 8·9802849	+ 8·8726938
(i) 45	− 9·8787748	− 9·8740388	− 9·2166880	− 9·2142830	+ 8·5494946
(h) 40	− 9·7298705	− 9·7231864	− 9·2617099	− 9·2543751	− 7·7310827
(g) 35	− 9·5389652	− 9·5303894	− 9·2012644	− 9·1904110	− 8·5018055
(f) 30	− 9·2973034	− 9·2869498	− 9·0545078	− 9·0407797	− 8·5819442
(e) 25	− 8·9906834	− 8·9787211	− 8·8190143	− 8·8028716	− 8·4734540
(d) 20	− 8·5947884	− 8·5814358	− 8·4764846	− 8·4583408	− 8·2134792
(c) 15	− 8·0635631	− 8·0490814	− 7·9842769	− 7·9645486	− 7·7769621
(b) 10	− 7·2931585	− 7·2778445	− 7·2405765	− 7·2196974	− 7·0696201
(a) 5	− 5·9516748	− 5·9358506	− 5·9146895	− 5·8931114	− 5·7644087
	For Y				
(s) 90°	0·5470008	+ 0·5470008	− 9·4330574	− 9·4330574	+ 8·6175819
(r) 85	0·6902513	+ 0·6901310	− 9·5318034	− 9·5316394	+ 8·6529153
(q) 80	0·6683373	+ 0·6678596	− 9·3419850	− 9·3413336	+ 8·0000800
(p) 75	0·6314069	+ 0·6303454	− 8·6637390	− 8·6622915	− 8·4895776
(o) 70	0·5788242	+ 0·5769696	+ 9·1666288	+ 9·1640997	− 8·7746230
(n) 65	0·5096441	+ 0·5068106	+ 9·5089703	+ 9·5051065	− 8·7979655
(m) 60	0·4225484	+ 0·4185795	+ 9·6551828	+ 9·6497706	− 8·5803105
(l) 55	0·3157473	+ 0·3105202	+ 9·7131009	+ 9·7059730	+ 7·7938709
(k) 50	0·1868269	+ 0·1802567	+ 9·7104777	+ 9·7015183	+ 8·7348596
(i) 45	0·0325115	+ 0·0245537	+ 9·6569269	+ 9·6460753	+ 8·9552090
(h) 40	9·8482803	+ 9·8389325	+ 9·5546877	+ 9·5419407	+ 9·0170874
(g) 35	9·6277228	+ 9·6170249	+ 9·4013461	+ 9·3867580	+ 8·9784933
(f) 30	9·3613860	+ 9·3494192	+ 9·1899976	+ 9·1736793	+ 8·8524161
(e) 25	9·0345607	+ 9·0214454	+ 8·9076402	+ 8·8897556	+ 8·6348340
(d) 20	8·6225580	+ 8·6084499	+ 8·5307617	+ 8·5115233	+ 8·3070025
(c) 15	8·0790521	+ 8·0641377	+ 8·0138557	+ 7·9935179	+ 7·8261828
(b) 10	7·3000019	+ 7·2844930	+ 7·2534441	+ 7·2322956	+ 7·0905743
(a) 5	5·9533797	+ 5·9375064	+ 5·9178667	+ 5·8962214	+ 5·7695215
	For Z				
(s) 90°	0·6261821	− 0·5470008	− 9·6371774	+ 9·5791854	+ 8·9186119
(r) 85	0·7678168	− 0·6884752	− 9·7343814	+ 9·6760285	+ 8·9524822
(q) 80	0·7410291	− 0·6612111	− 9·5400439	+ 9·4803271	+ 8·2968896
(p) 75	0·6958858	− 0·6152891	− 8·8572788	+ 8·7886091	− 8·7747365
(o) 70	0·6316093	− 0·5499554	+ 9·3423522	− 9·2855078	− 9·0486813
(n) 65	0·5470451	− 0·4640863	+ 9·6701397	− 9·6098714	− 9·0570453
(m) 60	0·4405824	− 0·3561102	+ 9·7972104	− 9·7345446	− 8·8210265
(l) 55	0·3100339	− 0·2238847	+ 9·8314870	− 9·7664717	+ 7·9967265
(k) 50	0·1524496	− 0·0645105	+ 9·8002633	− 9·7328467	+ 8·9200614
(i) 45	9·9638267	− 9·8740387	+ 9·7124483	− 9·6425981	+ 9·1067729
(h) 40	9·7386399	− 9·6469999	+ 9·5692858	− 9·4970218	+ 9·1279467
(g) 35	9·4690546	− 9·3756160	+ 9·3669419	− 9·2923462	+ 9·0404599
(f) 30	9·1435179	− 9·0483892	+ 9·0964140	− 9·0196344	+ 8·8552684
(e) 25	8·7440521	− 8·6473936	+ 8·7414327	− 8·6626805	+ 8·5651047
(d) 20	8·2404822	− 8·1425016	+ 8·2730004	− 8·1925454	+ 8·1457480
(c) 15	7·5761881	− 7·4771339	+ 7·6353163	− 7·5534802	+ 7·5441700
(b) 10	6·6240091	− 6·5241632	+ 6·7017833	− 6·6189289	+ 6·6354528
(a) 5	4·9781334	− 4·8778024	+ 5·0669567	− 4·9834788	+ 5·0151582

g_{-9}^{5} or h_{-9}^{5}	Log $(w^{\frac{1}{2}} x'_5)$ for g		Log $(w^{\frac{1}{2}} x'_5)$ for h		
	1845	1880	1845	1880	
0·0000000					(s)
− 8·7841985	8·2707413	8·14696	7·5775236	6·69671	(r)
− 8·9691516	8·6571217	8·03670	8·3888842	7·84657	(q)
− 8·9160252	8·6646936	7·92063	8·1585292	7·86627	(p)
− 8·5427706	8·7288877	8·10923	8·4544632	7·32780	(o)
+ 8·4235322	8·7441145	8·28945	8·2401528	− 6·57976	(n)
+ 8·7714726	8·5064845	8·49952	7·8131344	7·86013	(m)
+ 8·9552542	8·4140647	8·29063	− 7·9361862	8·20903	(l)
+ 8·8582927	8·2773300	7·79313	− 8·3913505	8·45107	(k)
+ 8·5256548	7·9681024	− 6·87890	− 8·2023493	8·50108	(i)
− 7·7780114	7·7070576	− 7·88655	− 7·8512272	8·43191	(h)
− 8·4922123	7·9516729	− 7·77742	− 7·7988690	8·33918	(g)
− 8·5657426	7·6631237	− ∞	− 7·9032887	− 7·15124	(f)
− 8·4534063	7·8611823	7·91771	− 8·0353216	6·85530	(e)
− 8·1906276	7·9000781		6·5810522		(d)
− 7·7520085	− 7·9407530		7·9212747		(c)
− 7·0431796					(b)
− 5·7370767					(a)

	Log $(w^{\frac{1}{2}} y'_5)$ for g		Log $(w^{\frac{1}{2}} y'_5)$ for h		
	1845	1880	1845	1880	
+ 8·6175819	− 8·6411224	− 8·72309	− 8·6276273	− 8·48251	(s)
+ 8·6527075	− 8·7416416	− 8·87453	− 8·7532337	− 8·56828	(r)
+ 7·9992549	− 8·6828380	− 8·87498	− 8·7117297	− 8·54798	(q)
− 8·4877440	− 8·6724531	− 8·79807	− 8·6769536	− 8·37860	(p)
− 8·7714195	− 8·6356854	− 8·65002	− 8·5570867	− 8·22091	(o)
− 8·7930713	− 8·6161910	− 8·45914	− 8·4058365	− 7·94149	(n)
− 8·5734550	− 8·5350628	− 8·31829	− 8·2410377	− 7·71623	(m)
+ 7·7848423	− 8·2451024	− 8·08651	− 7·6594742	− 7·54724	(l)
+ 8·7235110	7·2429021	− 7·78072	− 7·4859401	− 7·71278	(k)
+ 8·9414637	7·6059813	7·64074	7·3561039	− 7·60598	(i)
+ 9·0009411	7·6406804	7·79077	8·0029276	− 7·93366	(h)
+ 8·9600151	7·8797909	8·16982	7·2947642	− 8·13264	(g)
+ 8·8317462	7·8502103	8·32443	− 7·8609342	− 7·89553	(f)
+ 8·6121802	7·8267446	8·47478	− 7·7355939	− 7·97826	(e)
+ 8·2826339	8·2800222		− 7·6685059		(d)
+ 7·8004216	7·8460596		− 7·9212747		(c)
+ 7·0637862					(b)
+ 5·7421041					(a)

	Log $(w^{\frac{1}{2}} z'_5)$ for g		Log $(w^{\frac{1}{2}} z'_5)$ for h		
	1845	1880	1845	1880	
− 8·8728544	− 9·0564722	− 8·66692	− 6·8022733	− ∞	(s)
− 8·9061562	− 9·1855426	− 8·91995	− 8·0894070	− 7·85507	(r)
− 8·2453493	− 9·1784364	− 8·92728	− 8·4298779	7·47243	(q)
+ 8·7292705	− 8·6720005	− 8·97258	− 8·7953513	− 7·99981	(p)
+ 9·0003423	− 8·1743064	− 8·98581	− 8·7185752	− 8·39362	(o)
+ 9·0058468	8·2194971	− 8·92611	− 8·6903514	− 8·12999	(n)
+ 8·7653813	− 7·9829767	− 8·53033	− 8·8205164	− 8·29642	(m)
− 7·9678981	− 8·6721861	− 8·23034	− 8·7051457	− 8·19922	(l)
− 8·8656463	− 8·4981746	7·97930	− 8·3959544	− 7·95890	(k)
− 9·0481098	− 7·9479926	8·56022	7·5582086	− 8·14485	(i)
− 9·0659117	− 7·9554364	8·35912	8·2096353	− 8·25453	(h)
− 8·9753787	− 7·9442489	8·22809	8·0715213	7·39830	(g)
− 8·7874117	− 8·4495474	7·78471	8·4028785	8·16827	(f)
− 8·4947748	− 7·9125511	− 7·84329	8·2066044	8·04691	(e)
− 8·0732991	8·0658993		7·9985877		(d)
− 7·4700097	8·1671494		7·8132671		(c)
− 6·5600342					(b)
− 4·9389700					(a)

Co-latitude		g_6^5 or h_6^5	g_{-6}^5 or h_{-6}^5	g_8^5 or h_8^5	g_{-8}^5 or h_{-8}^5	g_{10}^5 or h_{10}^5
		For X				
(s)	90°	9·8480308	+ 9·8480308	− 9·1490608	− 9·1490608	+ 8·5149195
(r)	85	9·9714659	+ 9·9710282	− 9·2193393	− 9·2187329	+ 8·5105352
(q)	80	9·8846895	+ 9·8827917	− 8·8852868	− 8·8815783	− 6·9169466
(p)	75	9·7149869	+ 9·7098492	+ 8·5594681	+ 8·5674053	− 8·5235450
(o)	70	9·3730566	+ 9·3582294	+ 9·1310105	+ 9·1305393	− 8·6636643
(n)	65	− 8·5357440	− 8·6255116	+ 9·2769513	+ 9·2731480	− 8·5072251
(m)	60	− 9·4017610	− 9·4116343	+ 9·2687067	+ 9·2613193	− 6·7389666
(l)	55	− 9·5931084	− 9·5956362	+ 9·1241944	+ 9·1118191	+ 8·4824596
(k)	50	− 9·6492947	− 9·6478031	+ 8·7459365	+ 8·7216305	+ 8·6487281
(i)	45	− 9·6283299	− 9·6237012	− 8·2542427	− 8·2840662	+ 8·5708902
(h)	40	− 9·5463662	− 9·5389910	− 8·8219530	− 8·8172420	+ 8·2071875
(g)	35	− 9·4065397	− 9·3966815	− 8·9131451	− 8·9019286	− 7·7335035
(f)	30	− 9·2045879	− 9·1924902	− 8·8487138	− 8·8335535	− 8·2352723
(e)	25	− 8·9289140	− 8·9148395	− 8·6664272	− 8·6482813	− 8·2392757
(d)	20	− 8·5568113	− 8·5410555	− 8·3605746	− 8·3400631	− 8·0379213
(c)	15	− 8·0432565	− 8·0261493	− 7·8938405	− 7·8714962	− 7·6372279
(b)	10	− 7·2850675	− 7·2669705	− 7·1670341	− 7·1433738	− 6·9521277
(a)	5	− 5·9507610	− 5·9320598	− 5·8508350	− 5·8263799	− 5·6592397
		For Y				
(s)	90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r)	85	9·6276718	+ 9·6275296	− 8·9121285	− 8·9119426	+ 8·2566255
(q)	80	9·9052297	+ 9·9046652	− 9·1363670	− 9·1356288	+ 8·4059061
(p)	75	0·0417849	+ 0·0405304	− 9·1685792	− 9·1669387	+ 8·2462170
(o)	70	0·1104743	+ 0·1082824	− 9·0371468	− 9·0342805	− 7·1280566
(n)	65	0·1334582	+ 0·1301095	− 8·5022674	− 8·4978884	− 8·3199933
(m)	60	0·1196943	+ 0·1150037	+ 8·7970816	+ 8·7909478	− 8·4746201
(l)	55	0·0728580	+ 0·0666805	+ 9·1743172	+ 9·1662389	− 8·3502583
(k)	50	9·9937802	+ 9·9860154	+ 9·3170538	+ 9·3068998	− 7·0999038
(i)	45	9·8812609	+ 9·8718562	+ 9·3549544	+ 9·3426560	+ 8·3853072
(h)	40	9·7321777	+ 9·7211303	+ 9·3177254	+ 9·3032788	+ 8·6378699
(g)	35	9·5410987	+ 9·5284557	+ 9·2133328	+ 9·1967996	+ 8·6945846
(f)	30	9·2992738	+ 9·2851312	+ 9·0398179	+ 9·0213238	+ 8·6294603
(e)	25	8·9925064	+ 8·9770064	+ 8·7868766	+ 8·7666074	+ 8·4545757
(d)	20	8·5964842	+ 8·5798109	+ 8·4325989	+ 8·4107954	+ 8·1575516
(c)	15	8·0651556	+ 8·0475295	+ 7·9324797	+ 7·9094302	+ 7·6987284
(b)	10	7·2946750	+ 7·2763463	+ 7·1836764	+ 7·1597081	+ 6·9779572
(a)	5	5·9531447	+ 5·9343854	+ 5·8549215	+ 5·8303901	+ 5·6654972
		For Z				
(s)	90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r)	85	9·7717676	− 9·7055412	− 9·1654824	+ 9·1147486	+ 8·5972093
(q)	80	0·0444535	− 9·9776839	− 9·3849215	+ 9·3333629	+ 8·7417751
(p)	75	0·1727984	− 0·1051414	− 9·4090922	+ 9·3560788	+ 8·5744230
(o)	70	0·2297978	− 0·1609355	− 9·2664274	+ 9·2109516	− 7·4280542
(n)	65	0·2374021	− 0·1670526	− 8·7193658	+ 8·6552147	− 8·6193519
(m)	60	0·2042764	− 0·1322018	+ 8·9882183	− 8·9359911	− 8·7552865
(l)	55	0·1336986	− 0·0597125	+ 9·3430851	− 9·2855938	− 8·6080236
(k)	50	0·0259632	− 9·9499368	+ 9·4575015	− 9·3967755	− 7·3567593
(i)	45	9·8791427	− 9·8010085	+ 9·4612539	− 9·3976007	+ 8·5772358
(h)	40	9·6891104	− 9·6088651	+ 9·3831634	− 9·3167172	+ 8·7896899
(g)	35	9·4490098	− 9·3667143	+ 9·2298062	− 9·1607070	+ 8·7976660
(f)	30	9·1479908	− 9·0637686	+ 8·9971371	− 8·9255738	+ 8·6734926
(e)	25	8·7685881	− 8·6826221	+ 8·6715895	− 8·5978114	+ 8·4260609
(d)	20	8·2810032	− 8·1935299	+ 8·2257704	− 8·1500853	+ 8·0375321
(c)	15	7·6288902	− 7·5401931	+ 7·6048821	− 7·5276528	+ 7·4579635
(b)	10	6·6852835	− 6·5956837	+ 6·6829630	− 6·6045970	+ 6·5640914
(a)	5	5·0445013	− 4·9543486	+ 5·0549625	− 4·9759004	+ 4·9523939

g_{-10^5} or h_{-10^5}	Log $(w^{\frac{1}{2}}x'_5)$ for g 1845	Log $(w^{\frac{1}{2}}x'_5)$ for g 1880	Log $(w^{\frac{1}{2}}x'_5)$ for h 1845	Log $(w^{\frac{1}{2}}x'_5)$ for h 1880	
+ 8·5149195	− 7·8608680	7·66757	− 8·5228919	7·50124	(*s*)
+ 8·5097182	− 8·2613761	7·64119	− 8·7435952	6·77589	(*r*)
− 6·9750684	− 8·3513349	7·81485	− 8·7956824	7·85863	(*q*)
− 8·5230694	− 8·3443584	− 7·63466	− 8·4445302	7·45361	(*p*)
− 8·6600163	− 8·4167417	− 8·14374	− 8·3944653	− 6·83048	(*o*)
− 8·4990544	− 8·3298842	− 8·19781	− 8·6211544	− 7·85851	(*n*)
− 6·3609616	− 8·4924325	− 8·18552	− 8·3124287	− 8·30249	(*m*)
+ 8·4759379	− 8·4586099	− 7·87526	− 8·5376773	− 8·24621	(*l*)
+ 8·6363154	7·3967170	− 7·46038	− 8·1632863	− 7·83950	(*k*)
+ 8·5528625	− 7·9137447	7·32268	− 8·0403506	− 7·24696	(*i*)
+ 8·1787440	− 7·8362500	7·94960	− 7·6931590	7·67513	(*h*)
− 7·7423389	− 7·8575145	7·76060	7·7371234	7·95534	(*g*)
− 8·2189763	− 7·7252716	− 7·91467	7·7194420	7·38169	(*f*)
− 8·2174961	− 8·1727936	− 8·28813	− 6·9442412	8·09721	(*e*)
− 8·0127676	− 8·1528505		− 7·5843801		(*d*)
− 7·6096730	− 7·9795985		− 7·8299728		(*c*)
− 6·9229084					(*b*)
− 5·6290309					(*a*)

	Log $(w^{\frac{1}{2}}y'_5)$ for g 1845	Log $(w^{\frac{1}{2}}y'_5)$ for g 1880	Log $(w^{\frac{1}{2}}y'_5)$ for h 1845	Log $(w^{\frac{1}{2}}y'_5)$ for h 1880	
0·0000000					(*s*)
+ 8·2563958	− 8·1032502	− 8·10154	− 8·1423142	− 7·98001	(*r*)
+ 8·4049942	− 8·2154171	− 8·41361	− 8·4788963	− 8·28311	(*q*)
+ 8·2441904	− 7·8363099	− 8·54992	− 8·7138458	− 8·48675	(*p*)
− 7·1245159	7·2641315	− 8·63666	− 8·7404902	− 8·64052	(*o*)
− 8·3145839	− 7·3579129	− 8·65804	− 8·7728863	− 8·71885	(*n*)
− 8·4670430	− 7·9457600	− 8·60453	− 8·6844571	− 8·73445	(*m*)
− 8·3402793	− 8·3462343	− 8·56364	− 8·6432656	− 8·72408	(*l*)
− 7·0873606	− 8·4275935	− 8·55253	− 8·5902321	− 8·69083	(*k*)
+ 8·3701150	− 8·1800126	− 8·28267	− 8·6806150	− 8·63570	(*i*)
+ 8·6200240	− 8·0550403	− 8·29698	− 8·5627723	− 8·51066	(*h*)
+ 8·6741613	− 8·1653482	− 8·28119	− 8·3510826	− 8·19366	(*g*)
+ 8·6066146	− 8·0703184	− 8·24988	− 7·9453797	− 7·46299	(*f*)
+ 8·4295373	− 8·1904844	− 8·31360	7·7303613	7·98417	(*e*)
+ 8·1306179	− 8·1746790		− 7·0111768		(*d*)
+ 7·6702555	− 7·4195641		− 7·8200267		(*c*)
+ 6·9483493					(*b*)
+ 5·6351937					(*a*)

	Log $(w^{\frac{1}{2}}z'_5)$ for g 1845	Log $(w^{\frac{1}{2}}z'_5)$ for g 1880	Log $(w^{\frac{1}{2}}z'_5)$ for h 1845	Log $(w^{\frac{1}{2}}z'_5)$ for h 1880	
0·0000000					(*s*)
− 8·5560184	7·5162539	7·99338	− 8·6456118	− 8·49189	(*r*)
− 8·6994536	8·2034815	− 8·61231	− 9·0238803	− 8·44863	(*q*)
− 8·5296450	− 8·1629384	− 8·66607	− 8·9023695	− 8·57667	(*p*)
+ 7·4155568	8·1009884	− 8·81661	− 8·9223940	− 8·71047	(*o*)
+ 8·5743433	− 7·9755248	− 8·71964	− 8·8152901	− 8·58623	(*n*)
+ 8·7063346	− 8·5495309	− 8·42592	− 8·9684705	− 8·58082	(*m*)
+ 8·5544908	− 8·7589565	− 8·28046	− 8·8466038	− 8·62920	(*l*)
+ 7·2395497	− 8·2137137	7·83396	− 8·6557825	− 8·73566	(*k*)
− 8·5236400	8·1640396	8·69559	− 8·7616968	− 8·62717	(*i*)
− 8·7312277	8·0818204	8·55166	− 8·8708949	− 8·36065	(*h*)
− 8·7355930	− 8·1786440	8·46638	− 8·5257042	7·78828	(*g*)
− 8·6082796	− 8·1501532	8·19065	− 8·3185576	8·17655	(*f*)
− 8·3580963	− 7·9020435	8·18498	− 8·0469035	7·71700	(*e*)
− 7·9672272	7·9570673		− 7·8509242		(*d*)
− 7·3857752	8·3184171		− 7·3258050		(*c*)
− 6·4905217					(*b*)
− 4·8779801					(*a*)

Co-latitude		g_6^6 or h_6^6	g_{-6}^6 or h_{-6}^6	g_8^6 or h_8^6	g_{-8}^6 or h_{-8}^6	g_{10}^6 or h_{10}^6
		For X				
(s)	90°	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
(r)	85	− 9·7018271	− 9·7079634	+ 9·2956523	+ 9·2967115	− 8·7322230
(q)	80	− 9·9745342	− 9·9801063	+ 9·5341821	+ 9·5342654	− 8·9004529
(p)	75	− 0·1029135	− 0·1075639	+ 9·6001562	+ 9·5985284	− 8·8052011
(o)	70	− 0·1599596	− 0·1633578	+ 9·5547814	+ 9·5504610	− 8·2141514
(n)	65	− 0·1676216	− 0·1694749	+ 9·3894251	+ 9·3805005	+ 8·5768194
(m)	60	− 0·1345627	− 0·1346240	+ 8·9913836	+ 8·9692204	+ 8·8586838
(l)	55	− 0·0640588	− 0·0621348	− 8·5853185	− 8·6189258	+ 8·8677574
(k)	50	− 9·9564020	− 9·9523590	− 9·1195943	− 9·1208438	+ 8·6735589
(i)	45	− 9·8096628	− 9·8034309	− 9·2278415	− 9·2223484	+ 7·9953717
(h)	40	− 9·6197116	− 9·6112874	− 9·1979397	− 9·1881833	− 8·2923239
(g)	35	− 9·3796898	− 9·3691366	− 9·0718266	− 9·0586426	− 8·5010438
(f)	30	− 9·0787447	− 9·0661908	− 8·8562325	− 8·8401093	− 8·4488531
(e)	25	− 8·6994087	− 8·6850443	− 8·5420076	− 8·5233525	− 8·2380548
(d)	20	− 8·2118815	− 8·1959523	− 8·1038925	− 8·0831114	− 7·8709182
(c)	15	− 7·5598154	− 7·5426154	− 7·4882266	− 7·4657494	− 7·3045366
(b)	10	− 6·6162430	− 6·5981062	− 6·5696899	− 6·5459754	− 6·4187091
(a)	5	− 4·9754821	− 4·9567711	− 4·9435949	− 4·9191269	− 4·8113853
		For Y				
(s)	90°	0·6261821	+ 0·6261821	− 9·4500908	− 9·4500908	+ 8·5941008
(r)	85	0·7678096	+ 0·7676674	− 9·5399397	− 9·5397538	+ 8·6115220
(q)	80	0·7410006	+ 0·7404361	− 9·3081397	− 9·3074015	+ 7·6303079
(p)	75	0·6958221	+ 0·6945676	− 7·4050305	− 7·4033900	− 8·5551361
(o)	70	0·6314981	+ 0·6293062	+ 9·3214522	+ 9·3185859	− 8·7737477
(n)	65	0·5468753	+ 0·5435266	+ 9·5887161	+ 9·5843371	− 8·7323444
(m)	60	0·4403446	+ 0·4356540	+ 9·6983377	+ 9·6922039	− 8·3234122
(l)	55	0·3097209	+ 0·3035434	+ 9·7245965	+ 9·7165182	+ 8·4130251
(k)	50	0·1520563	+ 0·1442915	+ 9·6888812	+ 9·6787272	+ 8·8346005
(i)	45	9·9633505	+ 9·9539458	+ 9·5982508	+ 9·5859524	+ 8·9623771
(h)	40	9·7380809	+ 9·7270335	+ 9·4531922	+ 9·4387456	+ 8·9604329
(g)	35	9·4684151	+ 9·4557721	+ 9·2495091	+ 9·2329759	+ 8·8607832
(f)	30	9·1428028	+ 9·1286602	+ 8·9780068	+ 8·9595127	+ 8·6682513
(e)	25	8·7432686	+ 8·7277686	+ 8·6223061	+ 8·6020369	+ 8·3733229
(d)	20	8·2396397	+ 8·2229664	+ 8·1533437	+ 8·1315402	+ 7·9507637
(c)	15	7·5752978	+ 7·5576717	+ 7·5152786	+ 7·4922291	+ 7·3470302
(b)	10	6·6230834	+ 6·6047547	+ 6·5814877	+ 6·5575194	+ 6·4369222
(a)	5	4·9771862	+ 4·9584269	+ 4·9465122	+ 4·9219808	+ 4·8158457
		For Z				
(s)	90°	0·6931288	− 0·6261821	− 9·6261820	+ 9·5750295	+ 8·8573422
(r)	85	0·8331412	− 0·7660116	− 9·7144931	+ 9·6629504	+ 8·8733116
(q)	80	0·8014600	− 0·7337875	− 9·4782341	+ 9·4251421	+ 7·8920724
(p)	75	0·7480711	− 0·6795112	− 7·6383202	+ 7·4148970	− 8·8027871
(o)	70	0·6720567	− 0·6022920	+ 9·4697359	− 9·4181160	− 9·0101650
(n)	65	0·5720541	− 0·5008023	+ 9·7220846	− 9·6676437	− 8·9538464
(m)	60	0·4461612	− 0·3731847	+ 9·8124951	− 9·7556042	− 8·5276124
(l)	55	0·2917955	− 0·2169079	+ 9·8150873	− 9·7556765	+ 8·5879941
(k)	50	0·1054729	− 0·0285454	+ 9·7507603	− 9·6887322	+ 8·9825931
(i)	45	9·8824654	− 9·8034308	+ 9·6258607	− 9·5611617	+ 9·0764345
(h)	40	9·6162463	− 9·5351008	+ 9·4398768	− 9·3725198	+ 9·0337061
(g)	35	9·2975585	− 9·2143633	+ 9·1871911	− 9·1172615	+ 8·8851305
(f)	30	8·9127517	− 8·8276302	+ 8·8565100	− 8·7841683	+ 8·6334676
(e)	25	8·4405819	− 8·3537168	+ 8·4281864	− 8·3536646	+ 8·2659482
(d)	20	7·8453899	− 7·7570181	+ 7·8676712	− 7·7912664	+ 7·7518584
(c)	15	7·0602633	− 6·9706679	+ 7·1088293	− 7·0308967	+ 7·0273634
(b)	10	5·9349227	− 5·8444248	+ 6·0019176	− 5·9228589	+ 5·9441447
(a)	5	3·9897791	− 3·8987228	+ 4·0676935	− 3·9879448	+ 4·0238254

g_{-10}^{6} or h_{-10}^{6}	Log $(w^{\frac{1}{2}}x'_6)$ for g 1845	Log $(w^{\frac{1}{2}}x'_6)$ for g 1880	Log $(w^{\frac{1}{2}}x'_6)$ for h 1845	Log $(w^{\frac{1}{2}}x'_6)$ for h 1880	
0·0000000					(s)
− 8·7326575	8·3220225	− 7·56594	7·8365891	8·03517	(r)
− 8·8996328	8·5558134	7·52679	− 7·3474915	7·74350	(q)
− 8·8017936	8·5738433	7·58227	7·5290310	− 7·71549	(p)
− 8·1993340	8·1407139	8·30135	8·0106838	− 6·88847	(o)
+ 8·5764312	8·0797922	8·19781	− 7·4760123	7·25645	(n)
+ 8·8522183	8·4281822	7·97236	− 7·8818503	− 7·22331	(m)
+ 8·8563998	7·1322741	7·81950	− 7·9561828	− 6·80128	(l)
+ 8·6554042	7·6139700	− 7·51007	7·4189934	− 6·84496	(k)
+ 7·9453522	7·6844079	− 7·38714	− 7·5102008	− 7·90701	(i)
− 8·2892942	7·0346177	7·30222	− 7·8657050	− 8·08612	(h)
− 8·4868911	7·8597943	7·49257	− 7·7026125	− 7·49257	(g)
− 8·4296689	7·9853430	− 7·65639	7·9052062	7·90329	(f)
− 8·2152776	− 6·2118474	− 7·92785	7·9522101	7·24527	(e)
− 7·8453386	− 8·1270250		7·8633082		(d)
− 7·2767961	− 8·1243974		− 7·7759428		(c)
− 6·3894194					(b)
− 4·7811604					(a)

	Log $(w^{\frac{1}{2}}y'_6)$ for g 1845	Log $(w^{\frac{1}{2}}y'_6)$ for g 1880	Log $(w^{\frac{1}{2}}y'_6)$ for h 1845	Log $(w^{\frac{1}{2}}y'_6)$ for h 1880	
+ 8·5941008	− 8·2562708	− 7·40433	− 8·4885122	8·12906	(s)
+ 8·6112923	− 8·3381841	− 7·22819	− 8·5682829	8·26491	(r)
+ 7·6293960	− 8·0922190	7·90912	− 8·5658519	8·17429	(q)
− 8·5531095	− 7·7693632	8·09501	− 8·5383710	8·30296	(p)
− 8·7702070	7·4767396	8·20810	− 8·5155776	8·34331	(o)
− 8·7269350	7·7804754	8·22321	− 8·4935755	8·23055	(n)
− 8·3158351	7·8844903	8·00943	− 8·2829565	8·11105	(m)
+ 8·4030461	7·5946721	7·74857	− 7·9223245	7·81950	(l)
+ 8·8220573	7·2429021	7·58532	− 6·8961146	7·73426	(k)
+ 8·9471849	7·9720150	7·00392	7·8885279	7·52680	(i)
+ 8·9425870	7·6326377	− 7·05041	6·7171973	− 7·71719	(h)
+ 8·8403599	7·3496129	7·18082	7·7801580	− 7·56999	(g)
+ 8·6454056	7·8545317	7·94712	8·1501532	7·62836	(f)
+ 8·3482845	8·1708888	7·97528	8·1737429	7·73299	(e)
+ 7·9238300	7·9829826		7·6518002		(d)
+ 7·3185573	7·5273008		7·6346136		(c)
+ 6·4073143					(b)
+ 4·7855422					(a)

	Log $(w^{\frac{1}{2}}z'_6)$ for g 1845	Log $(w^{\frac{1}{2}}z'_6)$ for g 1880	Log $(w^{\frac{1}{2}}z'_6)$ for h 1845	Log $(w^{\frac{1}{2}}z'_6)$ for h 1880	
− 8·8159495	8·1057094	8·13583	− 8·4278144	7·83030	(s)
− 8·8313068	8·5076143	− 7·67898	− 8·6131640	8·11831	(r)
− 7·8390201	8·4455581	− 8·25299	− 8·6272598	8·58077	(q)
+ 8·7608936	8·7657289	− 8·62164	− 8·7970524	8·43369	(p)
+ 8·9655406	8·8528454	− 8·35829	− 7·8210685	8·66753	(o)
+ 8·9060575	8·8127578	7·93194	8·0151282	8·28733	(n)
+ 8·4729392	8·6405960	8·42288	8·0417548	8·32788	(m)
− 8·5420381	8·2732009	7·64638	8·3488798	− 7·51248	(l)
− 8·9301203	7·8422392	− 8·10622	− 8·3576798	− 7·97930	(k)
− 9·0200928	− 7·9520897	− 8·13157	− 8·2892911	− 8·18001	(i)
− 8·9739115	− 8·3900053	− 8·43576	− 8·0918046	− 8·10018	(h)
− 8·8221175	7·7988690	− 8·53493	6·8575145	− 8·13506	(g)
− 8·5674862	8·4990580	− 8·35536	− 7·4629942	− 8·23223	(f)
− 8·1973070	8·3902488	7·43716	− 8·1309255	− 8·37142	(e)
− 7·6809313	8·3569705		− 8·1317508		(d)
− 6·9545865	7·4966320		− 8·3015966		(c)
− 5·8700068					(b)
− 3·9488545					(a)

Co-latitude	For X: g_7^6 or h_7^6	g_{-7}^6 or h_{-7}^6	g_9^6 or h_9^6	g_{-9}^6 or h_{-9}^6	Log $(w^{\frac{1}{2}}x'_6)$ for g		Log $(w^{\frac{1}{2}}x'_6)$ for h	
					1845	1880	1845	1880
(s) 90°	9·8480308	+ 9·8480308	− 9·0947032	− 9·0947032	8·6188828	− 7·78253	7·1904535	8·31638
(r) 85	9·9664450	+ 9·9659374	− 9·1519988	− 9·1513111	8·7794954	− 7·96153	7·6967100	7·86106
(q) 80	9·8622756	+ 9·8600280	− 8·7313760	− 8·7265292	8·7191750	− 7·33773	8·4554548	8·46807
(p) 75	9·6503486	+ 9·6438687	+ 8·7232496	+ 8·7272896	8·7497453	− 8·00824	8·2642132	8·28347
(o) 70	9·1463944	+ 9·1204642	+ 9·1381567	+ 9·1367411	8·4135127	− 7·13151	8·5786640	7·88847
(n) 65	− 9·1283213	− 9·1514034	+ 9·2331623	+ 9·2282632	5·6766717	− 6·97770	8·4697633	− 7·27873
(m) 60	− 9·5213816	− 9·5270905	+ 9·1702472	+ 9·1611072	− 7·9702025	− 7·29026	8·2010325	− 7·67561
(l) 55	− 9·6360157	− 9·6363768	+ 8·9307401	+ 8·9143735	− 8·4902089	6·55824	8·3761385	− 7·74857
(k) 50	− 9·6432089	− 9·6398999	+ 8·0840897	+ 8·0068830	7·8503571	7·52165	8·2682080	6·78697
(i) 45	− 9·5774145	− 9·5709681	− 8·6461717	− 8·6475391	8·0233838	8·01816	7·9181763	− 7·58750
(h) 40	− 9·4481982	− 9·4388866	− 8·8451509	− 8·8361047	7·9668659	7·89992	8·0425866	− 7·71719
(g) 35	− 9·2552441	− 9·2432851	− 8·8255609	− 8·8114957	7·4872459	8·02280	7·9461168	− 7·63566
(f) 30	− 8·9912833	− 8·9769077	− 8·6758386	− 8·6580850	7·7223666	7·79469	7·5945033	− 7·58260
(e) 25	− 8·6409887	− 8·6244644	− 8·4068674	− 8·3861221	7·2204476	7·62009	− 7·7197033	− 8·07636
(d) 20	− 8·1759113	− 8·1575511	− 8·0010157	− 7·9778270	7·8633082		− 8·0525695	
(c) 15	− 7·5405838	− 7·5207438	− 7·4082118	− 7·3831009	8·1423258		− 7·8867338	
(b) 10	− 6·6086139	− 6·5876879	− 6·5050224	− 6·4785200				
(a) 5	− 4·9746816	− 4·9530922	− 4·8877906	− 4·8604442				

Co-latitude	For Y				Log $(w^{\frac{1}{2}}y'_6)$ for g		Log $(w^{\frac{1}{2}}y'_6)$ for h	
					1845	1880	1845	1880
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000				
(r) 85	9·7052301	+ 9·7050661	− 8·9330689	− 8·9328611	8·0147733	7·31996	− 8·0840998	8·08053
(q) 80	9·9778930	+ 9·9772416	− 9·1444282	− 9·1436031	8·0634949	7·97758	− 8·2383470	8·40524
(p) 75	0·1062001	+ 0·1047526	− 9·1490291	− 9·1471955	8·0846336	7·99553	− 8·2733807	8·39433
(o) 70	0·1631482	+ 0·1606191	− 8·9478840	− 8·9446805	8·1189168	7·91988	− 7·8661915	8·33949
(n) 65	0·1706894	+ 0·1668256	+ 6·4238659	+ 6·4189717	8·1609715	8·02692	− 7·8641924	7·84102
(m) 60	0·1374905	+ 0·1320783	+ 8·9896321	+ 8·9827766	7·3391043	8·02112	− 8·1455729	− 7·67561
(l) 55	0·0668316	+ 0·0597037	+ 9·2426252	+ 9·2335966	− 7·9407101	7·76909	− 8·3753121	− 7·72703
(k) 50	9·9590097	+ 9·9500503	+ 9·3284197	+ 9·3170711	− 8·0282319	7·72002	− 8·4218790	− 6·24290
(i) 45	9·8120999	+ 9·8012483	+ 9·3189423	+ 9·3051970	− 7·6992571	7·33971	− 7·8490194	− 5·92474
(h) 40	9·6219783	+ 9·6092313	+ 9·2333816	+ 9·2172353	− 8·0330062	6·90428	− 8·2063983	7·97983
(g) 35	9·3817910	+ 9·3672029	+ 9·0753267	+ 9·0568485	− 8·1165800	6·92118	− 8·1165800	8·00364
(f) 30	9·0806906	+ 9·0643723	+ 8·8395097	+ 8·8188398	− 8·1965633	6·99634	− 7·9418773	7·78471
(e) 25	8·7012143	+ 8·6833297	+ 8·5117761	+ 8·4891223	− 8·0327054	− 7·47667	− 8·1192588	− 7·37021
(d) 20	8·2135658	+ 8·1943274	+ 8·0644191	+ 8·0400505	− 7·5605305		− 8·0724139	
(c) 15	7·5614014	+ 7·5410636	+ 7·4424626	+ 7·4167014	− 7·7120783		− 6·8980886	
(b) 10	6·6177565	+ 6·5966080	+ 6·5198394	+ 6·4930513				
(a) 5	4·9769512	+ 4·9553059	+ 4·8914378	+ 4·8640204				

Co-latitude	For Z				Log $(w^{\frac{1}{2}}z'_6)$ for g		Log $(w^{\frac{1}{2}}z'_6)$ for h	
					1845	1880	1845	1880
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000				
(r) 85	9·8281892	− 9·7707737	− 9·1530328	+ 9·1075970	− 8·3734036	− 8·33219	− 7·2152239	7·55404
(q) 80	0·0959812	− 0·0379565	− 9·3595991	+ 9·3132615	− 8·2897752	− 8·18000	− 8·4614349	− 7·67655
(p) 75	0·2160798	− 0·1570598	− 9·3561847	+ 9·3082346	− 8·3677889	− 8·30296	− 8·0185615	7·92063
(o) 70	0·2613404	− 0·2009684	− 9·1439786	+ 9·0930671	− 7·8242270	− 8·12839	6·5294459	7·84271
(n) 65	0·2535051	− 0·1914647	− 5·5623292	− 6·8976726	− 7·8227997	8·14205	8·3597187	− 7·90198
(m) 60	0·2009482	− 0·1369725	+ 9·1484167	− 9·0985782	− 7·3572025	8·55010	8·3134101	7·04722
(l) 55	0·1065517	− 0·0404317	+ 9·3783063	− 9·3245381	7·8341298	8·41102	8·0389682	8·07675
(k) 50	9·9700764	− 9·9016677	+ 9·4356579	− 9·3786764	− 7·8081594	8·39671	8·2418150	8·25362
(i) 45	9·7888699	− 9·7180968	+ 9·3919890	− 9·3319243	7·0861081	− 6·82783	8·4412755	7·48104
(h) 40	9·5578035	− 9·4846621	+ 9·2655467	− 9·2024825	− 7·6601588	− 6·74938	8·5347118	− 7·68961
(g) 35	9·2685990	− 9·1931576	+ 9·0585168	− 8·9925796	− 7·9404887	− 7·76628	8·3214860	− 6·95897
(f) 30	8·9083085	− 8·8307058	+ 8·7635399	− 8·6949229	8·0070622	6·99634	8·4423871	7·53145
(e) 25	8·4562006	− 8·3766414	+ 8·3631968	− 8·2921656	8·1582997	7·99860	7·9458467	7·67124
(d) 20	7·8769925	− 7·7957425	+ 7·8242966	− 7·7511841	8·1729725		− 7·7126215	
(c) 15	7·1040462	− 7·0214232	+ 7·0815702	− 7·0067707	− 6·4481196		− 8·1431234	
(b) 10	5·9872770	− 5·9036415	+ 5·9858308	− 5·9097887				
(a) 5	4·0472211	− 3·9629652	+ 4·0581834	− 3·9813801				

Co-latitude	For X $g_7{}^7$ or $h_7{}^7$	$g_{-7}{}^7$ or $h_{-7}{}^7$	$g_9{}^7$ or $h_9{}^7$	$g_{-9}{}^7$ or $h_{-9}{}^7$	Log $(w^{\frac{1}{2}}x'_7)$ for g 1845	Log $(w^{\frac{1}{2}}x'_7)$ for g 1880	Log $(w^{\frac{1}{2}}x'_7)$ for h 1845	Log $(w^{\frac{1}{2}}x'_7)$ for h 1880
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000				
(r) 85	− 9·7672204	− 9·7732654	+ 9·2948460	+ 9·2958431	7·4601380	6·29877	− 8·1049500	− 7·03913
(q) 80	− 0·0350308	− 0·0404481	+ 9·5240844	+ 9·5239883	− 7·0367017	7·77346	− 7·5635107	5·99531
(p) 75	− 0·1551594	− 0·1595515	+ 9·5726957	+ 9·5706533	8·1051553	7·84854	− 8·1465479	− 7·60399
(o) 70	− 0·2004608	− 0·2034600	+ 9·4968727	+ 9·4916408	− 8·0326528	8·03460	− 8·3044840	− 7·60863
(n) 65	− 0·1926757	− 0·1939564	+ 9·2690992	+ 9·2576711	− 8·1736014	7·68527	7·0916451	− 7·71009
(m) 60	− 0·1401767	− 0·1394639	+ 8·5857014	+ 8·5362787	− 8·3202189	7·44516	8·0153113	− 7·87113
(l) 55	− 0·0458447	− 0·0429233	− 8·9119586	− 8·9220202	− 8·0929034	7·99757	− 8·3169655	− 7·81950
(k) 50	− 9·9094380	− 9·9041593	− 9·1691602	− 9·1667829	− 8·1322038	7·86095	− 8·5848310	− 7·62311
(i) 45	− 9·7283022	− 9·7205884	− 9·1966683	− 9·1888423	− 7·9024637	7·93334	− 8·1409060	− 7·28647
(h) 40	− 9·4973065	− 9·4871538	− 9·1064297	− 9·0945334	6·9254732	7·18303	− 7·8180978	7·26601
(g) 35	− 9·2081705	− 9·1956492	− 8·9208678	− 8·9054903	7·5830823	6·57876	− 7·6201536	7·56103
(f) 30	− 8·8479443	− 8·8331973	− 8·6397262	− 8·6212797	− 7·5662136	− 7·39428	6·5905730	7·41841
(e) 25	− 8·3958946	− 8·3791330	− 8·2487225	− 8·2275948	− 7·6301487	− 7·93448	− 7·1756352	7·13613
(d) 20	− 7·8167367	− 7·7982342	− 7·7162532	− 7·6928567	− 7·8615605		− 6·9722588	
(c) 15	− 7·0438311	− 7·0239148	− 6·9779215	− 6·9527063	7·7471710		7·8881695	
(b) 10	− 5·9270920	− 5·9061333	− 5·8850439	− 5·8584985				
(a) 5	− 3·9870544	− 3·9654569	− 3·9590140	− 3·9316574				

Co-latitude	For Y				Log $(w^{\frac{1}{2}}y'_7)$ for g 1845	Log $(w^{\frac{1}{2}}y'_7)$ for g 1880	Log $(w^{\frac{1}{2}}y'_7)$ for h 1845	Log $(w^{\frac{1}{2}}y'_7)$ for h 1880
(s) 90°	0·6931288	+ 0·6931288	− 9·4626799	− 9·4626799	− 8·4150572	− 8·26467	− 7·8165137	8·56320
(r) 85	0·8331334	+ 0·8329694	− 9·5435079	− 9·5433001	− 8·5201842	− 8·46312	− 7·6698379	8·70446
(q) 80	0·8014293	+ 0·8007779	− 9·2647984	− 9·2639733	− 8·4501539	− 8·40014	− 7·3377317	8·55522
(p) 75	0·7480027	+ 0·7465552	+ 8·6132826	+ 8·6114490	− 8·5156080	− 8·31136	− 7·6858171	8·52269
(o) 70	0·6719375	+ 0·6694084	+ 9·4264533	+ 9·4232498	− 8·3178164	− 8·18128	− 7·4003512	8·40698
(n) 65	0·5718719	+ 0·5680081	+ 9·6436286	+ 9·6387344	− 8·2436981	− 8·17360	− 7·4760123	8·21323
(m) 60	0·4459061	+ 0·4404939	+ 9·7223386	+ 9·7154831	− 8·2513376	− 8·09189	7·8844903	7·78758
(l) 55	0·2914598	+ 0·2843319	+ 9·7192733	+ 9·7102447	− 8·2825187	− 8·10847	8·3417891	− 7·13227
(k) 50	0·1050512	+ 0·0960918	+ 9·6516874	+ 9·6403388	− 8·1811716	− 8·31845	8·1432692	− 7·83396
(i) 45	9·8819549	+ 9·8711033	+ 9·5247054	+ 9·5109601	− 8·2976521	− 8·25515	7·7171318	− 8·02854
(h) 40	9·6156469	+ 9·6028999	+ 9·3372983	+ 9·3211520	− 8·0889753	− 8·05657	7·4421030	− 7·89992
(g) 35	9·2968728	+ 9·2822847	+ 9·0835942	+ 9·0651160	− 7·9643672	− 8·05005	7·7058657	− 7·74902
(f) 30	8·9119850	+ 8·8956667	+ 8·7521636	+ 8·7314937	− 7·6631237	− 7·94712	7·3752551	− 7·65639
(e) 25	8·4397419	+ 8·4218573	+ 8·3232810	+ 8·3006272	− 7·3324213	− 8·18313	7·1256613	− 7·93448
(d) 20	7·8444867	+ 7·8252483	+ 7·7623503	+ 7·7379817	− 7·6845927		− 7·0234113	
(c) 15	7·0593089	+ 7·0389711	+ 7·0032075	+ 6·9774463	− 7·9666335		7·0969230	
(b) 10	5·9339303	+ 5·9127818	+ 5·8960914	+ 5·8693033				
(a) 5	3·9887580	+ 3·9671127	+ 3·9617485	+ 3·9343311				

Co-latitude	For Z				Log $(w^{\frac{1}{2}}z'_7)$ for g 1845	Log $(w^{\frac{1}{2}}z'_7)$ for g 1880	Log $(w^{\frac{1}{2}}z'_7)$ for h 1845	Log $(w^{\frac{1}{2}}z'_7)$ for h 1880
(s) 90°	0·7511208	− 0·6931288	− 9·6175819	+ 9·5718244	− 8·8017905	− 7·81182	8·4692071	− 8·25965
(r) 85	0·8895107	− 0·8313136	− 9·6968759	+ 9·6506997	− 8·9227941	7·35947	8·6679859	− 8·19364
(q) 80	0·8529356	− 0·7941295	− 9·4137789	+ 9·3658437	− 8·7403838	8·15367	8·5085266	7·64852
(p) 75	0·7913003	− 0·7314989	+ 8·7486142	− 8·7111101	− 8·6932114	− 6·83631	− 8·2845745	7·83006
(o) 70	0·7035474	− 0·6423943	+ 9·5538634	− 9·5066210	− 8·3536648	7·67558	− 7·9939781	7·56516
(n) 65	0·5881051	− 0·5252838	+ 9·7559622	− 9·7060687	− 8·3101402	7·76303	− 7·6895089	7·33943
(m) 60	0·4427808	− 0·3780247	+ 9·8154225	− 9·7629582	− 8·2479314	8·05083	− 7·9850697	8·33540
(l) 55	0·2645966	− 0·1976965	+ 9·7886758	− 9·7334977	− 8·3417891	7·97737	− 8·2066028	8·21145
(k) 50	0·0495342	− 9·9803458	+ 9·6924718	− 9·6344493	− 7·9987770	− 8·01305	− 7·8888154	7·82836
(i) 45	9·7921409	− 9·7205884	+ 9·5312177	− 9·4702814	6·9459294	7·81683	− 8·0453140	− 8·04201
(h) 40	9·4848877	− 9·4109673	+ 9·3028846	− 9·2390423	8·5138783	8·35144	7·3514419	− 7·05041
(g) 35	9·1170959	− 9·0408760	+ 9·0001780	− 8·9335204	8·4813077	8·52324	7·9661507	8·23197
(f) 30	8·6730177	− 8·5946367	+ 8·6095693	− 8·5402702	8·2033568	8·27834	7·7252716	8·08574
(e) 25	8·1281427	− 8·0478056	+ 8·1080649	− 8·0363771	− 7·6097874	8·10394	8·0926610	8·08807
(d) 20	7·4413277	− 7·3593001	+ 7·4555824	− 7·3818314	− 8·2305368		− 6·9856227	
(c) 15	6·5353676	− 6·4519673	+ 6·5756040	− 6·5002379	− 7·6571469		7·8250282	
(b) 10	5·2368647	− 5·1524520	+ 5·2954276	− 5·2188673				
(a) 5	2·9924417	− 2·9074087	+ 3·0618367	− 2·9844198				

Co-latitude	For X g_8^7 or h_8^7	g_{-8}^7 or h_{-8}^7	g_{10}^7 or h_{10}^7	g_{-10}^7 or h_{-10}^7	Log $(w^{\frac{1}{2}}x'_7)$ for g 1845	Log $(w^{\frac{1}{2}}x'_7)$ for g 1880	Log $(w^{\frac{1}{2}}x'_7)$ for h 1845	Log $(w^{\frac{1}{2}}x'_7)$ for h 1880
(s) 90°	9·8480308	+ 9·8480308	− 9·0463985	− 9·0463985	6·4500908	8·12906	− 8·6361992	− 6·69313
(r) 85	9·9613975	+ 9·9608190	− 9·0904466	− 9·0896749	− 7·8489983	8·11501	− 8·6870489	− 7·29877
(q) 80	9·8392577	+ 9·8366397	− 8·5600040	− 8·5534067	7·9775802	7·62878	− 8·6837288	7·93483
(p) 75	9·5792314	+ 9·5710863	+ 8·8060166	+ 8·8083105	− 7·9454544	− 7·59327	− 7·2699655	7·82372
(o) 70	8·7374542	+ 8·6673583	+ 9·1304890	+ 9·1282637	− 8·2041759	− 7·73357	7·7137317	− 5·98847
(n) 65	− 9·3338549	− 9·3469401	+ 9·1798550	+ 9·1738559	− 8·1898893	− 7·20815	8·1623931	− 7·57976
(m) 60	− 9·5873711	− 9·5906560	+ 9·0581150	+ 9·0469523	− 8·0211148	6·26907	7·4378584	7·31046
(l) 55	− 9·6496123	− 9·6483001	+ 8·6827299	+ 8·6593948	7·5996355	− 6·55824	− 8·0513522	7·72703
(k) 50	− 9·6155916	− 9·6106893	− 8·1974496	− 8·2260920	− 7·9949505	− 7·41899	− 7·9891470	7·72002
(i) 45	− 9·5085977	− 9·5004578	− 8·7388458	− 8·7338571	− 8·2448864	6·87898	7·5921931	7·55821
(h) 40	− 9·3341583	− 9·3229857	− 8·8016630	− 8·7896897	− 8·1559221	7·20531	7·8073739	6·97446
(g) 35	− 9·0893356	− 9·0753238	− 8·7012983	− 8·6847691	− 7·9270658	7·85751	7·8926281	7·42386
(f) 30	− 8·7641971	− 8·7475741	− 8·4767489	− 8·4565557	− 7·6827192	8·00555	7·8856400	7·15124
(e) 25	− 8·3398509	− 8·3208960	− 8·1260915	− 8·1028188	7·9615837	7·95692	8·0880652	− 7·68314
(d) 20	− 7·7821944	− 7·7612413	− 7·6230122	− 7·5971814	7·8095315		8·0282102	
(c) 15	− 7·0253658	− 7·0027990	− 6·9057827	− 6·8779216	− 7·6922842		8·2937813	
(b) 10	− 5·9197925	− 5·8960403	− 5·8271872	− 5·7978491				
(a) 5	− 3·9863346	− 3·9618575	− 3·9094448	− 3·8792085				

Co-latitude	For Y g_8^7 or h_8^7	g_{-8}^7 or h_{-8}^7	g_{10}^7 or h_{10}^7	g_{-10}^7 or h_{-10}^7	Log $(w^{\frac{1}{2}}y'_7)$ for g 1845	Log $(w^{\frac{1}{2}}y'_7)$ for g 1880	Log $(w^{\frac{1}{2}}y'_7)$ for h 1845	Log $(w^{\frac{1}{2}}y'_7)$ for h 1880
(s) 90°	0·0000000	0.0000000	0·0000000	0·0000000				
(r) 85	9·7705540	+ 9·7703681	− 8·9478139	− 8·9475842	− 8·0230459	7·64119	− 7·2530125	7·75361
(q) 80	0·0383217	+ 0·0375835	− 9·1460600	− 9·1451481	− 8·0887307	7·95435	− 7·8874036	8·15966
(p) 75	0·1583807	+ 0·1567402	− 9·1211306	− 9·1191040	− 7·2583836	8·06309	− 7·1815436	8·31755
(o) 70	0·2035876	+ 0·2007213	− 8·8301992	− 8·8266585	7·5874379	8·19221	7·4167417	8·35645
(n) 65	0·1956860	+ 0·1913070	+ 8·4693142	+ 8·4639048	− 7·8498580	8·26773	7·4325466	8·39434
(m) 60	0·1430520	+ 0·1369182	+ 9·0960926	+ 9·0885155	− 7·9850697	8·28819	7·2580710	8·33911
(l) 55	0·0485705	+ 0·0404922	+ 9·2752060	+ 9·2652270	− 7·8482774	8·13227	7·6804587	8·26368
(k) 50	9·9120046	+ 9·9018506	+ 9·3146643	+ 9·3021211	− 7·3484123	7·54393	7·8111038	8·20194
(i) 45	9·7307044	+ 9·7184060	+ 9·2620394	+ 9·2468472	− 6·7028914	7·51580	7·8692228	8·15005
(h) 40	9·4995443	+ 9·4850977	+ 9·1303255	+ 9·1124796	8·0011939	7·55749	7·8413000	7·96498
(g) 35	9·2102488	+ 9·1937156	+ 8·9198933	+ 8·8994700	8·1362681	7·90510	7·0839109	7·43609
(f) 30	8·8498728	+ 8·8313787	+ 8·6225953	+ 8·5997496	8·2718142	8·11028	− 7·8816188	7·52231
(e) 25	8·3976876	+ 8·3774184	+ 8·2206191	+ 8·1955807	8·1256613	8·07398	− 7·7753285	7·92785
(d) 20	7·8184129	+ 7·7966094	+ 7·6805586	+ 7·6536249	8·0246160		− 7·8173568	
(c) 15	7·0454124	+ 7·0223629	+ 6·9370200	+ 6·9085471	7·6644055		− 8·0163213	
(b) 10	5·9286034	+ 5·9046351	+ 5·8407424	+ 5·8111345				
(a) 5	3·9885230	+ 3·9639916	+ 3·9127872	+ 3·8824837				

Co-latitude	For Z g_8^7 or h_8^7	g_{-8}^7 or h_{-8}^7	g_{10}^7 or h_{10}^7	g_{-10}^7 or h_{-10}^7	Log $(w^{\frac{1}{2}}z'_7)$ for g 1845	Log $(w^{\frac{1}{2}}z'_7)$ for g 1880	Log $(w^{\frac{1}{2}}z'_7)$ for h 1845	Log $(w^{\frac{1}{2}}z'_7)$ for h 1880
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000				
(r) 85	9·8777597	− 9·8270689	− 9·1422515	+ 9·1010978	7·9035359	7·83659	− 7·5292189	8·15610
(q) 80	0·1406574	− 0·0892915	− 9·3357105	+ 9·2935791	8·3133723	− 8·18564	− 8·7356717	8·24816
(p) 75	0·2525094	− 0·2000406	− 9·3027943	+ 9·2588873	8·7366771	− 8·05191	− 8·0685798	− 7·33363
(o) 70	0·2860307	− 0·2320638	− 9·0010767	+ 8·9534936	8·5136517	− 8·60656	− 8·3096604	7·79829
(n) 65	0·2627552	− 0·2069393	+ 8·6193811	− 8·5807958	8·2256750	− 8·77009	8·1191515	− 7·80377
(m) 60	0·1907659	− 0·1328055	+ 9·2297972	− 9·1825944	− 7·8401927	− 8·45940	8·1353537	8·38134
(l) 55	0·0725499	− 0·0122133	+ 9·3855494	− 9·3347365	− 8·5742309	− 8·44473	− 7·8906813	7·86999
(k) 50	9·9073340	− 9·8444611	+ 9·3965027	− 9·3423669	− 8·6382285	− 8·67587	− 7·5998835	− 7·81110
(i) 45	9·6917406	− 9·6262475	+ 9·3096616	− 9·2522454	− 8·6224053	− 8·48703	− 7·6571339	− 8·08611
(h) 40	9·4196392	− 9·3515218	+ 9·1370545	− 9·0764137	− 8·4433600	− 8·01149	8·2223472	− 8·35299
(g) 35	9·0813297	− 9·0106634	+ 8·8776411	− 8·8138976	7·0701226	7·11024	7·8388323	6·48185
(f) 30	8·6617667	− 8·5887054	+ 8·5211797	− 8·4545356	− 7·8302137	6·69531	− 7·8092517	− 8·12896
(e) 25	8·1369527	− 8·0617234	+ 8·0465921	− 7·9773306	− 8·0915166	− 7·80068	8·1017091	− 8·02610
(d) 20	7·4661209	− 7·3890177	+ 7·4149877	− 7·3434682	− 8·2580973		− 7·3861869	
(c) 15	6·5723405	− 6·4937157	+ 6·5506787	− 6·4773278	− 7·9330662		8·1470896	
(b) 10	5·2824087	− 5·2026619	+ 5·2812848	− 5·2065847				
(a) 5	3·0430785	− 2·9626441	+ 3·0540837	− 2·9785570				

Co-latitude	For X $g_8^{\ 8}$ or $h_8^{\ 8}$	$g_{-8}^{\ 8}$ or $h_{-8}^{\ 8}$	$g_{10}^{\ 8}$ or $h_{10}^{\ 8}$	$g_{-10}^{\ 8}$ or $h_{-10}^{\ 8}$	Log $(w^{\frac{1}{2}}x'_8)$ for g 1845	Log $(w^{\frac{1}{2}}x'_8)$ for g 1880	Log $(w^{\frac{1}{2}}x'_8)$ for h 1845	Log $(w^{\frac{1}{2}}x'_8)$ for h 1880
(s) 90°	0·0000000	0·0000000	0·0000000	0·0000000				
(r) 85	− 9·8236417	− 9·8296127	+ 9·2935761	+ 9·2945166	8·2672529	− 7·17383	7·7758913	− 7·62099
(q) 80	− 0·0865557	− 0·0918353	+ 9·5134469	+ 9·5131752	8·5762340	7·90912	8·2541856	− 7·55161
(p) 75	− 0·1984343	− 0·2025844	+ 9·5442327	+ 9·5417665	8·3346205	7·13734	8·4588200	− 7·85453
(o) 70	− 0·2319917	− 0·2346074	+ 9·4356633	+ 9·4294274	7·2641315	− 7·72574	8·4566696	− 7·40035
(n) 65	− 0·2087607	− 0·2094831	+ 9·1298135	+ 9·1148604	− 8·1638101	7·60095	8·5910149	− 7·52177
(m) 60	− 0·1368230	− 0·1353492	− 7·8713869	− 8·0338638	− 7·7673769	7·87653	8·3173139	− 7·08198
(l) 55	− 0·0186642	− 0·0147572	− 9·0355909	− 9·0390466	5·9561828	− 7·56896	7·9128314	− 7·71961
(k) 50	− 9·8535090	− 9·8470049	− 9·1743716	− 9·1695090	− 7·1971446	− 7·47335	8·1251419	− 7·26409
(i) 45	− 9·6379782	− 9·6287914	− 9·1409341	− 9·1311062	− 8·0057271	− 7·40186	8·1178647	− 7·15519
(h) 40	− 9·3659394	− 9·3540654	− 8·9971872	− 8·9832945	− 7·6326377	6·90428	7·6524719	5·90428
(g) 35	− 9·0276909	− 9·0132073	− 8·7554467	− 8·7379499	− 7·7801580	7·79887	7·1228289	− 7·24152
(f) 30	− 8·6081850	− 8·5912490	− 8·4105999	− 8·3898714	− 7·4184120	7·83698	− 7·8836341	− 7·87959
(e) 25	− 8·0834225	− 8·0642670	− 7·9439537	− 7·9203772	− 7·6334513	7·78704	− 7·6432112	− 7·62682
(d) 20	− 7·4126352	− 7·3915615	− 7·3178653	− 7·2918664	7·9667959		− 7·0903581	
(c) 15	− 6·5188911	− 6·4962595	− 6·4573485	− 6·4294021	7·1626018		− 7·5710798	
(b) 10	− 5·2289859	− 5·2052058	− 5·1904342	− 5·1610607				
(a) 5	− 2·9896719	− 2·9651879	− 2·9646378	− 2·9343929				
	For Y				Log $(w^{\frac{1}{2}}y'_8)$ for g 1845	1880	Log $(w^{\frac{1}{2}}y'_8)$ for h 1845	1880
(s) 90°	0·7511208	+ 0·7511208	− 9·4723672	− 9·4723672	7·8523522	− 7·45009	− 7·9049357	8·00032
(r) 85	0·8895026	+ 0·8893167	− 9·5440456	− 9·5438159	− 5·9977400	− 7·50289	− 7·6923452	8·18243
(q) 80	0·8529033	+ 0·8521651	− 9·2122013	− 9·2112894	− 7·9931321	6·99531	− 7·8434981	8·17715
(p) 75	0·7912286	+ 0·7895881	+ 8·9225075	+ 8·9204809	− 8·0286384	7·40618	− 7·7236057	7·65397
(o) 70	0·7034221	+ 0·7005558	+ 9·5028916	+ 9·4993509	− 7·8516652	7·82423	− 7·9298606	6·88847
(n) 65	0·5879138	+ 0·5835348	+ 9·6819141	+ 9·6765047	− 7·4895851	7·89151	7·5340042	− 6·82280
(m) 60	0·4425130	+ 0·4363792	+ 9·7329697	+ 9·7253926	7·6212489	7·69232	7·8629061	7·26907
(l) 55	0·2642441	+ 0·2561658	+ 9·7020064	+ 9·6920274	7·3363940	7·80128	8·0650859	7·58965
(k) 50	0·0490914	+ 0·0389374	+ 9·6033199	+ 9·5907767	− 7·8609502	7·74084	8·1308006	7·69774
(i) 45	9·7916047	+ 9·7793063	+ 9·4404542	+ 9·4252620	7·7998014	7·83038	7·6918960	7·93334
(h) 40	9·4842581	+ 9·4698115	+ 9·2110042	+ 9·1931583	7·6446466	7·92547	7·8019110	7·92547
(g) 35	9·1163760	+ 9·0998428	+ 8·9074892	+ 8·8870659	6·7828809	7·73105	− 7·8732271	7·32695
(f) 30	8·6722125	+ 8·6537184	+ 8·5162786	+ 8·4934329	− 7·4785992	7·29737	− 7·9978866	− 7·52231
(e) 25	8·1272605	+ 8·1069913	+ 8·0143209	+ 7·9892825	− 7·7197033	− 6·41597	− 7·6097874	− 7·77295
(d) 20	7·4403791	+ 7·4185756	+ 7·3614987	+ 7·3345650	5·7681388		7·3755938	
(c) 15	6·5343653	+ 6·5113158	+ 6·4813322	+ 6·4528593	7·8460596		7·5590152	
(b) 10	5·2358226	+ 5·2118543	+ 5·2009268	+ 5·1713189				
(a) 5	2·9913751	+ 2·9668437	+ 2·9672369	+ 2·9369334				
	For Z				Log $(w^{\frac{1}{2}}z'_8)$ for g 1845	1880	Log $(w^{\frac{1}{2}}z'_8)$ for h 1845	1880
(s) 90°	0·8022733	− 0·7511208	− 9·6106699	+ 9·5692772	− 6·7511208	8·22824	− 8·2545710	7·71135
(r) 85	0·9390406	− 0·8876609	− 9·6808178	+ 9·6389784	8·1665320	8·15911	− 8·5576466	7·58880
(q) 80	0·8975713	− 0·8455166	− 9·3446747	+ 9·3008306	8·6557001	7·88180	− 8·7774231	− 7·76616
(p) 75	0·8276894	− 0·7745318	+ 9·0436586	− 9·0050311	7·8996969	6·59327	− 8·4646989	7·75464
(o) 70	0·7281972	− 0·6735417	+ 9·6139013	− 9·5702607	8·2442545	− 7·72574	7·1157117	− 7·68435
(n) 65	0·5973146	− 0·5408105	+ 9·7777606	− 9·7314772	8·4526460	7·92218	8·1912195	− 7·81655
(m) 60	0·4325582	− 0·3739099	+ 9·8095427	− 9·7605349	7·9294575	− 7·17216	− 7·6212489	− 8·13536
(l) 55	0·2305545	− 0·1695303	+ 9·7548888	− 9·7029607	8·2314942	− 6·80128	− 7·9128314	6·85927
(k) 50	9·9867515	− 9·9231913	+ 9·6275802	− 9·5725746	8·1486980	8·03878	− 7·9483381	− 8·21138
(i) 45	9·6949712	− 9·6287912	+ 9·4304408	− 9·3722751	− 6·9459294	7·49294	7·8140418	− 7·58750
(h) 40	9·3466829	− 9·2778789	+ 9·1600646	− 9·0987432	− 7·5622953	− 7·57638	8·1029410	− 8·06264
(g) 35	8·9297865	− 8·8584340	+ 8·8075478	− 8·7431669	− 8·2030430	− 7·67218	7·9853011	− 8·07569
(f) 30	8·4264357	− 8·3526884	+ 8·3571597	− 8·2899073	− 8·2601434	− 8·46930	7·9090158	7·26518
(e) 25	7·8088547	− 7·7329396	+ 7·7825810	− 7·7127312	− 8·0557028	− 8·31360	8·0983381	7·72772
(d) 20	7·0304158	− 6·9526273	+ 7·0382081	− 6·9661143	− 6·8650488		8·2643763	
(c) 15	6·0036219	− 5·9243120	+ 6·0372666	− 5·9633511	7·5173166		7·9593951	
(b) 10	4·5319563	− 4·4515245	+ 4·5837419	− 4·5084833				
(a) 5	1·9882589	− 1·9071397	+ 2·0508043	− 1·9747228				

Co-latitude	For X: g_9^8 or h_9^8	g_{-9}^8 or h_{-9}^8	Log $(w^{\frac{1}{2}}x'_8)$ for g: 1845	1880	Log $(w^{\frac{1}{2}}x'_8)$ for h: 1845	1880
(s) 90°	9·8480308	+ 9·8480308	8·0935435	7·91622	7·9552408	7·67410
(r) 85	9·9563224	+ 9·9556724	8·4274923	7·77589	8·2457133	7·87855
(q) 80	9·8155880	+ 9·8125766	7·9446990	8·06350	7·9753124	7·87037
(p) 75	9·4996963	+ 9·4894048	8·6836181	− 7·59327	− 7·4610339	− 7·55941
(o) 70	− 8·3261700	− 8·4655921	8·4918829	− 7·97215	− 6·9853779	− 7·60863
(n) 65	− 9·4485729	− 9·4573464	8·4178233	7·79061	8·0171158	− 7·64046
(m) 60	− 9·6243515	− 9·6258991	8·5338842	− 7·60151	− 8·1541448	7·00943
(l) 55	− 9·6448444	− 9·6421029	8·5773591	6·43330	− 8·0783987	7·46133
(k) 50	− 9·5736250	− 9·5672510	8·0941604	− 7·05581	− 7·6451635	7·30360
(i) 45	− 9·4274479	− 9·4176849	7·2364940	− 7·30495	− 7·7635892	7·22577
(h) 40	− 9·2089604	− 9·1959710	7·6485769	− 7·67513	7·3356477	7·62856
(g) 35	− 8·9130166	− 8·8969808	6·8575145	− 7·38494	7·5564845	7·18082
(f) 30	− 8·5272023	− 8·5083508	7·3687242	− 7·66975	7·6495508	− 6·32733
(e) 25	− 8·0291527	− 8·0077789	− 7·1756352	− 7·88946	7·7608507	7·63345
(d) 20	− 7·3791606	− 7·3556215	7·3701988		7·7458624	
(c) 15	− 6·5009996	− 6·4757099	− 7·3752099		− 8·0599394	
(b) 10	− 5·2219334	− 5·1953565				
(a) 5	− 2·9890129	− 2·9616485				

Co-latitude	For Y		Log $(w^{\frac{1}{2}}y'_8)$ for g: 1845	1880	Log $(w^{\frac{1}{2}}y'_8)$ for h: 1845	1880
(s) 90°	0·0000000	0·0000000				
(r) 85	9·8269231	+ 9·8267153	− 6·9977400	7·98451	− 8·0751079	7·65095
(q) 80	0·0897958	+ 0·0889707	8·1008192	8·12241	− 8·2541856	8·01650
(p) 75	0·2016066	+ 0·1997730	7·7836036	8·11178	− 8·2007269	8·23918
(o) 70	0·2350722	+ 0·2318687	7·6799831	7·93962	− 8·0787996	8·22593
(n) 65	0·2117279	+ 0·2068337	− 7·1951856	7·77704	− 7·6678978	8·01989
(m) 60	0·1396589	+ 0·1328034	− 7·4151944	7·73889	− 6·6212489	7·75337
(l) 55	0·0213548	+ 0·0123262	7·5124853	7·31791	7·1992208	6·25721
(k) 50	9·8560448	+ 9·8446962	− 7·0558155	7·57534	7·9396952	− 7·52165
(i) 45	9·6403541	+ 9·6266088	7·5582086	7·44325	− 7·3479860	− 7·32268
(h) 40	9·3681555	+ 9·3520092	7·9128841	7·85852	− 6·5063439	− 7·60325
(g) 35	9·0297519	+ 9·0112737	7·9801614	7·90098	− 7·4978390	− 7·59579
(f) 30	8·6101003	+ 8·5894304	7·9202482	7·84585	− 6·9109081	− 7·31261
(e) 25	8·0852062	+ 8·0625524	7·5956628	7·13613	− 7·4422963	− 6·81391
(d) 20	7·4143052	+ 7·3899366	− 6·7223813		− 7·5201872	
(c) 15	6·5204689	+ 6·4947077	7·8881695		7·4930867	
(b) 10	5·2304957	+ 5·2037076				
(a) 5	2·9911402	+ 2·9637228				

Co-latitude	For Z		Log $(w^{\frac{1}{2}}z'_8)$ for g: 1845	1880	Log $(w^{\frac{1}{2}}z'_8)$ for h: 1845	1880
(s) 90°	0·0000000	0·0000000				
(r) 85	9·9219270	− 9·8765361	8·0640659	7·31996	7·8610629	− 7·87280
(q) 80	0·1799304	− 0·1337988	− 8·2493735	− 7·73567	8·1460654	7·37552
(p) 75	0·2835354	− 0·2361935	8·6206215	− 7·98244	− 7·4755117	8·23672
(o) 70	0·3053170	− 0·2563312	8·5032333	− 8·16147	− 7·3745440	8·18404
(n) 65	0·2666007	− 0·2155860	8·5059755	− 7·59048	7·7219947	7·79061
(m) 60	0·1751786	− 0·1218108	8·4400611	− 7·11417	8·3624881	7·69232
(l) 55	0·0331426	− 9·9771673	8·5307927	7·98556	8·0034577	8·10231
(k) 50	9·8391852	− 9·7804268	8·5753406	7·83950	7·7015399	7·90566
(i) 45	9·5892040	− 9·5275704	8·3455207	8·15519	− 7·9001719	7·20349
(h) 40	9·2760668	− 9·2115534	8·4959049	7·90428	− 8·2181511	7·50634
(g) 35	8·8886519	− 8·8213415	7·7962448	− 7·13506	− 6·4238589	7·11024
(f) 30	8·4098159	− 8·3398771	− 7·7692884	7·92209	8·1856682	− 8·21005
(e) 25	7·8122952	− 7·7399774	− 7·9393887	8·01257	7·4622674	− 7·70600
(d) 20	7·0498391	− 6·9754649	6·7458624		8·1326898	
(c) 15	6·0352245	− 5·9591805	7·8166600		7·6346136	
(b) 10	4·5721296	− 4·4948544				
(a) 5	2·0335251	− 1·9554952				

Co-latitude	For X: g_9^9 or h_9^9	g_{-9}^9 or h_{-9}^9	Log $(w^{\frac{1}{2}}x'_9)$ for g: 1845	Log $(w^{\frac{1}{2}}x'_9)$ for g: 1880	Log $(w^{\frac{1}{2}}x'_9)$ for h: 1845	Log $(w^{\frac{1}{2}}x'_9)$ for h: 1880
(s) 90°	0·0000000	0·0000000				
(r) 85	− 9·8732117	− 9·8791203	8·3052360	− 7·67898	− 7·8981071	− 7·31996
(q) 80	− 0·1312300	− 0·1363830	8·1025190	− 7·64852	− 8·3522904	− 6·47243
(p) 75	− 0·2348588	− 0·2387777	8·2305114	7·38915	− 7·6679055	6·46833
(o) 70	− 0·2566729	− 0·2589153	− 7·6034260	− 7·26413	6·9631015	5·98538
(n) 65	− 0·2179967	− 0·2181703	− 7·5340042	− 7·48285	− 6·8807917	− 7·33943
(m) 60	− 0·1266210	− 0·1243949	− 6·9892257	7·63080	7·7461877	7·00943
(l) 55	− 9·9846363	− 9·9797516	7·4680662	6·65515	8·1534634	7·31791
(k) 50	− 9·7907336	− 9·7830110	− 7·6782686	− 6·94187	− 8·3921212	6·98326
(i) 45	− 9·5408088	− 9·5301547	8·0057271	6·87898	− 7·9203753	5·92474
(h) 40	− 9·2277281	− 9·2141375	7·5855251	6·90428	− 7·9870693	− 7·01822
(g) 35	− 8·8403679	− 8·8239258	7·5030402	7·50304	− 7·4872459	− 6·78288
(f) 30	− 8·3615832	− 8·3424612	− 7·2973683	− 7·24815	7·6697542	7·54918
(e) 25	− 7·7641089	− 7·7425615	− 6·4159674	7·63998	6·1149374	7·38211
(d) 20	− 7·0016928	− 6·9780492	7·7481422		7·5352947	
(c) 15	− 5·9871107	− 5·9617647	− 7·0501796		− 7·0969230	
(b) 10	− 4·5240398	− 4·4974387				
(a) 5	− 1·9854500	− 1·9580797				

Co-latitude	For Y		Log $(w^{\frac{1}{2}}y'_9)$ for g: 1845	Log $(w^{\frac{1}{2}}y'_9)$ for g: 1880	Log $(w^{\frac{1}{2}}y'_9)$ for h: 1845	Log $(w^{\frac{1}{2}}y'_9)$ for h: 1880
(s) 90°	0·8022733	+ 0·8022733	− 7·4814993	7·67410	− 7·9586205	7·88145
(r) 85	0·9390321	+ 0·9388243	− 7·7420330	8·09116	− 8·1100098	8·22563
(q) 80	0·8975379	+ 0·8967128	− 7·7734603	8·12241	− 8·1856407	7·88180
(p) 75	0·8276150	+ 0·8257814	6·4683332	8·08812	− 8·1980378	− 6·59327
(o) 70	0·7280672	+ 0·7248637	7·6929481	8·09259	− 8·1629144	− 7·02677
(n) 65	0·5971162	+ 0·5922220	7·9926420	7·79061	− 8·2341789	7·40906
(m) 60	0·4322804	+ 0·4254249	7·4799198	7·63080	− 8·2292993	− 7·22331
(l) 55	0·2301888	+ 0·2211602	7·3955155	7·44754	− 7·7657425	7·50025
(k) 50	9·9862921	+ 9·9749435	7·4669169	− 6·98326	8·0025699	7·83950
(i) 45	9·6944149	+ 9·6806696	6·9247401	− 6·96613	6·4018614	7·82783
(h) 40	9·3460299	+ 9·3298836	− 7·1084039	6·20531	− 7·0504119	7·86807
(g) 35	8·9290395	+ 8·9105613	− 6·0558822	− 7·18082	7·1228289	7·76060
(f) 30	8·4256005	+ 8·4049306	− 7·3485209	6·89160	7·1926330	7·36872
(e) 25	7·8079396	+ 7·7852858	− 7·2982072	7·59924	7·5340667	8·03402
(d) 20	7·0294319	+ 7·0050633	7·7767390		− 6·7458624	
(c) 15	6·0025822	+ 5·9768210	7·8569760		− 7·6268350	
(b) 10	4·5308753	+ 4·4440872				
(a) 5	1·9871529	+ 1·9597355				

Co-latitude	For Z		Log $(w^{\frac{1}{2}}z'_9)$ for g: 1845	Log $(w^{\frac{1}{2}}z'_9)$ for g: 1880	Log $(w^{\frac{1}{2}}z'_9)$ for h: 1845	Log $(w^{\frac{1}{2}}z'_9)$ for h: 1880
(s) 90°	0·8480308	− 0·8022733	8·6390193	7·76711	8·2857814	7·89335
(r) 85	0·9831754	− 0·9371685	− 8·6416859	8·22305	8·2764936	7·74593
(q) 80	0·9368118	− 0·8900643	− 8·5835807	8·13832	− 8·1521609	− 7·61856
(p) 75	0·8586830	− 0·8107252	− 8·4513577	8·13422	6·7315746	− 7·43837
(o) 70	0·7474519	− 0·6978496	− 8·1672215	8·12839	− 8·2720589	7·90446
(n) 65	0·6011278	− 0·5494977	− 8·2353803	− 7·60095	− 7·7838817	7·42486
(m) 60	0·4169386	− 0·3629557	− 8·2984502	7·59129	8·0935177	7·11417
(l) 55	0·1911148	− 0·1345247	− 8·3769634	7·70437	7·7233387	7·57943
(k) 50	9·9185704	− 9·8591975	− 8·1811716	7·86095	7·1971446	− 8·03878
(i) 45	9·5924024	− 9·5301546	− 8·1192544	− 7·67293	− 8·1094315	− 7·37190
(h) 40	9·2030785	− 9·1379511	− 8·2808609	− 8·04729	− 8·4427320	− 7·89105
(g) 35	8·7370765	− 8·6691525	− 7·6506429	− 7·87102	− 8·0544321	− 7·92511
(f) 30	8·1744527	− 8·1039006	7·4985703	− 7·59057	7·6697542	− ∞
(e) 25	7·4841650	− 7·4112341	7·9244971	6·96004	− 7·9661957	− 7·80514
(d) 20	6·6141019	− 6·5391150	8·0246160		− 8·0305899	
(c) 15	5·4664737	− 5·3898172	8·1134447		− 7·4320328	
(b) 10	3·8216451	− 3·7437574				
(a) 5	0·9786736	− 0·9000314				

Co-latitude		For X: g_{10}^{9} or h_{10}^{9}	g_{-10}^{9} or h_{-10}^{9}	$\mathrm{Log}(w^{\frac{1}{2}}x'_9)$ for g 1845	$\mathrm{Log}(w^{\frac{1}{2}}x'_9)$ for g 1880	$\mathrm{Log}(w^{\frac{1}{2}}x'_9)$ for h 1845	$\mathrm{Log}(w^{\frac{1}{2}}x'_9)$ for h 1880
(s)	90°	9·8480308	+ 9·8480308	− 8·4126969	7·80707	− 8·5706647	− 7·46081
(r)	85	9·9512198	+ 9·9504974	− 8·5954352	7·79708	− 8·7667479	7·71374
(q)	80	9·7912130	+ 9·7877823	− 8·1383238	7·96844	− 8·5299696	7·41028
(p)	75	9·4087799	+ 9·3955787	− 8·3481933	− 7·24648	− 8·5894551	− 6·69018
(o)	70	− 8·9456110	− 8·9823143	− 8·2119778	− 6·98538	− 8·3664930	− 7·84870
(n)	65	− 9·5215139	− 9·5277269	− 8·2281217	8·04216	− 8·2182509	− 7·05688
(m)	60	− 9·6431152	− 9·6432666	− 8·0999757	7·70043	− 7·2467900	− 7·70043
(l)	55	− 9·6274800	− 9·6234472	7·5064112	7·51248	− 8·2794349	− 7·81950
(k)	50	− 9·5214059	− 9·5136334	− 7·6620314	7·82268	− 7·8051950	− 7·28429
(i)	45	− 9·3372819	− 9·3259399	7·1150718	7·88853	− 7·9247401	− 7·22577
(h)	40	− 9·0754870	− 9·0607092	7·4483519	7·66015	− 6·9456766	− 7·66771
(g)	35	− 8·7289021	− 8·7108608	6·6579422	− 6·95897	− 7·9535093	− 7·77742
(f)	30	− 8·2827386	− 8·2616706	− 7·6953083	− 7·81400	− 7·7194420	− 7·60608
(e)	25	− 7·7112138	− 7·6874290	− 7·6527565	− 7·01803	− 7·6334513	− 7·35798
(d)	20	− 6·9690469	− 6·9429266	7·3965277		− 6·8288366	
(c)	15	− 5·9696649	− 5·9416547	7·4636318		7·5680949	
(b)	10	− 4·5171795	− 4·4877790				
(a)	5	− 1·9848380	− 1·9545866				

Co-latitude		For Y		$\mathrm{Log}(w^{\frac{1}{2}}y'_9)$ for g 1845	$\mathrm{Log}(w^{\frac{1}{2}}y'_9)$ for g 1880	$\mathrm{Log}(w^{\frac{1}{2}}y'_9)$ for h 1845	$\mathrm{Log}(w^{\frac{1}{2}}y'_9)$ for h 1880
(s)	90°	0·0000000	0·0000000				
(r)	85	9·8764526	+ 9·8762229	7·2877746	7·62099	− 8·0929094	6·99774
(q)	80	0·1344303	+ 0·1335184	− 7·8586319	7·80822	− 8·1828297	6·47243
(p)	75	0·2379930	+ 0·2359664	− 7·7620639	7·73940	− 7·9824380	− 7·22166
(o)	70	0·2597173	+ 0·2561766	− 7·7707077	− 7·40035	− 7·8690393	− 7·78472
(n)	65	0·2209303	+ 0·2155209	− 8·0402837	− 7·09164	7·1390697	− 8·01909
(m)	60	0·1294264	+ 0·1218493	− 7·8791940	− 7·70840	7·2580710	− 7·63080
(l)	55	9·9872995	+ 9·9773205	− 7·5416435	− 7·80128	− 7·5946721	− 7·53596
(k)	50	9·7932455	+ 9·7807023	− 7·4796912	− 7·73426	6·9418721	− 5·94187
(i)	45	9·5431644	+ 9·5279722	− 7·6059813	− 7·10083	7·2257701	7·37190
(h)	40	9·2299273	+ 9·2120814	− 7·6751359	− 6·68243	− 7·3514419	7·49534
(g)	35	8·8424155	+ 8·8219922	− 7·1915448	− 7·34219	6·9404887	7·44799
(f)	30	8·3634883	+ 8·3406426	− 7·1402449	− 7·21194	− 7·2304215	− 7·58260
(e)	25	7·7658854	+ 7·7408470	− 5·5128774	− 7·54630	− 7·3517265	− 7·09266
(d)	20	7·0033581	+ 6·9764244	7·1105615		− 7·0353105	
(c)	15	5·9886857	+ 5·9602128	− 7·3705147		− 7·6667983	
(b)	10	4·5255484	+ 4·4959405				
(a)	5	1·9869179	+ 1·9566144				

Co-latitude		For Z		$\mathrm{Log}(w^{\frac{1}{2}}z'_9)$ for g 1845	$\mathrm{Log}(w^{\frac{1}{2}}z'_9)$ for g 1880	$\mathrm{Log}(w^{\frac{1}{2}}z'_9)$ for h 1845	$\mathrm{Log}(w^{\frac{1}{2}}z'_9)$ for h 1880
(s)	90°	0·0000000	0·0000000				
(r)	85	9·9617234	− 9·9206165	7·7497884	− 5·99774	7·1116834	5·99774
(q)	80	0·2148325	− 0·1729191	8·1898233	− 8·04063	− 7·9519576	7·98208
(p)	75	0·3101902	− 0·2669595	8·3759236	− 7·91029	− 8·4588200	8·21393
(o)	70	0·3202319	− 0·2752117	8·3137575	− 7·43254	− 8·2853208	8·03070
(n)	65	0·2660745	− 0·2188458	7·4828517	− 7·60095	− 8·3686368	− 7·42486
(m)	60	0·1552193	− 0·1054292	7·8601310	− 7·81314	− 8·1946363	7·95481
(l)	55	9·9893625	− 9·9367343	− 7·4680662	− 7·70437	6·6093953	8·32909
(k)	50	9·7666634	− 9·7110055	7·4921005	− ∞	7·6046299	8·03878
(i)	45	9·4822939	− 9·4235064	− 8·0302503	− 7·67293	8·2902281	7·67293
(h)	40	9·1281206	− 9·0661982	− 7·7036244	7·65247	8·2817723	7·52753
(g)	35	8·6916000	− 8·6266327	− 7·7186400	7·50304	7·7855868	7·32695
(f)	30	8·1534900	− 8·0856618	− 7·3620937	7·77449	7·7135332	7·76929
(e)	25	7·4832623	− 7·4128445	8·0180274	6·96004	− 7·6200874	7·65901
(d)	20	6·6291815	− 6·5565254	8·1474443		− 8·1814386	
(c)	15	5·4937372	− 5·4192582	8·2800475		− 7·6108469	
(b)	10	3·8574742	− 3·7816599				
(a)	5	1·0195952	− 0·9429596				

Co-latitude	For X: g_{10}^{10} or h_{10}^{10}	g_{-10}^{10} or h_{-10}^{10}	$\text{Log}(w^{\frac{1}{2}}x'_{10})$ for g 1845	$\text{Log}(w^{\frac{1}{2}}x'_{10})$ for g 1880	$\text{Log}(w^{\frac{1}{2}}x'_{10})$ for h 1845	$\text{Log}(w^{\frac{1}{2}}x'_{10})$ for h 1880
(s) 90°	0·0000000	0·0000000				
(r) 85	− 9·9173787	− 9·9232330	− 7·8814014	− 7·64119	7·6651930	− 7·22819
(q) 80	− 0·1705011	− 0·1755357	7·5576019	7·07449	8·3465253	7·33773
(p) 75	− 0·2658807	− 0·2695761	8·1716245	− 7·73940	8·4257808	7·07039
(o) 70	− 0·2759518	− 0·2778282	8·2885740	7·24065	8·4477759	− 7·51686
(n) 65	− 0·2218309	− 0·2214625	8·0882914	6·82280	7·9414895	− 7·12383
(m) 60	− 0·1110180	− 0·1080458	7·8283744	− 6·74619	7·6015049	− 7·19849
(l) 55	− 9·9452080	− 9·9393510	8·4133074	6·80128	7·9773721	7·52438
(k) 50	− 9·7225585	− 9·7136221	7·8283628	7·19714	8·0081980	− 7·19714
(i) 45	− 9·4382405	− 9·4261231	7·8332251	7·24696	8·1247694	− 7·22577
(h) 40	− 9·0841184	− 9·0688147	7·7676068	7·20531	8·0987982	− 6·68243
(g) 35	− 8·6476472	− 8·6292493	7·2222136	− 7·08391	7·4818509	6·92118
(f) 30	− 8·1095843	− 8·0882784	7·5577805	− 7·35536	7·2973683	7·62836
(e) 25	− 7·4393989	− 7·4154611	6·9899987	− 7·19412	7·6712399	− 7·62682
(d) 20	− 6·5853545	− 6·5591420	− 7·5498942		6·4213513	
(c) 15	− 5·4499348	− 5·4218749	7·3412254		7·5239982	
(b) 10	− 3·8136985	− 3·7842766				
(a) 5	− 0·9758329	− 0·9455762				
	For Y		$\text{Log}(w^{\frac{1}{2}}y'_{10})$ for g 1845	$\text{Log}(w^{\frac{1}{2}}y'_{10})$ for g 1880	$\text{Log}(w^{\frac{1}{2}}y'_{10})$ for h 1845	$\text{Log}(w^{\frac{1}{2}}y'_{10})$ for h 1880
(s) 90°	0·8480308	+ 0·8480308	7·7875501	− 7·48150	− 7·7774497	− 7·45009
(r) 85	0·9831667	+ 0·9829370	7·8669717	− 6·99774	− 7·9035359	− 7·46014
(q) 80	0·9367774	+ 0·9358655	− 7·1856407	7·72770	− 7·7661610	− 7·31753
(p) 75	0·8586064	+ 0·8565798	6·5352799	7·52269	− 7·4305446	− 7·59327
(o) 70	0·7473173	+ 0·7437766	− 6·7635292	7·57644	− 7·3471057	− 7·50389
(n) 65	0·6009236	+ 0·5955142	7·5797617	− 7·40906	7·5399946	− 7·66790
(m) 60	0·4166529	+ 0·4090758	7·3745766	7·24679	7·0983702	− 7·48655
(l) 55	0·1907386	+ 0·1807596	− 6·8312441	7·27840	− 7·0865166	− 7·23493
(k) 50	9·9180978	+ 9·9055546	7·7163891	7·46038	7·0558155	− 7·43323
(i) 45	9·5918302	+ 9·5766380	− 7·6844079	7·53752	− 7·0386835	− 7·64074
(h) 40	9·2024067	+ 9·1845608	− 6·6824352	7·40943	− 7·0656519	− 7·79637
(g) 35	8·7363081	+ 8·7158848	− 7·6394587	6·95897	7·7154815	− 7·48185
(f) 30	8·1735935	+ 8·1507478	− 7·1054828	7·35536	7·5744862	6·62836
(e) 25	7·4832238	+ 7·4581854	− 6·6268208	− 6·41597	7·5340667	6·59206
(d) 20	6·6130898	+ 6·5861561	− 7·4164988		− 7·6845927	
(c) 15	5·4654041	+ 5·4369312	− 8·0491914		7·4401507	
(b) 10	3·8205330	+ 3·7909251				
(a) 5	0·9775355	+ 0·9472320				
	For Z		$\text{Log}(w^{\frac{1}{2}}z'_{10})$ for g 1845	$\text{Log}(w^{\frac{1}{2}}z'_{10})$ for g 1880	$\text{Log}(w^{\frac{1}{2}}z'_{10})$ for h 1845	$\text{Log}(w^{\frac{1}{2}}z'_{10})$ for h 1880
(s) 90°	0·8894235	− 0·8480308	8·2873635	7·79742	− 8·2282420	− 7·05215
(r) 85	1·0229454	− 0·9812813	8·4974271	8·19364	− 8·3877915	7·44490
(q) 80	0·9716873	− 0·9292170	8·0939527	7·91439	− 8·4234438	− 6·59737
(p) 75	0·8853113	− 0·8415236	8·0793480	8·24163	− 7·9868471	− 7·60399
(o) 70	0·7623393	− 0·7167625	8·4979289	7·74125	7·4003512	7·18950
(n) 65	0·6005750	− 0·5527899	8·5603331	7·77009	8·1048065	− 8·15954
(m) 60	0·3969528	− 0·3466065	8·3211829	7·87113	8·3945477	− 6·74619
(l) 55	0·1473084	− 0·0941241	8·3711561	− 7·50025	8·1891789	7·33639
(k) 50	9·8460221	− 9·7898086	8·0972081	6·24290	7·9589054	− 7·46038
(i) 45	9·4854659	− 9·4261230	8·2091708	8·00392	8·0021080	7·33971
(h) 40	9·0551058	− 8·9926283	8·2447280	8·24472	8·2265032	7·26601
(g) 35	8·5399978	− 8·4744760	7·7401289	8·26896	8·1190904	8·15625
(f) 30	7·9181006	− 7·8497178	8·1775692	7·68272	7·2734562	8·04611
(e) 25	7·1551058	− 7·0841336	− 7·1256613	7·79618	− 7·3936910	7·43716
(d) 20	6·1934181	− 6·1202078	− 7·9722588		− 7·7271802	
(c) 15	4·9249553	− 4·8499274	− 8·0299762		− 7·9842187	
(b) 10	3·1069634	− 3·0305953				
(a) 5	9·9647173	− 9·8875280				

SECTION VIII.

FORMATION OF THE EQUATIONS OF CONDITION, FORMATION OF THE FINAL EQUATIONS AND THE DETERMINATION OF THE VALUES OF THE MAGNETIC CONSTANTS.

1. THE theory of Terrestrial Magnetism, as developed in Section V. for the Sphere, and in Section VI. for the Spheroidal surface, requires that the observations of the horizontal and vertical magnetic forces should be distributed uniformly over the whole surface of the Earth. In the absence of such observations, and especially when as yet there are very few trustworthy observations of the magnetic elements taken in high latitudes, it is necessary to adopt some other method of solution of the equations of condition just given at the end of the preceding Section, taking only those equations which relate to that portion of the Earth's surface over which fairly good observations have been taken. Under these circumstances it will be necessary to take into account the values of those terms of the equations which now no longer vanish when the integration is not taken over the whole surface of the Earth. We will assume that the Charts from which the observations are taken are trustworthy up to a latitude of $67°\frac{1}{2}$ either north or south of the Equator, *i.e.* taking a central belt 5° broad at the Equator and 13 belts of the same breadth on either side of it.

The *equations of condition* given in Section VII. and the *final equations* for X, Y and Z for the period 1845 are formed for each belt between latitudes $77°\frac{1}{2}$ N. and $77°\frac{1}{2}$ S., *i.e.* for 15 belts of 5° on either side of the equatorial belt, but the observations in extreme northern or southern latitudes between $67°\frac{1}{2}$ and $77°\frac{1}{2}$ have not been included in the solution of the equations except in a few special instances.

Regarding the Earth as a spheroid of revolution, we have seen that the values of μ' or $\cos\theta'$, where θ' is the geocentric colatitude, have been determined for every 5° of geographical colatitude. Also the values of $\cos\psi$, $\sin\psi$, $\frac{a}{r}$, G'^m_n and H'^m_n have been determined for every 5° of geographical colatitude and have been tabulated for all values of n and m from 0 to 10. The weights of the observations of the magnetic elements for these belts of latitude have also been determined on the assumption that the weight is proportional to the area of the corresponding portion of the Earth's surface. From the values of H'^m_n the values of X^m_n, X^m_{-n}, Y^m_n, Y^m_{-n}, Z^m_n and Z^m_{-n} have also been determined and recorded, and from these have been determined the values of X'^m_n or $(X^m_n \cos\psi + Z^m_n \sin\psi)$, X'^m_{-n}, Z'^m_n or $(-X^m_n \sin\psi + Z^m_n \cos\psi)$ and Z'^m_{-n}, which are the resolved parts of the expressions for the horizontal and vertical forces in the plane of the meridian on the spheroid, and so are the coefficients of the magnetic constants in the *equations of condition.*

2. In the preceding investigation Section VI. (p. 449) α_n has been taken to represent any magnetic constant depending on the action of magnetic forces in the interior of the Earth, and β_n has been taken to represent any magnetic constant depending on magnetic forces outside the Earth's surface. Thus α_n will represent the Gaussian constant g^m_n in the expression $g^m_n \cos m\lambda$ or the constant h^m_n in the expression $h^m_n \sin m\lambda$. These Gaussian constants g^m_n and h^m_n will have the same coefficients X'^m_n or Y'^m_n or Z'^m_n in the equations of condition for X, Y and Z respectively; and the corresponding external magnetic constants, which may be denoted by g^m_{-n} and h^m_{-n}, will have the same coefficients X'^m_{-n}, Y'^m_{-n} and Z'^m_{-n} in the same *equations of condition.*

As in the equations in Art. 8, p. 454, these equations will be of the form

$$X'^m_n g^m_n + X'^m_{-n} g^m_{-n} + X'^m_{n_1} g^m_{n_1} + X'^m_{-n_1} g^m_{-n_1} + \&c. = x'_m,$$
$$Y'^m_n g^m_n + Y'^m_{-n} g^m_{-n} + Y'^m_{n_1} g^m_{n_1} + Y'^m_{-n_1} g^m_{-n_1} + \&c. = y'_m,$$
$$Z'^m_n g^m_n + Z'^m_{-n} g^m_{-n} + Z'^m_{n_1} g^m_{n_1} + Z'^m_{-n_1} g^m_{-n_1} + \&c. = z'_m,$$

with similar equations for h^m_n, h^m_{-n} &c.

where
$$X'^m_n = X^m_n \cos\psi + Z^m_n \sin\psi,$$
$$Y'^m_n = Y^m_n,$$
$$Z'^m_n = -X^m_n \sin\psi + Z^m_n \cos\psi.$$

In these *equations of condition*, the theoretical expressions for the horizontal and vertical magnetic forces in terms of the magnetic constants will be of the same form for all periods of time. For a given period they are equated to x'_m, y'_m and z'_m, the absolute terms derived from the magnetic observations of the horizontal and vertical magnetic forces for that period, and by solving the equations the values of the magnetic constants for that period are determined. In the first solution, the *absolute terms* are derived from the observations of the magnetic elements given in Sabine's charts for the period about 1845 published in the *Philosophical Transactions* of the Royal Society. In the second solution they are derived from the much more complete and trustworthy observations recorded by Captain Creak in the Admiralty Charts of 1880. By the kind permission of the Lords of the Admiralty, reduced copies of these Charts are given at the end of this Volume.

The observations have been taken for every 10° of longitude and for belts of latitude 5° in breadth around the surface of the Earth.

The numerical values of the coefficients of the magnetic constants for the spheroidal figure of the Earth have been calculated and recorded in Section VII. for all values of n and m from 0 to 10 both for internal and external forces, and *the equations of condition* will contain two sets, each of 120 magnetic constants: the values of these two sets of constants may be found for any given period by substituting in these equations the values of x'_m, y'_m, z'_m derived from the observed values of the magnetic elements for that period.

Formation of the Absolute Terms of the Equations of Condition.

3. The terms x'_m, y'_m and z'_m as explained above are derived from the observations of the horizontal force in the plane of the meridian, of the horizontal force perpendicular to the meridian and of the vertical force in the equations for X, Y and Z respectively.

The values of the forces being given from the charts for $\lambda = 0$, $\lambda = 10°$, &c. to $\lambda = 350°$, they are analysed for each belt of latitude by a formula of the type

$$a_0 + a_1 \cos \lambda + b_1 \sin \lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda + \&c.$$

The values of the coefficients a_0, a_1, b_1, &c. for X, Y and Z for the different belts of latitude for the epochs 1845 and 1880 were obtained and tabulated.

The coefficients a_m of $\cos m\lambda$ and b_m of $\sin m\lambda$ in this series are the absolute terms of the equations for g_n^m and h_n^m respectively in that belt of latitude.

Let the forces whether X, Y or Z, when $\lambda = 0°$, $\lambda = 10°$, $\lambda = 20°$, &c. to $\lambda = 350°$, be denoted by

$$F_0, \quad F_1, \quad F_2, \quad \text{\&c. to } F_{35} \text{ respectively.}$$

Let $\Sigma(F)$ denote the sum of these quantities, and let $\Sigma_9(F_0)$ denote the sum of every ninth of these quantities beginning with F_0, and similarly let $\Sigma_9(F_1)$ denote the sum of every ninth of these quantities beginning with F_1, and so on.

Thus
$$\Sigma_9(F_0) = F_0 + F_9 + F_{18} + F_{27},$$
$$\Sigma_9(F_1) = F_1 + F_{10} + F_{19} + F_{28}.$$

Then we shall have

$$36a_0 = \Sigma(F);$$

$$18a_1 = F_0 - F_{18} + (F_1 - F_{17} - F_{19} + F_{35})\cos 10° + (F_2 - F_{16} - F_{20} + F_{34})\cos 20°$$
$$+ (F_3 - F_{15} - F_{21} + F_{33})\cos 30° + (F_4 - F_{14} - F_{22} + F_{32})\cos 40° + (F_5 - F_{13} - F_{23} + F_{31})\cos 50°$$
$$+ (F_6 - F_{12} - F_{24} + F_{30})\cos 60° + (F_7 - F_{11} - F_{25} + F_{29})\cos 70° + (F_8 - F_{10} - F_{26} + F_{28})\cos 80°;$$

$$18b_1 = (F_1 + F_{17} - F_{19} - F_{35})\sin 10° + (F_2 + F_{16} - F_{20} - F_{34})\sin 20° + (F_3 + F_{15} - F_{21} - F_{33})\sin 30°$$
$$+ (F_4 + F_{14} - F_{22} - F_{32})\sin 40° + (F_5 + F_{13} - F_{23} - F_{31})\sin 50° + (F_6 + F_{12} - F_{24} - F_{30})\sin 60°$$
$$+ (F_7 + F_{11} - F_{25} - F_{29})\sin 70° + (F_8 + F_{10} - F_{26} - F_{28})\sin 80° + F_9 - F_{27}.$$

Also let $\Sigma_9(\pm F_0) = F_0 - F_9 + F_{18} - F_{27}$, *i.e.* the sum of every ninth beginning with F_0 and changing the sign of the alternate terms, and let

$$\Sigma_9(\pm F_1) = F_1 - F_{10} + F_{19} - F_{28}, \quad \text{\&c.}$$

Then we shall have

$$18a_2 = \Sigma_9(\pm F_0) + [\Sigma_9(\pm F_1) - \Sigma_9(\pm F_8)]\cos 20° + [\Sigma_9(\pm F_2) - \Sigma_9(\pm F_7)]\cos 40°$$
$$+ [\Sigma_9(\pm F_3) - \Sigma_9(\pm F_6)]\cos 60° + [\Sigma_9(\pm F_4) - \Sigma_9(\pm F_5)]\cos 80°;$$

$$18b_2 = [\Sigma_9(\pm F_1) + \Sigma_9(\pm F_8)]\sin 20° + [\Sigma_9(\pm F_2) + \Sigma_9(\pm F_7)]\sin 40°$$
$$+ [\Sigma_9(\pm F_3) + \Sigma_9(\pm F_6)]\sin 60° + [\Sigma_9(\pm F_4) + \Sigma_9(\pm F_5)]\sin 80°;$$

$$18a_3 = \Sigma_6(\pm F_0) + [\Sigma_6(\pm F_1) - \Sigma_6(\pm F_5)]\cos 30° + [\Sigma_6(\pm F_2) - \Sigma_6(\pm F_4)]\cos 60°;$$

$$18b_3=[\Sigma_6(\pm F_1)+\Sigma_6(\pm F_5)]\sin 30^\circ+[\Sigma_6(\pm F_2)+\Sigma_6(\pm F_4)]\sin 60^\circ+\Sigma_6(\pm F_3);$$

$$18a_4=\Sigma_9(F_0)+[\Sigma_9(F_1)+\Sigma_9(F_8)]\cos 40^\circ+[\Sigma_9(F_2)+\Sigma_9(F_7)]\cos 80^\circ$$
$$-[\Sigma_9(F_3)+\Sigma_9(F_6)]\cos 60^\circ-[\Sigma_9(F_4)+\Sigma_9(F_5)]\cos 20^\circ;$$

$$18b_4=[\Sigma_9(F_1)-\Sigma_9(F_8)]\sin 40^\circ+[\Sigma_9(F_2)-\Sigma_9(F_7)]\sin 80^\circ$$
$$+[\Sigma_9(F_3)-\Sigma_9(F_6)]\sin 60^\circ+[\Sigma_9(F_4)-\Sigma_9(F_5)]\sin 20^\circ;$$

$$18a_5=F_0-F_{18}+(F_1-F_{17}-F_{19}+F_{35})\cos 50^\circ-(F_2-F_{16}-F_{20}+F_{34})\cos 80^\circ$$
$$-(F_3-F_{15}-F_{21}+F_{33})\cos 30^\circ-(F_4-F_{14}-F_{22}+F_{32})\cos 20^\circ-(F_5-F_{13}-F_{23}+F_{31})\cos 70^\circ$$
$$+(F_6-F_{12}-F_{24}+F_{30})\cos 60^\circ+(F_7-F_{11}-F_{25}+F_{29})\cos 10^\circ+(F_8-F_{10}-F_{26}+F_{28})\cos 40^\circ;$$

$$18b_5=(F_1+F_{17}-F_{19}-F_{35})\sin 50^\circ+(F_2+F_{16}-F_{20}-F_{34})\sin 80^\circ+(F_3+F_{15}-F_{21}-F_{33})\sin 30^\circ$$
$$-(F_4+F_{14}-F_{22}-F_{32})\sin 20^\circ-(F_5+F_{13}-F_{23}-F_{31})\sin 70^\circ-(F_6+F_{12}-F_{24}-F_{30})\sin 60^\circ$$
$$-(F_7+F_{11}-F_{23}-F_{29})\sin 10^\circ+(F_8+F_{10}-F_{26}-F_{28})\sin 40^\circ+F_9-F_{27};$$

$$18a_6=\Sigma_3(\pm F_0)+[\Sigma_3(\pm F_1)-\Sigma_3(\pm F_2)]\cos 60^\circ;$$

$$18b_6=\qquad [\Sigma_3(\pm F_1)+\Sigma_3(\pm F_2)]\sin 60^\circ;$$

$$18a_7=F_0-F_{18}+(F_1-F_{17}-F_{19}+F_{35})\cos 70^\circ-(F_2-F_{16}-F_{20}+F_{34})\cos 40^\circ$$
$$-(F_3-F_{15}-F_{21}+F_{33})\cos 30^\circ+(F_4-F_{14}-F_{22}+F_{32})\cos 80^\circ+(F_5-F_{13}-F_{23}+F_{31})\cos 10^\circ$$
$$+(F_6-F_{12}-F_{24}+F_{30})\cos 60^\circ-(F_7-F_{11}-F_{25}+F_{29})\cos 50^\circ-(F_8-F_{10}-F_{26}+F_{28})\cos 20^\circ;$$

$$18b_7=(F_1+F_{17}-F_{19}-F_{35})\sin 70^\circ+(F_2+F_{16}-F_{20}-F_{34})\sin 40^\circ-(F_3+F_{15}-F_{21}-F_{33})\sin 30^\circ$$
$$-(F_4+F_{14}-F_{22}-F_{32})\sin 80^\circ-(F_5+F_{13}-F_{23}-F_{31})\sin 10^\circ+(F_6+F_{12}-F_{24}-F_{30})\sin 60^\circ$$
$$+(F_7+F_{11}-F_{25}-F_{29})\sin 50^\circ-(F_8+F_{10}-F_{26}-F_{28})\sin 20^\circ-(F_9-F_{27});$$

$$18a_8=\Sigma_9(F_0)+[\Sigma_9(F_1)+\Sigma_9(F_8)]\cos 80^\circ-[\Sigma_9(F_2)+\Sigma_9(F_7)]\cos 20^\circ-[\Sigma_9(F_3)+\Sigma_9(F_6)]\cos 60^\circ$$
$$+[\Sigma_9(F_4)+\Sigma_9(F_5)]\cos 40^\circ;$$

$$18b_8=[\Sigma_9(F_1)-\Sigma_9(F_8)]\sin 80^\circ+[\Sigma_9(F_2)-\Sigma_9(F_7)]\sin 20^\circ-[\Sigma_9(F_3)-\Sigma_9(F_6)]\sin 60^\circ$$
$$-[\Sigma_9(F_4)-\Sigma_9(F_5)]\sin 40^\circ;$$

$$18a_9=\Sigma_2(\pm F_0)=F_0-F_2+F_4-F_6+F_8-F_{10}+\&c.;$$

$$18b_9=\Sigma_2(\pm F_1)=F_1-F_3+F_5-F_7+F_9-\&c.;$$

$$18a_{10}=\Sigma_9(\pm F_0)-[\Sigma_9(\pm F_1)-\Sigma_9(\pm F_8)]\cos 80^\circ-[\Sigma_9(\pm F_2)-\Sigma_9(\pm F_7)]\cos 20^\circ$$
$$+[\Sigma_9(\pm F_3)-\Sigma_9(\pm F_6)]\cos 60^\circ+[\Sigma_9(\pm F_4)-\Sigma_9(\pm F_5)]\cos 40^\circ;$$

$$18b_{10}=[\Sigma_9(\pm F_1)+\Sigma_9(\pm F_8)]\sin 80^\circ-[\Sigma_9(\pm F_2)+\Sigma_9(\pm F_7)]\sin 20^\circ$$
$$-[\Sigma_9(\pm F_3)+\Sigma_9(\pm F_6)]\sin 60^\circ+[\Sigma_9(\pm F_4)+\Sigma_9(\pm F_5)]\sin 40^\circ;$$

$$18a_{11}=F_0-F_{18}-(F_1-F_{17}-F_{19}+F_{35})\cos 70^\circ-(F_2-F_{16}-F_{20}+F_{34})\cos 40^\circ$$
$$+(F_3-F_{15}-F_{21}+F_{33})\cos 30^\circ+(F_4-F_{14}-F_{22}+F_{32})\cos 80^\circ-(F_5-F_{13}-F_{23}+F_{31})\cos 10^\circ$$
$$+(F_6-F_{12}-F_{24}+F_{30})\cos 60^\circ+(F_7-F_{11}-F_{25}+F_{29})\cos 50^\circ-(F_8-F_{10}-F_{26}+F_{28})\cos 20^\circ;$$

$$18b_{11}=(F_1+F_{17}-F_{19}-F_{35})\sin 70^\circ-(F_2+F_{16}-F_{20}-F_{34})\sin 40^\circ-(F_3+F_{15}-F_{21}-F_{33})\sin 30^\circ$$
$$+(F_4+F_{14}-F_{22}-F_{32})\sin 80^\circ-(F_5+F_{13}-F_{23}-F_{31})\sin 10^\circ-(F_6+F_{12}-F_{24}-F_{30})\sin 60^\circ$$
$$+(F_7+F_{11}-F_{25}-F_{29})\sin 50^\circ+(F_8+F_{10}-F_{26}-F_{28})\sin 20^\circ-(F_9-F_{27});$$

$$18a_{12}=\Sigma_3(F_0)-[\Sigma_3(F_1)+\Sigma_3(F_2)]\cos 60^\circ;$$

$$18b_{12}=[\Sigma_3(F_1)-\Sigma_3(F_2)]\sin 60^\circ.$$

The same groups of terms enter into the expressions for a_1 & b_1, a_5 & b_5, a_7 & b_7, a_{11} & b_{11}, a_{13} & b_{13}, and a_{17} & b_{17}, only multiplied by different coefficients in each case.

Also the same groups of terms with different coefficients enter into the expressions for a_2 & b_2, for a_{10} & b_{10} and for a_{14} & b_{14}; and the same groups enter into the expressions for a_4 & b_4, a_8 & b_8 and a_{16} & b_{16}.

The above equations give the values of a_m and b_m for different values of m for each belt of latitude, and it will not be necessary to go beyond the terms above found.

Formation of the Equations of Condition.

4. Take $\Sigma(X'^m_n g^m_n + X'^m_{-n} g^m_{-n})$ to denote the series of terms

$$X'^m_n g^m_n + X'^m_{-n} g^m_{-n} + X'^m_{n_1} g^m_{n_1} + X'^m_{-n_1} g^m_{-n_1} + \&\text{c.};$$

and employ a similar notation for Y and Z as well as for the h constants; then the equations of condition for each belt of latitude are

$$\Sigma(X'^m_n g^m_n + X'^m_{-n} g^m_{-n}) = a_m, \text{ and } \Sigma(X'^m_n h^m_n + X'^m_{-n} h^m_{-n}) = b_m,$$
$$\Sigma(Y'^m_n g^m_n + Y'^m_{-n} g^m_{-n}) = b_m, \text{ and } \Sigma(Y'^m_n h^m_n + Y'^m_{-n} h^m_{-n}) = -a_m,$$
$$\Sigma(Z'^m_n g^m_n + Z'^m_{-n} g^m_{-n}) = a_m, \text{ and } \Sigma(Z'^m_n h^m_n + Z'^m_{-n} h^m_{-n}) = b_m.$$

The values of the coefficients X'^m_n, X'^m_{-n}, Y'^m_n &c. in these equations are the values derived for the spheroidal surface of the Earth from the formulae given in Section VI. (p. 452); their logarithms are recorded in tables in Section VII. (see pp. 482—519).

When $n-m$ is even, the value of X'^m_n contains only odd powers of μ, and the values of Y'^m_n and Z'^m_n contain only even powers of μ.

Similarly, when $n-m$ is odd, the value of X'^m_n contains only even powers of μ, and the values of Y'^m_n and Z'^m_n contain only odd powers of μ.

Hence if the coefficient of $\cos m\lambda$ in either of the quantities X, Y or Z be denoted by a_m and the coefficient of $\sin m\lambda$ by b_m for a given north latitude, and if a'_m, b'_m denote the similar quantities for the corresponding south latitude, then combining the equations for these two belts together we have, when $n-m$ is even,

$$\Sigma\left(X'^m_n g^m_n + X'^m_{-n} g^m_{-n}\right) = \frac{1}{2}(a_m - a'_m), \text{ and } \Sigma\left(X'^m_n h^m_n + X'^m_{-n} h^m_{-n}\right) = \frac{1}{2}(b_m - b'_m) \quad \ldots(1),$$

$$\Sigma\left(Y'^m_n g^m_n + Y'^m_{-n} g^m_{-n}\right) = \frac{1}{2}(b_m + b'_m), \text{ and } \Sigma\left(Y'^m_n h^m_n + Y'^m_{-n} h^m_{-n}\right) = -\frac{1}{2}(a_m + a'_m),$$

$$\Sigma\left(Z'^m_n g^m_n + Z'^m_{-n} g^m_{-n}\right) = \frac{1}{2}(a_m + a'_m), \text{ and } \Sigma\left(Z'^m_n h^m_n + Z'^m_{-n} h^m_{-n}\right) = \frac{1}{2}(b_m + b'_m);$$

and, when $n-m$ is odd, we have

$$\Sigma\left(X'^m_n g^m_n + X'^m_{-n} g^m_{-n}\right) = \frac{1}{2}(a_m + a'_m), \text{ and } \Sigma\left(X'^m_n h^m_n + X'^m_{-n} h^m_{-n}\right) = \frac{1}{2}(b_m + b'_m),$$

$$\Sigma\left(Y'^m_n g^m_n + Y'^m_{-n} g^m_{-n}\right) = \frac{1}{2}(b_m - b'_m), \text{ and } \Sigma\left(Y'^m_n h^m_n + Y'^m_{-n} h^m_{-n}\right) = -\frac{1}{2}(a_m - a'_m),$$

$$\Sigma\left(Z'^m_n g^m_n + Z'^m_{-n} g^m_{-n}\right) = \frac{1}{2}(a_m - a'_m), \text{ and } \Sigma\left(Z'^m_n h^m_n + Z'^m_{-n} h^m_{-n}\right) = \frac{1}{2}(b_m - b'_m).$$

Thus the equations for the magnetic constants, when $n-m$ is even, are separated from the equations for the constants when $n-m$ is odd, and each equation contains only half the number of unknown magnetic constants to be determined.

Also the equations for the quantities h^m_n will be found from the equations for g^m_n by substituting,

when $n-m$ is even,

$$\frac{1}{2}(b_m-b'_m) \text{ for } \frac{1}{2}(a_m-a'_m) \text{ in the equations for } X,$$

$$-\frac{1}{2}(a_m+a'_m) \text{ for } \frac{1}{2}(b_m+b'_m) \text{ in the equations for } Y,$$

$$\frac{1}{2}(b_m+b'_m) \text{ for } \frac{1}{2}(a_m+a'_m) \text{ in the equations for } Z;$$

and when $n-m$ is odd, by substituting

$$\frac{1}{2}(b_m+b'_m) \text{ for } \frac{1}{2}(a_m+a'_m) \text{ in the equations for } X,$$

$$-\frac{1}{2}(a_m-a'_m) \text{ for } \frac{1}{2}(b_m-b'_m) \text{ in the equations for } Y,$$

and $$\frac{1}{2}(b_m-b'_m) \text{ for } \frac{1}{2}(a_m-a'_m) \text{ in the equations for } Z.$$

Thus we have in the equations for X

$$x'_m=\frac{1}{2}(a_m-a'_m) \text{ or } \frac{1}{2}(a_m+a'_m) \text{ in the equation for } g_n^m,$$

and $$x'_m=\frac{1}{2}(b_m-b'_m) \text{ or } \frac{1}{2}(b_m+b'_m) \text{ in the equation for } h_n^m,$$

according as $n-m$ is even or odd.

Also we have in the equations for Y

$$y'_m=\ \frac{1}{2}(b_m+b'_m) \text{ or } \ \frac{1}{2}(b_m-b'_m) \text{ in the equation for } g_n^m,$$

and $$y'_m=-\frac{1}{2}(a_m+a'_m) \text{ or } -\frac{1}{2}(a_m-a'_m) \text{ in the equation for } h_n^m,$$

according as $n-m$ is even or odd.

Similarly we have in the equations for Z

$$z'_m=\frac{1}{2}(a_m+a'_m) \text{ or } \frac{1}{2}(a_m-a'_m) \text{ in the equation for } g_n^m,$$

and $$z'_m=\frac{1}{2}(b_m+b'_m) \text{ or } \frac{1}{2}(b_m-b'_m) \text{ in the equation for } h_n^m,$$

according as $n-m$ is even or odd.

Thus the values of x'_m, y'_m and z'_m in the equations of condition given in the previous Section (see pp. 520—553) have been determined.

Each system of equations of condition will involve a single value of m combined either with all even values of n or with all odd values of n. There will be one system for the coefficients X'^m_n, X'^m_{-n}, another for the coefficients Y'^m_n, Y'^m_{-n} and a third for the coefficients Z'^m_n, Z'^m_{-n}.

The belts of latitude, 5° in breadth, starting from $87°\frac{1}{2}$ N. latitude, are distinguished by the letters (a), (b), (c) &c., the centres of the belts being 5°, 10°, 15° &c. from the pole respectively, and each belt of north latitude combined with the corresponding belt of south latitude will contribute an equation to each system.

Thus for latitude 60° N. combined with 60° S., the set (f) will furnish, for the g constants, the three following equations to the respective systems X, Y, Z corresponding to $m=4$ and n even:

$$* \quad -[9{\cdot}6479108]g^4_4 - [9{\cdot}6397698]g^4_{-4} - [9{\cdot}3739435]g^4_6 - [9{\cdot}3627519]g^4_{-6} - \&c. = \frac{1}{2}(a_4 - a'_4) \text{ for } X,$$

$$[9{\cdot}7120302]g^4_4 + [9{\cdot}7022392]g^4_{-4} + [9{\cdot}5314878]g^4_6 + [9{\cdot}5173452]g^4_{-6} + \&c. = \frac{1}{2}(b_4 + b'_4) \text{ for } Y,$$

$$[9{\cdot}5118188]g^4_4 - [9{\cdot}4012092]g^4_{-4} + [9{\cdot}4766723]g^4_6 - [9{\cdot}3934121]g^4_{-6} + \&c. = \frac{1}{2}(a_4 + a'_4) \text{ for } Z,$$

and the three following equations to the similar systems corresponding to odd values of n:

$$-[9{\cdot}5471920]g^4_5 - [9{\cdot}5374280]g^4_{-5} - [9{\cdot}1267947]g^4_7 - [9{\cdot}1145742]g^4_{-7} - \&c. = \frac{1}{2}(a_4 + a'_4) \text{ for } X,$$

$$[9{\cdot}6499180]g^4_5 + [9{\cdot}6379512]g^4_{-5} + [9{\cdot}3653414]g^4_7 + [9{\cdot}3490233]g^4_{-7} + \&c. = \frac{1}{2}(b_4 - b'_4) \text{ for } Y,$$

$$[9{\cdot}5284778]g^4_5 - [9{\cdot}4344144]g^4_{-5} + [9{\cdot}3682450]g^4_7 - [9{\cdot}2923736]g^4_{-7} + \&c. = \frac{1}{2}(a_4 - a'_4) \text{ for } Z.$$

* where [9·6479108] is employed to express the number of which 9·6479108 is the logarithm.

Formation of the Final Equations.

5. Each equation of condition, as above found, will give rise to as many final equations as there are magnetic constants in the equation to be determined. The equations of condition are multiplied by the weights w_a, w_b &c. of the observations for their respective belts of latitude. In the following solutions of the equations the weight of the observations for any belt of latitude has been taken to be proportional to the area of that belt. The weight of each equation from the equatorial belt (s) is taken as $\frac{1}{2}w_s$, since this belt extends only $2^\circ\frac{1}{2}$ on each side of the equator.

Then the *final* equation for each magnetic constant g_n^m is formed by multiplying each equation so formed by the coefficient of g_n^m in the corresponding equation of condition, and adding together the resulting coefficients of g_n^m from the different belts of latitude (a), (b), (c) &c.

Thus the type of the final equation for g_n^m is

$$\Sigma\left[(X'^m_n)^2 w\right] g_n^m + \Sigma\left[X'^m_n X'^m_{-n} w\right] g^m_{-n} + \Sigma\left[X'^m_n X'^m_{n_1} w\right] g^m_{n_1} + \&c. = \Sigma\left[X'^m_n w x'_m\right],$$

with similar equations for Y and Z. The absolute term $\Sigma\left[X'^m_n w x'_m\right]$ is different, according as $n-m$ is even or odd, the value of x'_m being arrived at as indicated in the last article.

For the convenience of the ready calculation of the coefficients in the final equations, a series of numerical equations has been formed from the *equations of condition* by multiplying the equation of condition for each latitude by the square root of the weight of the observations in that latitude. These equations so formed are given in the previous Section (see pp. 554—587) and are of the types

$$\Sigma\left(X'^m_n w^{\frac{1}{2}} g_n^m + X'^m_{-n} w^{\frac{1}{2}} g^m_{-n}\right) = w^{\frac{1}{2}} x'_m, \text{ and } \Sigma\left(X'^m_n w^{\frac{1}{2}} h_n^m + X'^m_{-n} w^{\frac{1}{2}} h^m_{-n}\right) = w^{\frac{1}{2}} x'_m \ldots (2),$$

with similar equations for Y and Z.

6. From the above equations (2) the final equations for any magnetic constant g_n^m or h_n^m for a given latitude are formed by multiplying by $(X'^m_n w^{\frac{1}{2}})$, $(Y'^m_n w^{\frac{1}{2}})$, and $(Z'^m_n w^{\frac{1}{2}})$, i.e., by the coefficient of g_n^m or h_n^m in the above equations for X, Y and Z respectively.

This will give for g_n^m an equation of the type

$$(X'^m_n)^2 wg_n^m + X'^m_n X'^m_{-n} wg^m_{-n} + X'^m_n X'^m_{n_1} wg^m_{n_1} + \&c. = X'^m_n wx'_m \;\ldots\ldots\ldots(3),$$

with similar equations for Y and Z.

Then integrating or adding together the equations in X or Y or Z for the different latitudes we get the final equations for g_n^m of the types

$$\Sigma\left[(X'^m_n)^2 w\right] g_n^m + \Sigma\left[X'^m_n X'^m_{-n} w\right] g^m_{-n} + \Sigma\left[X'^m_n X'^m_{n_1} w\right] g^m_{n_1} + \&c. = \Sigma\left[X'^m_n wx'_m\right] \ldots(4),$$

$$\Sigma\left[(Y'^m_n)^2 w\right] g_n^m + \Sigma\left[Y'^m_n Y'^m_{-n} w\right] g^m_{-n} + \Sigma\left[Y'^m_n Y'^m_{n_1} w\right] g^m_{n_1} + \&c. = \Sigma\left[Y'^m_n wy'_m\right],$$

$$\Sigma\left[(Z'^m_n)^2 w\right] g_n^m + \Sigma\left[Z'^m_n Z'^m_{-n} w\right] g^m_{-n} + \Sigma\left[Z'^m_n Z'^m_{n_1} w\right] g^m_{n_1} + \&c. = \Sigma\left[Z'^m_n wz'_m\right].$$

The changes in the values of x'_m, y'_m and z'_m according as $(n-m)$ is even or odd have been above explained, and their values in the equations for determining h_n^m instead of g_n^m have also been given (see p. 594).

We shall have a separate final equation for each value of n; thus the final equation for $g^m_{n_1}$ for a given latitude from the equations for X is

$$X'^m_n X'^m_{n_1} wg_n^m + X'^m_{-n} X'^m_{n_1} wg^m_{-n} + (X'^m_{n_1})^2 wg^m_{n_1} + \&c. = X'^m_{n_1} wx'_m,$$

where the coefficient of g_n^m is the same as the coefficient of $g^m_{n_1}$, in equation (3).

Then adding up, for the constant $g^m_{n_1}$, the coefficients in the final equations for all the different belts of latitude we have the final equation from the series (X), which may be represented by the form

$$\Sigma\left[X'^m_n X'^m_{n_1} w\right] g_n^m + \Sigma\left[X'^m_{-n} X'^m_{n_1} w\right] g^m_{-n} + \Sigma\left[(X'^m_{n_1})^2 w\right] g^m_{n_1} + \&c. = \Sigma\left[X'^m_{n_1} wx'_m\right],$$

where x'_m stands for $\frac{1}{2}(\alpha_m - \alpha'_m)$ when $n_1 - m$ is even and for $\frac{1}{2}(\alpha_m + \alpha'_m)$ when $n_1 - m$ is odd.

Equations similar to the above will be derived from the series (Y) and from the series (Z).

These equations may be solved separately, and the values of the magnetic constants determined from each series; taking series (X), series (Y) and series (Z) separately.

7. For another and more satisfactory determination of the magnetic constants the series (X) and the series (Y) may also be conveniently combined into one equation in the same way as the above equations for

different latitudes in X have been combined, in which case the coefficient of $g^m_{n_1}$ in the final equation for g^m_n will be

$$\Sigma\,[X'^m_n\,X'^m_{n_1}w]+\Sigma\,[Y'^m_n\,Y'^m_{n_1}w].$$

And the coefficient of g^m_n in the final equation for g^m_n will be

$$\Sigma\,(X'^m_n)^2w+\Sigma\,(Y'^m_n)^2w.$$

We have seen above that in the case of *a sphere* the coefficients of each of the magnetic constants in this final equation except the coefficient of g^m_n will vanish; but this will only be the case when the summation is taken all over the Earth's surface. The corresponding coefficients on the spheroid will be small quantities depending on the value of the square of the eccentricity.

The right-hand side of the equation (4) becomes under these conditions

$$\Sigma\,[X'^m_n\,x'_m w]+\Sigma\,[Y'^m_n\,y'_m w].$$

If for a first approximation small terms be neglected, the value of g^m_n will be given by the equation

$$\{\Sigma\,[(X'^m_n)^2\,w]+\Sigma\,[(Y'^m_n)^2\,w]\}\,g^m_n=\Sigma\,[X'^m_n\,x'_m\,w]+\Sigma\,[Y'^m_n\,y'_m\,w].$$

When the belts of latitude which are employed in giving the equations extend over the whole of the Earth's surface, and when the successive belts are sufficiently narrow, the coefficient of $(g^m_n+g^m_{-n})$ in the final equation for g^m_n is approximately

$$n\,(n+1)\int_{-1}^{1}(H'^m_n)^2d\mu \quad\text{or}\quad \frac{n\,(n+1)}{2n+1}\times 2\,\frac{(n-m)!\,(n+m)!}{[1.3.5\,\ldots\ldots\,(2n-1)]^2},$$

and, as before (see p. 439 above), the right-hand side of the equation becomes

$$\int_{-1}^{1}X'^m_n\,x'_m\,d\mu+\int_{-1}^{1}Y'^m_n\,y'_m\,d\mu.$$

It will be seen that $\Sigma\,[X'^m_n X'^m_{n_1}w]$, which is the coefficient of $g^m_{n_1}$ in the final equation for g^m_n [equation (4)], is also the coefficient of g^m_n in the final equation for $g^m_{n_1}$.

A similar interchange of coefficients will also hold good in the equations for Y and in the equations for Z.

This interchange of coefficients will also hold good when the series of equations for (X) and for (Y) are combined.

The final equations for some of the more important magnetic constants are given at the end of this volume. We propose to solve the final equations derived from the series for X, Y and Z combined, taking into account the data obtained from magnetic observations over the portion of the surface of the Earth between latitudes $67°\frac{1}{2}$ N. and $67°\frac{1}{2}$ S., taking only the equations of condition for belts between these latitudes, and taking only those terms in these equations for values of m from 0 to 6 and for values of n from 1 to 6 inclusive. These equations will give values for 48 constants, and no equation will contain more than three unknown quantities.

The Table on p. 605 gives the values of these 48 constants thus obtained from observations of the magnetic elements for 1845 and for 1880.

Solution of the Equations.

8. The coefficients on the left-hand side of the equations of condition (and therefore also on the left-hand side of the final equations) will be the same for g_n^m and for h_n^m, but the right-hand members of the equations or the *absolute* terms will be different. Taking a_n^m to stand for either g_n^m or h_n^m, the equations for solution may be conveniently arranged as follows:

From the equations for (X) taken separately

$$\Sigma\,[X'^m_n X'^m_{n_1} w]\, a_n^m + \Sigma\,[(X'^m_{n_1})^2 w]\, a_{n_1}^m = \begin{pmatrix}\text{absolute term}\\ \text{for } g_{n_1}^m\end{pmatrix} \text{ or } \begin{pmatrix}\text{absolute term}\\ \text{for } h_{n_1}^m\end{pmatrix}.$$

From the series of equations for (X) combined with those for (Y) we may also solve the equations, of which the type will be as follows:

$$\{\Sigma\,[X'^m_n X'^m_{n_1} w] + \Sigma\,[Y'^m_n Y'^m_{n_1} w]\}\, a_n^m$$
$$+\,\Sigma\,[(X'^m_{n_1})^2 w] + \Sigma\,[(Y'^m_{n_1})^2 w]\, a_{n_1}^m + \&c. = \begin{pmatrix}\text{absolute term}\\ \text{for } g_{n_1}^m\end{pmatrix} \text{ or } \begin{pmatrix}\text{absolute term}\\ \text{for } h_{n_1}^m\end{pmatrix},$$

the absolute terms being derived in this case from the series for (X) and for (Y) combined.

The values of the same constant derived from these equations taken separately or taken together will differ somewhat from one another.

The final equation for any magnetic constant derived from the series for X, Y and Z combined will be formed by adding together the coefficients of the same magnetic constants in the three final equations for X, for Y and for Z, each taken separately. As there is a separate final equation for each magnetic constant, there will be as many combined final equations as there are magnetic constants in them to be determined.

Let us illustrate the mode of solving these final equations by taking the case given above (see p. 596) in which $m=4$ and n is odd, taking the equations up to latitude $77°\tfrac{1}{2}$ inclusive, and combining the equations for X, Y and Z, supposing the magnetic constants corresponding to negative values of n to be non-existent. We will include the terms involving $n=7$.

The coefficients for g_5^4 and h_5^4 being the same, the final equations for g_5^4 and h_5^4 for the period 1845 may be written thus:

$$\begin{array}{lrl} & & \qquad\qquad\text{(for } g\text{)} \qquad \text{(for } h\text{)} \\ \text{from } (X) & 3{\cdot}4034960\,\alpha_5^4 - {\cdot}3898572\,\alpha_7^4 = & {\cdot}2416593 \text{ or } -{\cdot}0159063, \\ \text{,,} \quad (Y) & 9{\cdot}4158541\,\alpha_5^4 + {\cdot}4092903\,\alpha_7^4 = & {\cdot}0589245 \text{ or } {\cdot}3418323, \\ \text{,,} \quad (Z) & 15{\cdot}3871472\,\alpha_5^4 + {\cdot}0223528\,\alpha_7^4 = & {\cdot}4657356 \text{ or } {\cdot}1824818. \end{array}$$

Adding these together we have

$$28{\cdot}2064973\,\alpha_5^4 + {\cdot}0417859\,\alpha_7^4 = {\cdot}7663194 \text{ or } {\cdot}5084078 \ldots\ldots (1).$$

Similarly the final equations for g_7^4 and h_7^4 may be written thus:

$$\begin{array}{lrl} \text{from } (X) & -{\cdot}3898572\,\alpha_5^4 + {\cdot}2637326\,\alpha_7^4 = & {\cdot}0204205 \text{ or } {\cdot}0140404, \\ \text{,,} \quad (Y) & {\cdot}4092903\,\alpha_5^4 + {\cdot}3081774\,\alpha_7^4 = & {\cdot}0454171 \text{ or } {\cdot}0373065, \\ \text{,,} \quad (Z) & {\cdot}0223528\,\alpha_5^4 + {\cdot}6536612\,\alpha_7^4 = & {\cdot}0056358 \text{ or } {\cdot}0882180. \end{array}$$

Adding these together we have

$${\cdot}0417859\,\alpha_5^4 + 1{\cdot}2255712\,\alpha_7^4 = {\cdot}0714734 \text{ or } {\cdot}1395649 \ldots\ldots\ldots (2).$$

Eliminating α_5^4 from the equations (1) and (2) we get

$$1{\cdot}2255093\,\alpha_7^4 = {\cdot}0703382 \text{ or } {\cdot}1388117.$$

Hence $$g_7^4 = {\cdot}0573951 \text{ and } h_7^4 = {\cdot}1132686.$$

Substituting in the first equation, we get

$$g_5^4 = {\cdot}0270832 \text{ and } h_5^4 = {\cdot}0178567.$$

Thus the values of g_7^4 and h_7^4 are more important than the values of g_5^4 and h_5^4.

Similarly in solving the equations with $m=4$ and n even, it is found that

$$g_4^4 = \cdot 0029684, \quad h_4^4 = \cdot 0217744, \quad g_6^4 = \cdot 0642604 \text{ and } h_6^4 = \cdot 0603230;$$

so that g_6^4 and h_6^4 are more important than g_4^4 and h_4^4.

9. The relative importance of magnetic constants of different orders is well shewn by the solution of the final equations for h_3^2, h_5^2 and h_7^2 for the period 1880.

Keeping in the terms containing h_7^2, the final equations derived by combining the equations for X, Y and Z are

$$24\cdot 1400624\, h_3^2 - \cdot 2579706\, h_5^2 - \cdot 1213933\, h_7^2 = \cdot 19111,$$

$$-\cdot 2579706\, h_3^2 + 2\cdot 1784697\, h_5^2 - \cdot 0719819\, h_7^2 = \cdot 13841,$$

$$-\cdot 1213933\, h_3^2 - \cdot 0719819\, h_5^2 + \cdot 1887180\, h_7^2 = -\cdot 02852.$$

The solution of these equations gives the values

$$h_3^2 = \cdot 00794, \quad h_5^2 = \cdot 06041, \quad h_7^2 = -\cdot 12298 \text{ (British units)}.$$

Converting these into C.G.S. units, we get

$$h_3^2 = \cdot 000366, \quad h_5^2 = \cdot 0027855, \quad h_7^2 = -\cdot 00567.$$

Comparison with the tables of values of magnetic constants given below (p. 605) shews that the effect of keeping in the constant h_7^2 is to make a considerable change in the values of the constants h_3^2 and h_5^2.

The corresponding equations for g_3^2, g_5^2 and g_7^2 are

$$24\cdot 1400624\, g_3^2 - \cdot 2579706\, g_5^2 - \cdot 1213933\, g_7^2 = -14\cdot 62295,$$

$$-\cdot 2579706\, g_3^2 + 2\cdot 1784697\, g_5^2 - \cdot 0719819\, g_7^2 = -1\cdot 11044,$$

$$-\cdot 1213933\, g_3^2 - \cdot 0719819\, g_5^2 + \cdot 1887180\, g_7^2 = \cdot 05785,$$

and the solution of these equations gives the values

$$g_3^2 = -\cdot 613670, \quad g_5^2 = -\cdot 592789, \quad g_7^2 = -\cdot 314308 \text{ (British units)},$$

or $$g_3^2 = -\cdot 028295, \quad g_5^2 = -\cdot 027332, \quad g_7^2 = -\cdot 014492 \text{ (C.G.S. units)}.$$

These values of g_3^2 and g_5^2 do not differ much from the values obtained and recorded in the Table (see p. 605), when g_7^2 is neglected.

10. Let us further illustrate the mode of solving these final equations by taking the case of $m=0$ and n odd from the equations for X, and also from the equations for Z, taken separately, for the period 1845.

We will form the equations of condition taking into account the data only up to $67°\frac{1}{2}$ N. and S. latitudes. The formation of the final equations for g_1^0, g_3^0 and g_5^0 will then be as follows:

From equations for (X),

$$\begin{aligned} 7{\cdot}6331952\,g_1^0 - {\cdot}1138565\,g_3^0 - {\cdot}0886747\,g_5^0 &= 53{\cdot}575026, \\ -{\cdot}1138565\,g_1^0 + 2{\cdot}8880836\,g_3^0 - {\cdot}1765112\,g_5^0 &= -2{\cdot}456863, \\ -{\cdot}0886747\,g_1^0 - {\cdot}1765112\,g_3^0 + {\cdot}3955108\,g_5^0 &= -{\cdot}4538875. \end{aligned}$$

And we have from equations for (Z),

$$\begin{aligned} 12{\cdot}0636234\,g_1^0 - 2{\cdot}1413469\,g_3^0 - {\cdot}7000106\,g_5^0 &= 85{\cdot}065860, \\ -2{\cdot}1413469\,g_1^0 + 2{\cdot}7856531\,g_3^0 - {\cdot}4744250\,g_5^0 &= -16{\cdot}292662, \\ -{\cdot}7000106\,g_1^0 - {\cdot}4744250\,g_3^0 + {\cdot}4394974\,g_5^0 &= -4{\cdot}6678164. \end{aligned}$$

Solving the equations for (X), we get

$$g_1^0 = 7{\cdot}01229, \qquad g_3^0 = -{\cdot}56367, \qquad g_5^0 = {\cdot}17302.$$

These values agree almost exactly with those found from the whole of the equations for (X) up to latitude $77°\frac{1}{2}$.

Solving the equations for (Z), we get

$$g_1^0 = 6{\cdot}951666, \qquad g_3^0 = -{\cdot}524544, \text{ and } g_5^0 = -{\cdot}11476.$$

These values agree very closely with those found from the whole of the equations for (Z) up to latitude $77°\frac{1}{2}$.

The values of g_1^0 and g_3^0 derived from the equations for (Z) agree fairly well with those found from the equations for (X), but the values of g_5^0 have opposite signs. Probably the neglected term in g_7^0 may have some influence on this result.

Taking the magnetic constants depending upon external forces into account, let us find *approximately* what values of g_{-1}^0, g_{-3}^0, g_{-5}^0 will bring the two sets of results into harmony. This may be done by substituting $g_n^0 + g_{-n}^0$ for g_n^0 in the equations for (X), and $g_n^0 - \frac{n}{n+1} g_{-n}^0$ for g_n^0 in the

equations for (Z). Hence we get in the cases treated above

$$g_1^0 = 6{\cdot}971874, \quad g_{-1}^0 = {\cdot}040416, \quad g_3^0 = -{\cdot}541312, \quad g_{-3}^0 = -{\cdot}022358,$$
$$g_5^0 = {\cdot}01605, \text{ and } g_{-5}^0 = {\cdot}15697.$$

Hence the constant g_{-5}^0 seems to be of great importance.

The values found for the two first of the above constants are in British units

	by Gauss (1830)	by Erman (1829)
	$g_1^0 = 7{\cdot}0155$	$g_1^0 = 6{\cdot}9417$
	$g_3^0 = -{\cdot}1430$	$g_3^0 = -{\cdot}4069.$

The values of these constants, derived from the above series of equations for (X), (Y) and (Z) combined, for all latitudes from 67°½ S. to 67°½ N., are

$$g_1^0 = 6{\cdot}98081 \text{ and } g_3^0 = -{\cdot}523986$$

for the period 1845 from Sabine's charts.

The values derived for the above constants from the above equations of condition, taking m from 0 to 4 and n from 1 to 4 only, and neglecting the other terms, i.e., taking those constants only which were determined by Gauss, are

$$g_1^0 = 6{\cdot}9777 \text{ and } g_3^0 = -{\cdot}5310$$

for the same period.

11. The values of the constants given in the two following tables are derived from the combined equations for (X), (Y) and (Z) to equations (e) inclusive (i.e. between latitudes 67°½ S. and 67°½ N.), supposing the constants corresponding to negative values of n to be non-existent. The equations for the belts outside these latitudes have not been included because of the scarcity of observations of the magnetic elements.

The second of these tables gives the values of the constants when we include in the equations only those 24 constants which are taken into account by Gauss himself. This table also includes the values (in British units) of these constants as determined by Gauss and by Erman.

*The values of the Magnetic Constants are derived from

(1) Sabine's Charts for 1845 in the *Philosophical Transactions* of the Royal Society.

(2) The Admiralty Charts for 1880.

* (The values given in the *British Association Reports* for 1898 (p. 128) for h_2', h_4' and h_6' for the period 1845 should be replaced by the values here given to those constants.)

TABLE of the Values of the Magnetic Constants.

	1845		1880	
	British	C. G. S.	British	C. G. S.
g_1^0	6·98081	·321871	6·87176	·316843
g_2^0	− ·0275845	− ·00127187	·158464	·0073065
g_3^0	− ·523986	− ·0241595	− ·58113	− ·026795
g_4^0	− ·67352	− ·0310546	− ·73195	− ·033749
g_5^0	·0513465	·00236748	·27987	·012904
g_6^0	− ·30013	− ·0138385	− ·07904	− ·0036446
g_1^1	·602567	·0277832	·52644	·024273
g_2^1	−1·065495	− ·0491279	−1·11386	− ·051358
g_3^1	·678817	·0312989	·91030	·041972
g_4^1	− ·712584	− ·0328558	− ·79880	− ·036831
g_5^1	− ·784390	− ·0361666	− ·61614	− ·028409
g_6^1	− ·272348	− ·0125575	− ·59068	− ·027235
g_2^2	− ·007649	− ·0003527	− ·11506	− ·0053054
g_3^2	− ·607671	− ·0280185	− ·61198	− ·0282172
g_4^2	− ·331346	− ·0152777	− ·41928	− ·019332
g_5^2	− ·661354	− ·0304937	− ·58220	− ·026844
g_6^2	·300535	·012932	·15864	·0073145
g_3^3	− ·044994	− ·0020746	− ·06274	− ·0028928
g_4^3	·076635	·0035335	·12123	·0055896
g_5^3	− ·023517	− ·0010843	− ·011915	− ·0005494
g_6^3	·241583	·011139	·37369	·0172301
g_4^4	·002980	·0001374	− ·02346	− ·0010818
g_5^4	·02713	·0012508	·00433	·0001996
g_6^4	·064652	·0029810	·07682	·0035421
g_5^5	− ·01512	− ·0006970	− ·01435	− ·0006615
g_6^5	− ·009531	− ·0004394	− ·02154	− ·0009933
g_6^6	·003132	·0001444	− ·00047	− ·0000218
h_1^1	−1·254179	− ·0578277	−1·30780	− ·0602998
h_2^1	·039173	·0018062	·28051	·0129335
h_3^1	·297611	·0137222	·16224	·0074808
h_4^1	− ·119440	− ·0055071	− ·23026	− ·010617
h_5^1	·5291705	·0243990	·58114	·026795
h_6^1	− ·139605	− ·0064369	− ·06162	− ·002841
h_2^2	− ·254829	− ·0116484	− ·27960	− ·0128917
h_3^2	− ·088692	− ·0040894	·00861	·0003968
h_4^2	·214592	·0098944	·11316	·0052175
h_5^2	− ·025210	− ·0011624	·06455	·0029737
h_6^2	− ·069335	− ·0031969	− ·17877	− ·0082428
h_3^3	− ·146981	− ·0067770	− ·10697	− ·0049321
h_4^3	·084794	·0039097	·09392	·0043306
h_5^3	·009588	·0004421	·02851	·0013145
h_6^3	·123986	·0057167	·11457	·0052826
h_4^4	·021780	·00100425	·02029	·0009355
h_5^4	·01799	·0008295	·02489	·0011477
h_6^4	·060462	·0027878	·03984	·0018370
h_5^5	− ·00864	− ·0003984	− ·00414	− ·0001908
h_6^5	− ·049244	− ·0022705	− ·02920	− ·0013465
h_6^6	− ·005664	− ·0002612	·00390	·0001799

Comparison of the Values of the Magnetic Constants in British Units, as determined—(1) by Gauss, (2) by Erman, (3) by Adams for 1845, (4) by Adams for 1880, with their yearly rate of increase from 1845 to 1880.

Constants	Gauss	Erman	Adams, 1845	Adams, 1880	Yearly rate of change
g_1^0	7·0155	6·9417	6·9777	6·8558	– ·00325
g_2^0	– ·1672	+ ·0262	– ·0124	+ ·1624	·00466
g_3^0	– ·1430	– ·4069	– ·5310	– ·6194	– ·00236
g_4^0	– ·8249	– ·5937	– ·6309	– ·7207	– ·00239
g_1^1	·6746	·6149	·6145	·5358	– ·00210
h_1^1	–1·3545	–1·3036	–1·2622	–1·3166	– ·00145
g_2^1	–1·0981	– ·9659	–1·0598	–1·1014	– ·00111
h_2^1	– ·0457	+ ·0156	+ ·0421	+ ·2818	·00639
g_3^1	·9316	·6477	·7300	·9505	·00588
h_3^1	·3622	·3567	·2631	·1243	– ·00370
g_4^1	–1·1563	– ·8330	– ·6904	– ·7507	– ·00161
h_4^1	+ ·4858	– ·0693	– ·1081	– ·2252	– ·00312
g_2^2	+ ·0037	+ ·0271	– ·0083	– ·1154	– ·00286
h_2^2	– ·2956	– ·2741	– ·2547	– ·2792	– ·00065
g_3^2	– ·5546	– ·6664	– ·6006	– ·6057	– ·00014
h_3^2	– ·1725	– ·1347	– ·0884	+ ·0079	·00257
g_4^2	– ·3470	– ·3382	– ·3376	– ·4226	– ·00227
h_4^2	·3226	·2353	·2160	·1169	– ·00264
g_3^3	+ ·0106	– ·0276	– ·0450	– ·0627	– ·00047
h_3^3	– ·1421	– ·1572	– ·1470	– ·1070	·00107
g_4^3	·1498	·1455	·0764	·1209	·00119
h_4^3	– ·0013	+ ·0654	+ ·0847	+ ·0938	·00024
g_4^4	+ ·0313	+ ·0194	+ ·0030	– ·0234	– ·00070
h_4^4	·0241	·0240	·0218	·0203	– ·00004

The multiplier for the conversion from British units into C.G.S. units is 0·046108.

It will be seen on examining these Tables for the period 1845—

(1) That g_4^0 and g_6^0 are numerically very much larger than g_2^0.

(2) That the values of g_2^0 from the same equations differ greatly according as g_6^0 is or is not included, the value of g_2^0 being $-·0276$ when g_6^0 is included, and $-·0124$ when g_6^0 is excluded.

(3) It also appears from the comparison of the solutions when the equations of condition are included up to $77°\frac{1}{2}$ latitude with the solutions above (i.e. stopping at $67°\frac{1}{2}$ latitude), that $g_2^0 = -·0126$ in the first case, and $-·0276$ in the second case, and that this discrepancy is partly due to the fact that the sum of the absolute terms in the final equation for g_2^0 is $+·08815$ when we stop at latitude $67°\frac{1}{2}$, and $-·07184$ when we proceed to latitude $77°\frac{1}{2}$. Hence a wide variation in the value of g_2^0 is to be expected in the two cases, even when g_6^0 is included in both sets of equations.

(4) It also appears from the above Tables that those constants in the values of which Gauss and Erman greatly differ are those which have undergone the greatest changes in the interval from 1845 to 1880, and that the values for 1845 now determined for the most part agree more nearly with those of Erman than with those of Gauss*.

12. The values of the magnetic constants have been determined from the equations for (X) and (Y) combined, and from the equations for (Z) separately, as well as from the equations for (X), (Y) and (Z) combined, and their values have been compared. Also their values have been determined (1) by including all the equations up to (e), i.e. between latitudes $67°\frac{1}{2}$ N. and $67°\frac{1}{2}$ S., as given in the following table, and (2) by including all the equations up to (c), i.e. between latitudes $77°\frac{1}{2}$ N. and $77°\frac{1}{2}$ S.

Comparative Values of the Magnetic Constants in British Units as deduced from different magnetic elements.

	1845		1880			1845		1880	
	From X and Y	Z	From X and Y	Z		From X and Y	Z	From X and Y	Z
g_1^0	7·012	6·952	6·869	6·877					
g_2^0	− ·009	− ·089	+ ·159	+ ·179					
g_3^0	− ·564	− ·525	− ·578	− ·574					
g_4^0	− ·596	− ·846	− ·800	− ·630					
g_5^0	+ ·173	− ·115	+ ·245	+ ·329					
g_6^0	− ·148	− ·646	− ·056	− ·005					
g_1^1	·597	·605	·497	·536	h_1^1	−1·287	−1·240	−1·273	−1·321
g_2^1	−1·089	−1·052	−1·097	−1·122	h_2^1	·043	·035	·229	·309
g_3^1	·682	·675	·807	·973	h_3^1	·285	·299	·190	·152
g_4^1	− ·704	− ·726	− ·709	− ·873	h_4^1	− ·192	− ·063	− ·284	− ·192
g_5^1	− ·858	− ·722	− ·620	− ·643	h_5^1	·534	·503	·724	·478
g_6^1	− ·260	− ·299	− ·343	− ·842	h_6^1	− ·110	− ·168	− ·366	+ ·223
g_2^2	·000	− ·013	− ·098	− ·126	h_2^2	− ·247	− ·260	− ·265	− ·289
g_3^2	− ·596	− ·617	− ·590	− ·628	h_3^2	− ·095	− ·083	− ·029	− ·035
g_4^2	− ·353	− ·313	− ·413	− ·423	h_4^2	·176	·245	·133	·100
g_5^2	− ·678	− ·647	− ·574	− ·586	h_5^2	+ ·068	− ·098	+ ·037	+ ·078
g_6^2	·416	·206	·156	·165	h_6^2	− ·011	− ·120	− ·172	− ·176
g_3^3	− ·037	− ·051	− ·078	− ·051	h_3^3	− ·150	− ·145	− ·117	− ·099
g_4^3	·093	·064	·126	·117	h_4^3	·077	·091	·092	·096
g_5^3	− ·028	− ·019	− ·032	+ ·005	h_5^3	+ ·029	− ·007	·031	·027
g_6^3	·221	·259	·242	·483	h_6^3	·026	·205	·135	·098
g_4^4	− ·001	+ ·006	− ·018	− ·028	h_4^4	·021	·022	·030	·012
g_5^4	·023	·030	·029	− ·016	h_5^4	·025	·012	·051	·003
g_6^4	·055	·073	·073	·081	h_6^4	·050	·070	·054	·028
g_5^5	− ·013	− ·017	− ·013	− ·016	h_5^5	− ·009	− ·008	− ·006	− ·002
g_6^5	− ·011	− ·009	− ·025	− ·019	h_6^5	− ·041	− ·056	− ·029	− ·029
g_6^6	− ·002	+ ·008	·001	− ·002	h_6^6	− ·006	− ·005	+ ·003	+ ·004

[* It should be remembered that before the excellent Admiralty Charts of 1880, prepared by Captain Creak, the magnetic charts of the world were based on observations which were insufficient and not distributed widely or regularly over the Earth's surface.]

13. In order to test the accuracy of the work in the determination of the magnetic constants we may substitute their values in the theoretical expressions for X, Y and Z and compare the results with the values of X, Y and Z as taken from the charts.

For this purpose we have to form for each parallel of latitude the value of the expression

$$\begin{aligned}\frac{1}{2}(a_m + a'_m) = {} & X_1^0 g_1^0 + X_3^0 g_3^0 + X_5^0 g_5^0 \\ & + (X_2^1 g_2^1 + X_4^1 g_4^1 + X_6^1 g_6^1)\cos\lambda + (X_2^1 h_2^1 + X_4^1 h_4^1 + X_6^1 h_6^1)\sin\lambda \\ & + (X_3^2 g_3^2 + X_5^2 g_5^2)\cos 2\lambda \qquad + (X_3^2 h_3^2 + X_5^2 h_5^2)\sin 2\lambda \\ & + (X_4^3 g_4^3 + X_6^3 g_6^3)\cos 3\lambda \qquad + (X_4^3 h_4^3 + X_6^3 h_6^3)\sin 3\lambda \\ & + X_5^4 g_5^4 \cos 4\lambda + X_6^5 g_6^5 \cos 5\lambda + X_5^4 h_5^4 \sin 4\lambda + X_6^5 h_6^5 \sin 5\lambda,\end{aligned}$$

and also the value of the expression

$$\begin{aligned}\frac{1}{2}(a_m - a'_m) = {} & X_2^0 g_2^0 + X_4^0 g_4^0 + X_6^0 g_6^0 \\ & + (X_1^1 g_1^1 + X_3^1 g_3^1 + X_5^1 g_5^1)\cos\lambda + (X_1^1 h_1^1 + X_3^1 h_3^1 + X_5^1 h_5^1)\sin\lambda \\ & + (X_2^2 g_2^2 + X_4^2 g_4^2 + X_6^2 g_6^2)\cos 2\lambda + (X_2^2 h_2^2 + X_4^2 h_4^2 + X_6^2 h_6^2)\sin 2\lambda \\ & + (X_3^3 g_3^3 + X_5^3 g_5^3)\cos 3\lambda \qquad + (X_3^3 h_3^3 + X_5^3 h_5^3)\sin 3\lambda \\ & + (X_4^4 g_4^4 + X_6^4 g_6^4)\cos 4\lambda \qquad + (X_4^4 h_4^4 + X_6^4 h_6^4)\sin 4\lambda \\ & + X_5^5 g_5^5 \cos 5\lambda + X_6^6 g_6^6 \cos 6\lambda + X_5^5 h_5^5 \sin 5\lambda + X_6^6 h_6^6 \sin 6\lambda,\end{aligned}$$

and then to form the sum and difference of these expressions for the values of X in northern or southern latitudes respectively, which may then be directly compared with the charts.

Similar expressions must be formed in the same way for Y and Z for each parallel of latitude, and their sums and differences taken as in the case of X.

When the values of the magnetic constants had been determined, they were substituted in the equations of condition for each belt of latitude, the terms of which when added up gave the theoretical value of the *absolute term* for that latitude. This calculated value of the absolute term

was then compared with the value of the corresponding absolute term derived from the observations which had been used in the solution of the equations.

The following table gives some of these comparisons between the calculated and observed values of the absolute terms of the equations of condition for the period 1880 for $m=0$, $m=1$ and $m=2$, for odd and even values of n, and for $m=3$, for odd values of n, *i.e.* for all the more important magnetic constants.

The observed values are taken from the Admiralty Charts and are the values used in the solution of the equations, and it will be seen by the comparison of the calculated and observed values that a chart drawn to give the results of the calculations would not differ much from the Admiralty Charts from which the observations have been taken.

In the equations of condition and the final equations which have been used above no account is taken of observations within the area of a portion of the surface of the Earth immediately around the poles, *i.e.* within an area bounded by a small circle of radius $2^\circ 30'$ round the pole when we take all the above equations, or of radius $12^\circ 30'$ when we stop at the belt (c) given by latitude $77^\circ\frac{1}{2}$. This area will be bounded by a circle of radius $22^\circ 30'$ when we stop at equations (e) or latitude $67^\circ\frac{1}{2}$, as in the determinations of the magnetic constants given in the above tables (pp. 605—607).

COMPARISON BETWEEN CALCULATED AND OBSERVED VALUES OF THE ABSOLUTE TERM

	$m=0$				$m=1$							
	n odd		n even		for g, n odd		for h, n odd		for g, n even		for h, n even	
X	calculated	observed	calculated	observed	calculated	observed	calculated	observed	calculated	observed	calculated	observed
(*a*) 85°	0·540		− ·129		− 1·026		·979		1·681		− ·138	
(*b*) 80	1·075		− ·241		− 1·004		1·031		1·504		− ·153	
(*c*) 75	1·600		− ·318		− ·963		1·102		1·240		− ·171	
(*d*) 70	2·114		− ·353		− ·896		1·182		·919		− ·189	
(*e*) 65	2·615	2·610	− ·339	− ·382	− ·794	− ·668	1·249	1·421	·576	·442	− ·202	− ·276
(*f*) 60	3·091	3·099	− ·278	− ·314	− ·657	− ·569	1·288	1·410	·249	·195	− ·204	− ·260
(*g*) 55	3·585	3·578	− ·181	− ·212	− ·485	− ·443	1·289	1·331	− ·038	− ·015	− ·191	− ·243
(*h*) 50	4·058	4·050	− ·058	− ·081	− ·283	− ·288	1·244	1·236	− ·270	− ·200	− ·161	− ·206
(*i*) 45	4·526	4·516	·074	·058	− ·069	− ·114	1·153	1·123	− ·442	− ·356	− ·116	− ·133
(*k*) 40	4·985	4·991	·198	·201	·144	·076	1·025	1·001	− ·562	− ·511	− ·056	− ·039
(*l*) 35	5·432	5·443	·301	·319	·326	·250	·871	·849	− ·642	− ·636	·014	·023
(*m*) 30	5·857	5·867	·370	·387	·466	·390	·705	·686	− ·702	− ·746	·088	·083
(*n*) 25	6·249	6·250	·400	·422	·547	·486	·542	·533	− ·741	− ·833	·163	·152
(*o*) 20	6·598	6·606	·396	·406	·556	·507	·392	·377	− ·775	− ·871	·232	·228
(*p*) 15	6·888	6·895	·332	·358	·495	·452	·265	·244	− ·809	− ·855	·289	·266
(*q*) 10	7 106	7·101	·242	·277	·369	·342	·161	·141	− ·835	− ·847	·333	·276
(*r*) 5	7·241	7·222	·127	·153	·197	·184	·075	·064	− ·854	− ·833	·361	·290
(*s*) 0	7·288	7·255							− ·862	− ·832	·370	·298
Y												
(*a*) 85°					1·031		− ·969		− 1·724		·136	
(*b*) 80					1·029		− ·991		− 1·670		·142	
(*c*) 75					1·022		− 1·024		− 1·586		·149	
(*d*) 70					1·012		− 1·066		− 1·476		·159	
(*e*) 65					·994	·859	− 1·114	− 1·003	− 1·350	− 1·193	·170	·035
(*f*) 60					·968	·872	− 1·164	− 1·059	− 1·213	− 1·103	·179	·088
(*g*) 55					·931	·851	− 1·210	− 1·121	− 1·071	− 1·013	·185	·119
(*h*) 50					·884	·832	− 1·253	− 1·176	− ·935	− ·916	·190	·141
(*i*) 45					·824	·804	− 1·287	− 1·219	− ·806	− ·813	·190	·155
(*k*) 40					·757	·749	− 1·312	− 1·256	− ·688	− ·696	·186	·161
(*l*) 35					·672	·687	− 1·329	− 1·288	− ·579	− ·502	·176	·157
(*m*) 30					·606	·611	− 1·336	− 1·302	− ·480	− ·484	·161	·152
(*n*) 25					·531	·527	− 1·337	− 1·310	− ·389	− ·380	·142	·142
(*o*) 20					·460	·445	− 1·332	− 1·311	− ·306	− ·290	·120	·134
(*p*) 15					·400	·372	− 1·325	− 1·311	− ·227	− ·212	·092	·110
(*q*) 10					·353	·313	− 1·319	− 1·303	− ·150	− ·138	·063	·073
(*r*) 5					·325	·274	− 1·314	− 1·298	− ·076	− ·071	·031	·038
(*s*) 0					·315	·255	− 1·312	− 1·287				
Z												
(*a*) 85°	13·110		− ·538		·197		− ·071		− ·585		·008	
(*b*) 80	12·974		− ·448		·402		− ·161		− 1·117		·022	
(*c*) 75	12·752		− ·307		·694		− ·314		− 1·552		·047	
(*d*) 70	12·448		− ·136		·925		− ·465		− 1·863		·085	
(*e*) 65	12·067	12·099	·038	·091	1·149	1·332	− ·690	− ·780	− 2·038	− 2·271	·139	·299
(*f*) 60	11·613	11·615	·201	·238	1·344	1·444	− ·954	− 1·033	− 2·085	− 2·151	·202	·292
(*g*) 55	11·086	11·073	·328	·340	1·499	1·526	− 1·246	− 1·288	− 2·026	− 2·010	·279	·302
(*h*) 50	10·486	10·463	·406	·394	1·596	1·605	− 1·542	− 1·571	− 1·889	− 1·854	·337	·336
(*i*) 45	9·810	9·804	·428	·369	1·621	1·644	− 1·824	− 1·814	− 1·706	− 1·672	·397	·375
(*k*) 40	9·051	9·040	·393	·340	1·568	1·617	− 2·070	− 2·048	− 1·505	− 1·488	·438	·454
(*l*) 35	8·205	8·213	·311	·253	1·443	1·530	− 2·273	− 2·248	− 1·303	− 1·298	·460	·476
(*m*) 30	7·268	7·294	·185	·171	1·246	1·329	− 2·421	− 2·415	− 1·109	− 1·094	·457	·475
(*n*) 25	6·238	6·253	·039	·048	1·009	1·057	− 2·516	− 2·540	− ·926	− ·930	·428	·453
(*o*) 20	5·117	5·117	− ·113	− ·096	·756	·749	− 2·568	− 2·630	− ·748	− ·763	·376	·407
(*p*) 15	3·918	3·895	− ·252	− ·235	·517	·481	− 2·587	− 2·623	− ·567	− ·555	·303	·340
(*q*) 10	2·652	2·658	− ·364	− ·337	·322	·302	− 2·587	− 2·618	− ·384	− ·377	·212	·253
(*r*) 5	1·338	1·387	− ·435	− ·435	·194	·149	− 2·583	− 2·610	− ·195	− ·173	·109	·174
(*s*) 0			− ·460	− ·465	·149	·098	− 2·580	− 2·590				

IN THE EQUATIONS OF CONDITION FOR 1880, FOR $m=0$, $m=1$, AND $m=2$.

$m=2$						$m=3$				
for g, n even		for h, n even		for g, n odd		for g, n odd		for h, n odd		
calculated	observed	calculated	observed	calculated	observed	calculated	observed	calculated	observed	
·070		·046		·175		·002		·002		(a)
·135		·091		·327		·007		·007		(b)
·190		·128		·438		·015		·017		(c)
·227		·158		·492		·024		·029		(d)
·244	·312	·181	·075	·486	·366	·034	·039	·045	·045	(e)
·233	·288	·207	·181	·425	·335	·046	·064	·062	·046	(f)
·208	·236	·218	·227	·318	·246	·054	·082	·081	·063	(g)
·156	·169	·238	·250	·183	·128	·062	·096	·098	·078	(h)
·085	·097	·257	·274	·035	·013	·068	·092	·113	·097	(i)
·006	·007	·279	·282	− ·105	− ·100	·070	·100	·124	·106	(k)
− ·070	− ·079	·295	·287	− ·226	− ·211	·069	·089	·131	·118	(l)
− ·137	− ·150	·303	·300	− ·318	− ·316	·065	·073	·132	·138	(m)
− ·182	− ·183	·298	·306	− ·379	− ·393	·058	·066	·125	·153	(n)
− ·198	− ·199	·273	·283	− ·412	− ·429	·050	·051	·111	·142	(o)
− ·184	− ·152	·229	·228	− ·425	− ·435	·039	·025	·090	·123	(p)
− ·141	− ·101	·165	·153	− ·424	− ·425	·027	·018	·063	·084	(q)
− ·076	− ·044	·087	·076	− ·420	− ·412	·014	·001	·033	·024	(r)
				− ·418	− ·416					(s)
− ·072		− ·048		− ·178		− ·002		− ·003		(a)
− ·140		− ·094		− ·344		− ·006		− ·008		(b)
− ·204		− ·138		− ·492		− ·015		− ·017		(c)
− ·261		− ·177		− ·612		− ·026		− ·031		(d)
− ·309	− ·259	− ·217	− ·233	− ·699	− ·680	− ·040	− ·094	− ·048	− ·085	(e)
− ·347	− ·322	− ·247	− ·257	− ·751	− ·722	− ·054	− ·112	− ·068	− ·127	(f)
− ·370	− ·358	− ·278	− ·276	− ·769	− ·739	− ·072	− ·107	− ·092	− ·148	(g)
− ·381	− ·367	− ·311	− ·286	− ·755	− ·733	− ·086	− ·116	− ·118	− ·165	(h)
− ·376	− ·361	− ·344	− ·305	− ·713	− ·702	− ·103	− ·129	− ·146	− ·181	(i)
− ·346	− ·330	− ·378	− ·332	− ·650	− ·643	− ·118	− ·142	− ·176	− ·199	(k)
− ·328	− ·289	− ·415	− ·366	− ·573	− ·569	− ·133	− ·155	− ·205	− ·216	(l)
− ·289	− ·251	− ·451	− ·406	− ·487	− ·479	− ·146	− ·169	− ·234	− ·234	(m)
− ·245	− ·211	− ·490	− ·452	− ·398	− ·384	− ·158	− ·183	− ·260	− ·258	(n)
− ·201	− ·165	− ·526	− ·496	− ·311	− ·281	− ·167	− ·198	− ·284	− ·295	(o)
− ·162	− ·117	− ·558	− ·533	− ·228	− ·188	− ·175	− ·213	− ·304	− ·332	(p)
− ·128	− ·077	− ·581	− ·559	− ·148	− ·115	− ·180	− ·228	− ·319	− ·357	(q)
− ·109	− ·038	− ·597	− ·581	− ·073	− ·058	− ·183	− ·240	− ·327	− ·377	(r)
− ·101	− ·024	·603	− ·582			− ·184	− ·245	− ·330	− ·385	(s)
− ·013		− ·008		− ·037		− ·000		− ·000		(a)
− ·050		− ·028		− ·143		− ·001		− ·001		(b)
− ·109		− ·062		− ·303		− ·006		− ·005		(c)
− ·185		− ·100		− ·495		− ·012		− ·012		(d)
− ·272	− ·221	− ·143	− ·084	− ·690	− ·882	− ·024	− ·058	− ·024	− ·111	(e)
− ·312	− ·328	− ·185	− ·173	− ·864	− ·970	− ·038	− ·038	− ·041	− ·101	(f)
− ·445	− ·439	− ·227	− ·261	− ·995	− 1·053	− ·057	− ·039	− ·064	− ·106	(g)
− ·511	− ·555	− ·276	− ·331	− 1·069	− 1·062	− ·077	− ·054	− ·094	− ·112	(h)
− ·549	− ·612	− ·328	− ·393	− 1·080	− 1·035	− ·101	− ·070	− ·131	− ·130	(i)
− ·555	− ·619	− ·390	− ·427	− 1·033	− 1·000	− ·125	− ·066	− ·173	− ·192	(k)
− ·524	− ·569	− ·464	− ·465	− ·936	− ·925	− ·149	− ·108	− ·218	− ·215	(l)
− ·462	− ·459	− ·549	− ·556	− ·807	− ·818	− ·171	− ·143	− ·266	− ·219	(m)
− ·375	− ·359	− ·641	− ·676	− ·659	− ·683	− ·191	− ·152	− ·313	− ·257	(n)
− ·274	− ·278	− ·736	− ·763	− ·508	− ·552	− ·209	− ·172	− ·356	− ·319	(o)
− ·173	− ·219	− ·822	− ·826	− ·364	− ·412	− ·225	− ·169	− ·393	− ·374	(p)
− ·088	− ·144	− ·895	− ·907	− ·232	− ·285	− ·234	− ·185	− ·422	− ·398	(q)
− ·032	− ·072	− ·941	− ·964	− ·112	− ·153	− ·241	− ·219	− ·442	− ·437	(r)
− ·012	− ·030	− ·958	− 1·008			− ·263	− ·244	− ·447	− ·483	(s)

Formation of Equations for the Polar regions.

14. To complete the investigation this polar area should be taken into account by the addition of theoretical terms involving the magnetic constants on the left-hand side of the equations of condition and the final equations, and by the addition on the right-hand side of these equations of quantities derived from observations of the magnetic elements over this polar area. In the absence or the uncertainty with regard to these polar observations we can scarcely do more than prepare the way for the time when our knowledge of these elements shall be more extensive by making the equations as complete as possible.

To find the values of X, Y, Z at the poles for $m=0$ and $m=1$ for the several values of n, we have $r=(1-e^2)^{\frac{1}{2}}$,

also $X_n{}^0=0$, $Y_n{}^0=0$, for all values of n,

$$Z_1{}^0=\frac{2}{r^3},\quad Z_2{}^0=\frac{2}{r^4},\quad Z_3{}^0=\frac{8}{5}\frac{1}{r^5},\quad Z_4{}^0=\frac{8}{7}\frac{1}{r^6},\quad Z_5{}^0=\frac{16}{21}\frac{1}{r^7},$$

$$Z_6{}^0=\frac{16}{33}\frac{1}{r^8},\quad Z_7{}^0=\frac{128}{429}\frac{1}{r^9},\quad Z_8{}^0=\frac{128}{715}\frac{1}{r^{10}},\quad Z_9{}^0=\frac{256}{2431}\frac{1}{r^{11}},\quad Z_{10}{}^0=\frac{256}{4199}\frac{1}{r^{12}}.$$

Next let $m=1$,

then $$Y_n{}^1=-X_n{}^1=\frac{1}{2}Z_n{}^0,\ \text{for all values of } n.$$

For X these coefficients are to be multiplied by $-g_n{}^1\cos\lambda-h_n{}^1\sin\lambda$, and for Y by $g_n{}^1\sin\lambda-h_n{}^1\cos\lambda$. Also $Z_n{}^1=0$, for all values of n.

To find the logarithms of the coefficients of g and h for the pole,

$$\log(1-e^2)=9{\cdot}9970916,\quad \log r=9{\cdot}9985458,\quad \log\frac{1}{r}={\cdot}0014542.$$

Taking [·0014542] to represent the number of which ·0014542 is the logarithm, we have $Y_1^1=[\cdot 0043626]$, $Y_2^1=[\cdot 0058168]$, &c. as in the following equations, and $Z_1^0=2Y_1^1=[\cdot 3053926]$, &c.

Also

$$Z_{-1}^0=-1,\qquad Z_{-2}^0=-\frac{4}{3}r,\qquad Z_{-3}^0=-\frac{6}{5}r^2,\qquad Z_{-4}^0=-\frac{32}{35}r^3,$$

$$Z_{-5}^0=-\frac{40}{63}r^4,\qquad Z_{-6}^0=-\frac{32}{77}r^5,\qquad Z_{-7}^0=-\frac{112}{429}r^6,\qquad Z_{-8}^0=-\frac{1024}{6435}r^7,$$

$$Z_{-9}^0=-\frac{1152}{12155}r^8,\qquad Z_{-10}^0=-\frac{2560}{46189}r^9,$$

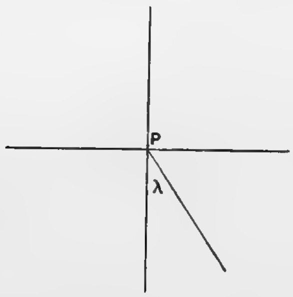
and $Y_{-n}^1=-X_{-n}^1=Y_n^1\times r^{2n+1}$. Therefore $Y_{-1}^1=1=-Z_{-1}^0$.

Let the magnetic forces at the North pole, considered as a point in the zero meridian, be $X=a$, $Y=b$, $Z=c$, where a is positive when directed towards *the north in that meridian*, b is positive when in the direction perpendicular to that meridian and towards the west, and c is positive when directed downwards.

Then if, instead of as above, we consider the pole to belong to the meridian whose east longitude is λ, we shall have

$$X=\quad a\cos\lambda+b\sin\lambda,$$

$$Y=-a\sin\lambda+b\cos\lambda.$$



Similarly let a', b', c' denote the magnetic forces at the South pole.

Then we shall have, expressing X, Y, Z in terms of the Gaussian constants,

$$-[\cdot 0043626]g_1^1-[\cdot 0058168]g_2^1-[9\cdot 9103610]g_3^1-[9\cdot 7656872]g_4^1-[9\cdot 5910501]g_5^1$$
$$-[9\cdot 3962097]g_6^1-[9\cdot 1868105]g_7^1-[8\cdot 9664160]g_8^1-[8\cdot 7374212]g_9^1-[8\cdot 5015145]g_{10}^1=a,$$

$$-[\cdot 0043626]h_1^1-[\cdot 0058168]h_2^1-[9\cdot 9103610]h_3^1-[9\cdot 7656872]h_4^1-[9\cdot 5910501]h_5^1$$
$$-[9\cdot 3962097]h_6^1-[9\cdot 1868105]h_7^1-[8\cdot 9664160]h_8^1-[8\cdot 7374212]h_9^1-[8\cdot 5015145]h_{10}^1=b,$$

and

$$[0\cdot 3053926]g_1^0+[0\cdot 3068468]g_2^0+[0\cdot 2113910]g_3^0+[0\cdot 0667172]g_4^0+[9\cdot 8920801]g_5^0$$
$$+[9\cdot 6972397]g_6^0+[9\cdot 4878405]g_7^0+[9\cdot 2674460]g_8^0+[9\cdot 0384512]g_9^0+[8\cdot 8025445]g_{10}^0=c.$$

Also

$$[\ \cdot0043626]g_1^1-[\ \cdot0058168]g_2^1+[9\cdot9103610]g_3^1-[9\cdot7656872]g_4^1+[9\cdot5910501]g_5^1$$
$$-[9\cdot3962097]g_6^1+[9\cdot1868105]g_7^1-[8\cdot9664160]g_8^1+[8\cdot7374212]g_9^1-[8\cdot5015145]g_{10}^1=a',$$

$$[\ \cdot0043626]h_1^1-[\ \cdot0058168]h_2^1+[9\cdot9103610]h_3^1-[9\cdot7656872]h_4^1+[9\cdot5910501]h_5^1$$
$$-[9\cdot3962097]h_6^1+[9\cdot1868105]h_7^1-[8\cdot9664160]h_8^1+[8\cdot7374212]h_9^1-[8\cdot5015145]h_{10}^1=b',$$

and

$$-[0\cdot3053926]g_1^0+[0\cdot3068468]g_2^0-[0\cdot2113910]g_3^0+[0\cdot0667172]g_4^0-[9\cdot8920801]g_5^0$$
$$+[9\cdot6972397]g_6^0-[9\cdot4878405]g_7^0+[9\cdot2674460]g_8^0-[9\cdot0384512]g_9^0+[8\cdot8025445]g_{10}^0=c'.$$

Hence the equations for the polar element to be added to the equations of condition previously found are

$$-[\ \cdot0058168]g_2^1-[9\cdot7656872]g_4^1-[9\cdot3962097]g_6^1-[8\cdot9664160]g_8^1-[8\cdot5015145]g_{10}^1=\frac{1}{2}(a+a')$$

$$-[\ \cdot0043626]g_1^1-[9\cdot9103610]g_3^1-[9\cdot5910501]g_5^1-[9\cdot1868105]g_7^1-[8\cdot7374212]g_9^1=\frac{1}{2}(a-a')$$

$$-[\ \cdot0058168]h_2^1-[9\cdot7656872]h_4^1-[9\cdot3962097]h_6^1-[8\cdot9664160]h_8^1-[8\cdot5015145]h_{10}^1=\frac{1}{2}(b+b')$$

$$-[\ \cdot0043626]h_1^1-[9\cdot9103610]h_3^1-[9\cdot5910501]h_5^1-[9\cdot1868105]h_7^1-[8\cdot7374212]h_9^1=\frac{1}{2}(b-b')$$

$$[0\cdot3068468]g_2^0+[0\cdot0667172]g_4^0+[9\cdot6972397]g_6^0+[9\cdot2674460]g_8^0+[8\cdot8025445]g_{10}^0=\frac{1}{2}(c+c')$$

$$[0\cdot3053926]g_1^0+[0\cdot2113910]g_3^0+[9\cdot8920801]g_5^0+[9\cdot4878405]g_7^0+[9\cdot0384512]g_9^0=\frac{1}{2}(c-c')$$

The weight of each of these equations taken on the same scale as before is proportional to the area of the circular segment of the surface over which the observations extend. When the observations extend over a zone of 5° in breadth, the area of the zone will contain the factor $2\sin\frac{\omega}{2}$, where ω is the circular measure of 5°. We must divide by this factor to find the weight, as we have done in our former calculations.

Now the area bounded by the small circle of radius 2° 30′ round the pole is very approximately

$$2\pi\times\frac{N^4}{a^2}(1-e^2)\left(1-\cos\frac{\omega}{2}\right)=4\pi\frac{N^4}{a^2}(1-e^2)\sin^2\frac{\omega}{4}.$$

Dividing this by 2π and by $2\sin\frac{\omega}{2} = 4\sin\frac{\omega}{4}\cos\frac{\omega}{4}$ we have for the weight corresponding to this small area

$$\frac{1}{2}\,\frac{N^4}{a^2}(1-e^2)\tan\frac{\omega}{4}.$$

Making $a=1$, we have at the pole $\log N = \cdot 0014542$, $\log(1-e^2) = 9\cdot 9970916$,

and
$$\log\tan\frac{\omega}{4} = 8\cdot 3388563.$$

Hence log (weight) $= 8\cdot 0407347$, and the weight $= \cdot 01098335$.

To find the weight corresponding to segment of area bounded by a small circle round the pole of radius 12° 30′, we must add the areas of the belts for latitudes 85° and 80° to the small circle of radius 2° 30′. Thus the weight will be,

for the pole,	·01098335	...... $\log(w) = 8\cdot 0407347$,
for belt (a),	·08773241	...... $\log(w) = 8\cdot 9431600$,
for belt (b),	·17474415	...... $\log(w) = 9\cdot 2424026$,
weight for 12° ½ =	·27345991	...... $\log(w) = 9\cdot 4368937$.

Next suppose the polar segment to have a radius of 22° 30′. Then the areas of the next two belts (c) and (d) must be added to the above, and we get

	·27345991	
for belt (c),	·26032372	...... $\log(w) = 9\cdot 4155137$,
for belt (d),	·34377762	...... $\log(w) = 9\cdot 5362776$,
weight for 22° ½ =	·87756125	...... $\log(w) = 9\cdot 9432774$.

From the above equations of condition the coefficients of the final equations for $m=0$ and $m=1$ for all values of n from 0 to 10 have been determined for these polar elements. These have been added to the coefficients of the corresponding final equations from (X) and (Z) or from (X), (Y) and (Z) for the several belts of latitude so as, in each of the cases taken, to include the whole surface of the Earth. The final equations for values of n from 0 to 6 are given below.

In these final equations as given below, α stands for $\frac{a-a'}{2}$ and α' for

$\frac{a+a'}{2}$, similarly β stands for $\frac{b-b'}{2}$ and β' for $\frac{b+b'}{2}$, also γ stands for $\frac{c-c'}{2}$ and γ' for $\frac{c+c'}{2}$.

15. Formation of the final equations for $m=0$, and n odd, taking into account the equations of condition, stopping at (e) inclusive, and also the equations corresponding to the polar segments with radius 22° 30′.

The final equations will be as follows:

	g_1^0	g_3^0	g_5^0	Absolute terms 1845	Absolute terms 1880
		For g_1^0			
From X ...	7·6331952	− ·1138565	− ·0886747	53·575026	52·47937
From Z ...	12·0636234	− 2·1413469	− ·7000106	85·065860	83·96225
	3·5814808	2·8844368	1·3827704	1·772842 γ	1·772842 γ
	23·2782994	·6292334	·5940851	138·640886 +	136·44162 +
		also for g_3^0			
From X ...	− ·1138565	2·8880836	− ·1765112	− 2·456863	− 2·49548
From Z ...	− 2·1413469	2·7856531	− ·4744250	− 16·292662	− 16·48110
	2·8844368	2·3230544	1·1136496	1·4278032 γ	1·42780 γ
	·6292334	7·9967911	·4627134	− 18·749525 +	− 18·97658 +
		and for g_5^0			
From X ...	− ·0886747	− ·1765112	·3955108	− ·4538875	− ·41034
From Z ...	− ·7000106	− ·4744250	·4394974	− 4·6678164	− 4·39735
	1·3827704	1·1136496	·5338724	·6844748 γ	·68447 γ
	·5940851	·4627134	1·3688806	− 5·1217039	− 4·80769

Also for $m=0$, and n even, the final equations will be

	g_2^0	g_4^0	g_6^0	Absolute terms 1845	Absolute terms 1880
		For g_2^0			
From X ...	5·9353003	− ·2100908	− ·1084497	·088152	1·11998
From Z ...	6·4063186	− 1·2109824	− ·3156105	·655802	1·90941
	3·6055460	2·0741556	·8858576	1·7787882 γ'	1·77879 γ'
	15·9471649	·6530824	·4617974	·743954	3·02939
		also for g_4^0			
From X ...	− ·2100908	1·1321535	− ·0999649	− ·657740	− ·93417
From Z ...	− 1·2109824	1·1328694	− ·1498668	− ·753613	− ·92917
	2·0741556	1·1931960	·5096056	1·023280 γ'	1·02328 γ'
	·6530824	3·4582189	·2597739	− 1·411353	− 1·86334
		and for g_6^0			
From X ...	− ·1084497	− ·0999649	·1311547	·041058	·05543
From Z ...	− ·3156105	− ·1498668	·1604164	·051397	·03719
	·8858576	·5096056	·2176492	·437036 γ'	·437036 γ'
	·4617974	·2597739	·5092203	·092455	·09262

For $m=1$ and n odd, the final equations will be

	g_1^1 or h_1^1	g_3^1 or h_3^1	g_5^1 or h_5^1	Absolute term for g 1845	Absolute term for g 1880	Absolute term for h 1845	Absolute term for h 1880
	For g_1^1 and h_1^1						
(X)...	2·9862372	− 2·0046256	− ·4049405	·6460745	·06856	− 4·7000693	− 4·53794
(Y)...	10·6203049	·9080995	·0375997	7·0375320	6·03140	− 13·3237434	− 13·25900
(Z)...	30·4174073	− ·3256928	− ·2100608	18·3312516	16·13701	− 37·9189145	− 40·34372
(P)...	·8953702	·7211092	·3456926	− ·886421 α	− ·88642 α	− ·886421 β	− ·88642 β
	44·9193196	− ·7011097	− ·2317090	26·0148581	22·23697	− 55·9427272	− 58·14066
	For g_3^1 and h_3^1						
(X)...	− 2·0046256	2·7010559	− ·3038483	·9192013	1·34058	3·1891606	2·78482
(Y)...	·9080995	·7688186	·0479263	1·0138712	1·07387	− ·9252286	− ·91569
(Z)...	− ·3256928	5·1262964	− ·2832645	3·4681067	4·99417	1·7927788	1·07219
(P)...	·7211092	·5807636	·2784124	− ·7139016 α	− ·71390 α	− ·7139016 β	− ·71390 β
	− ·7011097	9·1769345	− ·2607741	5·4011792	7·40862	4·0567108	2·94132
	For g_5^1 and h_5^1						
(X)...	− ·4049405	− ·3038483	·4729736	− ·8469248	− ·72421	·6884089	·79895
(Y)...	·0375997	·0479263	·0357094	·0168797	·01964	− ·0166866	− ·01151
(Z)...	− ·2100608	− ·2832645	·5690702	− ·7292658	− ·75427	·4622901	·50652
(P)...	·3456926	·2784124	·1334681	− ·3422374 α	− ·34224 α	− ·3422374 β	− ·34224 β
	− ·2317090	− ·2607741	1·2112213	− 1·5593109	− 1·45884	1·1340124	1·29396

For $m=1$ and n even, the final equations will be

	g_2^1 or h_2^1	g_4^1 or h_4^1	g_6^1 or h_6^1	Absolute term for g 1845	Absolute term for g 1880	Absolute term for h 1845	Absolute term for h 1880
	For g_2^1 and h_2^1						
(X)...	4·7252731	− ·8834794	− ·1654593	− 4·6983874	− 4·57101	·5335877	1·40581
(Y)...	3·0362275	·2553293	− ·0068366	− 3·2648495	− 3·43537	− ·0587768	·61340
(Z)...	13·3230622	− ·3986114	− ·1888655	− 13·6722508	− 14·44531	·5241924	4·15380
(P)...	·9013865	·5185389	·2214644	− ·8893941 α'	− ·88939 α'	− ·8893941 β'	− ·88939 β'
	21·9859493	− ·5082226	− ·1396970	− 21·6354877	− 22·45169	·9990033	6·17301
	For g_4^1 and h_4^1						
(X)...	− ·8834794	1·2083344	− ·0940175	·1065771	·12159	− ·2536825	− ·48670
(Y)...	·2553293	·1729017	·0051341	− ·3723795	− ·38209	− ·0291936	− ·01651
(Z)...	− ·3986114	1·7670285	− ·1458594	− ·8196584	− ·97200	− ·1006035	− ·49527
(P)...	·5185389	·2982990	·1274014	− ·5116401 α'	− ·51164 α'	− ·5116401 β'	− ·51164 β'
	− ·5082226	3·4465636	− ·1073414	− 1·0854608	− 1·23250	− ·3834796	− ·99848
	For g_6^1 and h_6^1						
(X)...	− ·1654593	− ·0940175	·1679758	·2040272	·18532	− ·0081682	− ·07374
(Y)...	− ·0068366	·0051341	·0072219	·0005837	·00658	·0000802	− ·00470
(Z)...	− ·1888655	− ·1458594	·1783999	·2511771	·18903	− ·0273859	·00940
(P)...	·2214644	·1274014	·0544123	− ·2185179 α'	− ·21852 α'	− ·2185179 β'	− ·21852 β'
	− ·1396970	− ·1073414	·4080099	·4557880	·38093	− ·0354739	− ·06904

16. Formation of the final equations for the period 1845, taking into account the equations of condition for (X), (Y) and (Z) to (c) inclusive, and also the equations (P) corresponding to the polar segments with radius 12° 30′.

The final equations will be as follows:

For $m=0$ and n odd,

	g_1^0	g_3^0	g_5^0	Absolute term 1845
		For g_1^0		
(X)...	7·6940384	·0130398	− ·0110762	53·941576
(Z)...	14·2852236	− ·7829229	− ·3528180	99·737391
(P)...	1·1160376	·8988296	·4308900	·5524416 γ
	23·0952996	·1289465	·0669958	153·678967
		also for g_3^0		
(X)...	·0130398	3·1530372	− ·0141019	− 1·692265
(Z)...	− ·7829229	3·6220022	− ·2560720	− 7·325830
(P)...	·8988296	·7238952	·3470284	·4449228 γ
	·1289465	7·4989346	·0768545	− 9·018095
		and for g_5^0		
(X)...	− ·0110762	− ·0141019	·4955695	·0138187
(Z)...	− ·3528180	− ·2560720	·5001678	− 2·3794415
(P)...	·4308900	·3470284	·1663620	·2032916 γ
	·0669958	0768545	1·1620993	− 2·3656228

For $m=0$ and n even,

	g_2^0	g_4^0	g_6^0	Absolute term 1845
		For g_2^0		
(X)...	6·1521685	− ·0061571	− ·0156015	− ·071836
(Z)...	8·2177097	− ·5245692	− ·2056823	·282131
(P)...	1·1235368	·6463352	·2760452	·554294 γ'
	15·4934150	·1156089	·0547614	·210295
		also for g_4^0		
(X)...	− ·0061571	1·3244086	− ·0119941	− ·808850
(Z)...	− ·5245692	1·3982534	− ·1039114	− ·899464
(P)...	·6463352	·3718160	·1588000	·318868 γ'
	·1156089	3·0944780	·0428945	− 1·708314
		and for g_6^0		
(X)...	− ·0156015	− ·0119941	·1718058	− ·028342
(Z)...	− ·2056823	− ·1039114	·1705926	·025255
(P)...	·2760452	·1588000	·0678224	·136186 γ'
	·0547614	·0428945	·4102208	− ·003087

For $m=1$ and n odd, the final equations will be

	For g_1^1 and h_1^1			1845	
	g_1^1 or h_1^1	g_3^1 or h_3^1	g_5^1 or h_5^1	For g	For h
(X)...	3·5405705	− 1·7194983	− ·3793161	1·1283716	− 5·3512098
(Y)...	11·2354744	1·3434531	·2008277	7·5412902	− 13·8957675
(Z)...	30·6564773	·0089039	− ·0257077	18·6082131	− 38·0228819
(P)...	·2790094	·2247074	·1077225	− ·2762208 α	− ·2762208 β
	45·7115316	− ·1424339	− ·0964736	27·2778749	− 57·2698592
	For g_3^1 and h_3^1				
(X)...	− 1·7194983	2·8509357	− ·2873597	1·1677598	2·8583076
(Y)...	1·3434531	1·0773108	·1638961	1·3703002	− 1·3296299
(Z)...	·0089039	5·5951168	− ·0243385	3·8561444	1·6480024
(P)...	·2247074	·1809738	·0867571	− ·2224614 α	− ·2224614 β
	− ·1424339	9·7043371	− ·0610450	6·3942044	3·1766801
	For g_5^1 and h_5^1				
(X)...	− ·3793161	− ·2873597	·4775556	− ·8241335	·6624873
(Y)...	·2008277	·1638961	·0795428	·1504548	− ·1679845
(Z)...	− ·0257077	− ·0243385	·7128006	− ·5149878	·3833853
(P)...	·1077225	·0867571	·0415905	− ·1066458 α	− ·1066458 β
	− ·0964736	− ·0610450	1·3114895	− 1·1886665	·8778881

For $m=1$ and n even, the final equations will be

	For g_2^1 and h_2^1			1845	
	g_2^1 or h_2^1	g_4^1 or h_4^1	g_6^1 or h_6^1	For g	For h
(X)...	5·1280310	− ·7756170	− ·1810288	− 5·1551077	·5575605
(Y)...	3·5953618	·5228428	·0767312	− 3·9864211	− ·0955385
(Z)...	13·8072555	− ·0183365	− ·0271653	− 14·5454971	·5495836
(P)...	·2808842	·1615838	·0690113	− ·2771472 α'	− ·2771472 β'
	22·8115325	− ·1095269	− ·0624516	− 23·6870259	1·0116056
	For g_4^1 and h_4^1				
(X)...	− ·7756170	1·2404733	− ·0956215	− ·0189260	− ·2479337
(Y)...	·5228428	·3012508	·0454432	− ·7180144	− ·0467206
(Z)...	− ·0183365	2·0664442	− ·0179021	− 1·5067400	− ·0800582
(P)...	·1615838	·0929540	·0397000	− ·1594340 α'	− ·1594340 β'
	− ·1095269	3·7011223	− ·0283804	− 2·2436804	− ·3747125
	For g_6^1 and h_6^1				
(X)...	− ·1810288	− ·0956215	·1706017	·2191669	− ·0096245
(Y)...	·0767312	·0454432	·0200090	− ·1076297	− ·0053584
(Z)...	− ·0271653	− ·0179021	·2336239	− ·0420438	− ·0181385
(P)...	·0690113	·0397000	·0169556	− ·0680931 α'	− ·0680931 β'
	− ·0624516	− ·0283804	·4411902	·0694934	− ·0331214

17. Formation of the theoretical coefficients for the final equations for g_n^m for $m=0$ and for all positive and negative values of n from 1 to 8 inclusive, taking into account all the equations for belts of latitude of 5° up to 87° ½, and also the equations (P) corresponding to the polar segments of radius 2° 30′—*i.e.* integrating all over the Earth's surface.

(1) When n is odd.

		g_1^0	g_{-1}^0	g_3^0	g_{-3}^0	g_5^0	g_{-5}^0	g_7^0	g_{-7}^0
g_1^0	(X)... (Z)... (P)...	7·7003306 15·3312564 ·0448248 23·0764118	7·6542768 − 7·6501978 − ·0221885 − ·0281095	·0275805 ·0112863 ·0361008 ·0749676	·0149877 ·0036833 − ·0264484 − ·0077774	− ·0002119 − ·0115687 ·0173064 ·0055258	− ·0002538 ·0093341 − ·0139005 − ·0048202	·0000178 − ·0046001 ·0068228 ·0022405	·0000165 ·0038278 − ·0056776 − ·0018333
g_{-1}^0	(X)... (Z)... (P)...	7·6542768 − 7·6501978 − ·0221885 − ·0281095	7·6087902 3·9152845 ·0109833 11·5350580	− ·0035156 ·0222637 − ·0178700 ·0008781	− ·0156707 − ·0219730 ·0130920 − ·0245517	·0000136 ·0471209 − ·0085667 ·0385678	·0000418 − ·0383464 ·0068808 − ·0314238	·0000055 ·0057897 − ·0033773 ·0024179	·0000051 − ·0049577 ·0028104 − ·0021422
g_3^0	(X)... (Z)... (P)...	·0275805 ·0112863 ·0361008 ·0749676	− ·0035156 ·0222637 − ·0178700 ·0008781	3·1866427 4·2255137 ·0290748 7·4412312	3·1484722 − 3·1346454 − ·0213009 − ·0074741	·0110111 ·0036675 ·0139380 ·0286166	·0030657 ·0042680 − ·0111951 − ·0038614	·0000344 − ·0036917 ·0054948 ·0018375	·0000127 ·0030911 − ·0045726 − ·0014688
g_{-3}^0	(X)... (Z)... (P)...	·0149877 ·0036833 − ·0264484 − ·0077774	− ·0156707 − ·0219730 ·0130920 − ·0245517	3·1484722 − 3·1346454 − ·0213009 − ·0074741	3·1110440 2·371803 ·015605 5·498452	− ·0009898 ·0078808 − ·0102112 − ·0033202	− ·0086237 − ·0118448 ·0082016 − ·0122669	·0000170 ·0026641 − ·0040257 − ·0013446	·0000328 − ·0022498 ·0033499 ·0011329
g_5^0	(X)... (Z)... (P)...	− ·0002119 − ·0115687 ·0173064 ·0055258	·0000136 ·0471209 − ·0085667 ·0385678	·0110111 ·0036675 ·0139380 ·0286166	− ·0009898 ·0078808 − ·0102112 − ·0033202	·5143421 ·6123189 ·0066820 1·1333430	·5048439 − ·5011986 − ·0053667 − ·0017214	·0021105 ·0006029 ·0026344 ·0053478	·0004593 ·0010459 − ·0021920 − ·0006878
g_{-5}^0	(X)... (Z)... (P)...	− ·0002538 ·0093341 − ·0139005 − ·0048202	·0000418 − ·0383464 ·0068808 − ·0314238	·0030657 ·0042680 − ·0111951 − ·0038614	− ·0086237 − ·0118448 ·0082016 − ·0122669	·5048439 − ·5011986 − ·0053667 − ·0017214	·4956187 ·4103186 ·0043105 ·9102478	− ·0001063 ·0015776 − ·0021158 − ·0006445	− ·0016723 − ·0026245 ·0017606 − ·0025362
g_7^0	(X)... (Z)... (P)...	·0000178 − ·0046001 ·0068228 ·0022405	·0000055 ·0057897 − ·0033773 ·0024179	·0000344 − ·0036917 ·0054948 ·0018375	·0000170 ·0026641 − ·0040257 − ·0013446	·0021105 ·0006029 ·0026344 ·0053478	− ·0001063 ·0015776 − ·0021158 − ·0006445	·0611503 ·0691404 ·0010384 ·1313291	·0596145 − ·0590206 − ·0008642 − ·0002703
g_{-7}^0	(X)... (Z)... (P)...	·0000165 ·0038278 − ·0056776 − ·0018333	·0000051 − ·0049577 ·0028104 − ·0021422	·0000127 ·0030911 − ·0045726 − ·0014688	·0000328 − ·0022498 ·0033499 ·0011329	·0004593 ·0010459 − ·0021920 − ·0006878	− ·0016723 − ·0026245 ·0017606 − ·0025362	·0596145 − ·0590206 − ·0008642 − ·0002703	·0581365 ·0523984 ·0007191 ·1112540

Type of equation—

$$\Sigma\left[(X'^m_n)^2 w\right] g^m_n + \Sigma\left[(X'^m_n . X'^m_{n_1}) w\right] g^m_{n_1} + \Sigma\left[(Z'^m_n)^2 w\right] g^m_n + \Sigma\left[(Z'^m_n . Z'^m_{n_1}) w\right] g^m_{n_1}$$
$$= \Sigma\left[(x_m' . X'^m_n) w\right] + \Sigma\left[(z_m' . Z'^m_n) w\right].$$

(2) When n is even.

		g_2^0	g_{-2}^0	g_4^0	g_{-4}^0	g_6^0	g_{-6}^0	g_8^0	g_{-8}^0
g_2^0	(X)...	6·1764738	6·1222621	·0203434	·0080301	·0000467	·0000101	·0000088	·0000083
	(Z)...	9·2226421	− 6·1044053	·0074258	·0059439	− ·0074261	·0061265	− ·0027891	·0023418
	(P)...	·0451260	− ·0295840	·0259596	− ·0201510	·0110872	− ·0090985	·0041212	− ·0034606
		15·4442419	− ·0117272	·0537288	− ·0061770	·0037078	− ·0029619	·0013409	− ·0011105
g_{-2}^0	(X)...	6·1222621	6·0688968	− ·0022139	− ·0140836	·0000174	·0000424	·0000085	·0000080
	(Z)...	− 6·1044053	4·0406704	·0136435	− ·0185690	·0048900	− ·0039942	·0018285	− ·0015353
	(P)...	− ·0295840	·0193950	− ·0170190	·0132110	− ·0072686	·0059648	− ·0027018	·0022687
		− ·0117272	10·1289622	− ·0055894	− ·0194416	− ·0023612	·0020130	− ·0008648	·0007414
g_4^0	(X)...	·0203434	− ·0022139	1·3533148	1·3326863	·0050778	·0012915	·0000201	·0000092
	(Z)...	·0074258	·0136435	1·6803805	− 1·3249820	·0015592	·0022885	− ·0015992	·0013525
	(P)...	·0259596	− ·0170190	·0149336	− ·0115920	·0063780	− ·0052340	·0023708	− ·0019908
		·0537288	− ·0055894	3·0486289	− ·0038877	·0130150	− ·0016540	·0007917	− ·0006291
g_{-4}^0	(X)...	·0080301	− ·0140836	1·3326863	1·3125545	− ·0003786	− ·0039908	·0000103	·0000186
	(Z)...	·0059439	− ·0185690	− 1·3249820	·9448833	·0037393	− ·0059747	·0012495	− ·0010419
	(P)...	− ·0201510	·0132110	− ·0115920	·0089985	− ·0049510	·0040629	− ·0018403	·0015453
		− ·0061770	− ·0194416	− ·0038877	2·2664363	− ·0015903	− ·0059026	− ·0005805	·0005220
g_6^0	(X)...	·0000467	·0000174	·0050778	− ·0003786	·1818936	·1779238	·0008157	·0001560
	(Z)...	− ·0074261	·0048900	·0015592	·0037393	·2102425	− ·1764031	·0002174	·0004318
	(P)...	·0110872	− ·0072686	·0063780	− ·0049510	·0027240	− ·0022354	·0010124	− ·0008502
		·0037078	− ·0023612	·0130150	− ·0015903	·3948601	− ·0007147	·0020455	− ·0002624
g_{-6}^0	(X)...	·0000101	·0000424	·0012915	− ·0039908	·1779238	·1740863	− ·0000422	− ·0006640
	(Z)...	·0061265	− ·0039942	·0022885	− ·0059747	− ·1764031	·1480471	·0006151	− ·0010518
	(P)...	− ·0090985	·0059648	− ·0052340	·0040629	− ·0022354	·0018345	− ·0008309	·0006977
		− ·0029619	·0020130	− ·0016540	− ·0059026	− ·0007147	·3239679	− ·0002580	− ·0010181
g_8^0	(X)...	·0000088	·0000085	·0000201	·0000103	·0008157	− ·0000422	·0198007	·0192380
	(Z)...	− ·0027891	·0018285	− ·0015992	·0012495	·0002174	·0006151	·0220050	− ·0190182
	(P)...	·0041212	− ·0027018	·0023708	− ·0018403	·0010124	− ·0008309	·0003764	− ·0003160
		·0013409	− ·0008648	·0007917	− ·0005805	·0020455	− ·0002580	·0421821	− ·0000962
g_{-8}^0	(X)...	·0000083	·0000080	·0000092	·0000186	·0001560	− ·0006640	·0192380	·0186993
	(Z)...	·0023418	− ·0015353	·0013525	− ·0010419	·0004318	− ·0010518	− ·0190182	·0164437
	(P)...	− ·0034606	·0022687	− ·0019908	·0015453	− ·0008502	·0006977	− ·0003160	·0002654
		− ·0011105	·0007414	− ·0006291	·0005220	− ·0002624	− ·0010181	− ·0000962	·0354084

Formation of the theoretical coefficients for the final equations for g_n^m or h_n^m for $m=1$, (1) for n odd and (2) for n even, for all positive and negative values of n from 1 to 6 inclusive, taking all the equations for belts of latitude up to $87°\frac{1}{2}$, and also the equations corresponding to the polar segments of radius 2° 30′.

(1) When n is odd.

		g_1^1 or h_1^1	g_{-1}^1 or h_{-1}^1	g_3^1 or h_3^1	g_{-3}^1 or h_{-3}^1	g_5^1 or h_5^1	g_{-5}^1 or h_{-5}^1
g_1^1	(X)...	3·8019594	3·8096047	− 1·5270651	− 1·5076599	− ·3017824	− ·2912836
	(Y)...	11·5031546	11·4646148	1·5528311	1·5168987	·2960352	·2853310
	(Z)...	30·6811646	− 15·2788251	·0472046	− ·0187276	·0000781	− ·0000120
	(P)...	·0112062	·0110942	·0090252	·0088162	·0043266	·0041701
		45·9974848	·0064886	·0819958	− ·0006726	− ·0013425	− ·0017945
g_{-1}^1	(X)...	3·8096047	3·8175078	− 1·7446347	− 1·5343784	− ·2988183	− ·2879802
	(Y)...	11·4646148	11·4263058	1·5373650	1·5016970	·2930764	·2824781
	(Z)...	− 15·2788251	7·6088010	·0046836	− ·0116029	− ·0000163	·0000320
	(P)...	·0110942	·0109833	·0089350	·0087280	·0042834	·0041285
		·0064886	22·8635979	− ·1936511	− ·0355563	− ·0014748	− ·0013412
g_3^1	(X)...	− 1·5270651	− 1·7446347	2·9928863	2·9673672	− ·2299197	− ·2302655
	(Y)...	1·5528311	1·5373650	1·2411170	1·2204998	·2384083	·2297958
	(Z)...	·0472046	·0046836	5·6545429	− 4·1978186	·0156768	− ·0039429
	(P)...	·0090252	·0089350	·0072687	·0071003	·0034845	·0033585
		·0819958	− ·1936511	9·8958149	− ·0028513	·0276499	− ·0010541
g_{-3}^1	(X)...	− 1·5076599	− 1·5343784	2·9673672	2·9423058	− ·2386728	− ·2388522
	(Y)...	1·5168987	1·5016970	1·2204998	1·2002722	·2328938	·2244633
	(Z)...	− ·0187276	− ·0116029	− 4·1978186	3·1166058	·0011900	− ·0075783
	(P)...	·0088162	·0087280	·0071003	·0069358	·0034038	·0032807
		− ·0006726	− ·0355563	− ·0028513	7·2661196	− ·0011852	− ·0186865
g_5^1	(X)...	− ·3017824	− ·2988183	− ·2299197	− ·2386728	·5010132	·4934494
	(Y)...	·2960352	·2930764	·2384083	·2328938	·1134598	·1100925
	(Z)...	·0000781	− ·0000163	·0156768	·0011900	·7397539	− ·6057704
	(P)...	·0043266	·0042834	·0034845	·0034038	·0016705	·0016100
		− ·0013425	− ·0014748	·0276499	− ·0011852	1·3558974	− ·0006185
g_{-5}^1	(X)...	− ·2912836	− ·2879802	− ·2302655	− ·2388522	·4934494	·4860509
	(Y)...	·2853310	·2824781	·2297958	·2244633	·1100925	·1068322
	(Z)...	− ·0000120	·0000320	− ·0039429	− ·0075783	− ·6057704	·4961398
	(P)...	·0041701	·0041285	·0033585	·0032807	·0016100	·0015518
		− ·0017945	− ·0013412	− ·0010541	− ·0186865	− ·0006185	1·0905747

Formation of the theoretical coefficients for the final equations for $m=1$.

(2) When n is even.

		g_2^1 or h_2^1	g_{-2}^1 or h_{-2}^1	g_4^1 or h_4^1	g_{-4}^1 or h_{-4}^1	g_6^1 or h_6^1	g_{-6}^1 or h_{-6}^1
g_2^1	(X)...	5·3732443	5·3455962	− ·6515436	− ·6468403	− ·1386266	− ·1327617
	(Y)...	3·8586274	3·8200228	·6682295	·6484134	·1349333	·1291857
	(Z)...	13·8614767	− 9·1817777	·0310519	− ·0094915	·0000698	− ·0000185
	(P)...	·0112815	·0110942	·0064899	·0062972	·0027718	·0026537
		23·1046299	− ·0050645	·0542277	− ·0016212	− ·0008517	− ·0009408
g_{-2}^1	(X)...	5·3455962	5·3184296	− ·6685050	− ·6635806	− ·1363538	− ·1305189
	(Y)...	3·8200228	3·7818764	·6571564	·6376276	·1326939	·1272032
	(Z)...	− 9·1817777	6·0824656	·0027669	− ·0121256	− ·0000214	·0000381
	(P)...	·0110942	·0109100	·0063822	·0061927	·0027258	·0026097
		− ·0050645	15·1936816	− ·0021995	− ·0318859	− ·0009555	− ·0006679
g_4^1	(X)...	− ·6515436	− ·6685050	1·3035371	1·2880783	− ·0738726	− ·0748904
	(Y)...	·6682295	·6571564	·3815712	·3727300	·0776182	·0743152
	(Z)...	·0310519	·0027669	2·1114387	− 1·6658350	·0069175	− ·0014760
	(P)...	·0064899	·0063822	·0037334	·0036226	·0015945	·0015266
		·0542277	− ·0021995	3·8002804	− ·0014041	·0122576	− ·0005246
g_{-4}^1	(X)...	− ·6468403	− ·6635806	1·2880783	1·2729526	− ·0778697	− ·0787857
	(Y)...	·6484135	·6376276	·3727300	·3641162	·0753171	·0729062
	(Z)...	− ·0094915	− ·0121256	− 1·6658350	1·3144282	·0004417	− ·0037548
	(P)...	·0062972	·0061927	·0036226	·0035151	·0015472	·0014813
		− ·0016212	− ·0318859	− ·0014041	2·9550121	− ·0005637	− ·0081530
g_6^1	(X)...	− ·1386266	− ·1363538	− ·0738726	− ·0778697	·1782377	·1749490
	(Y)...	·1349333	·1326939	·0776182	·0753171	·0329115	·0317222
	(Z)...	·0000698	− ·0000214	·0069175	·0004417	·2473219	− ·2075771
	(P)...	·0027718	·0027258	·0015945	·0015472	·0006810	·0006520
		− ·0008517	− ·0009555	·0122576	− ·0005637	·4591521	− ·0002539
g_{-6}^1	(X)...	− ·1327617	− ·1305189	− ·0748904	− ·0787857	·1749490	·1717610
	(Y)...	·1291857	·1272032	·0743152	·0729062	·0317222	·0305787
	(Z)...	− ·0000185	·0000381	− ·0014760	− ·0037548	− ·2075771	·1742616
	(P)...	·0026537	·0026097	·0015266	·0014813	·0006520	·0006242
		− ·0009408	− ·0006679	− ·0005246	− ·0081530	− ·0002539	·3772255

The above tables of the coefficients of the final equations for $m=0$ and for $m=1$ take into account all the equations to (a) inclusive and also the equations corresponding to the polar segments with radius $2°\ 30'$.

It will be seen (1) that all the terms, except the principal term in each equation, are very small; (2) that the coefficients of the principal terms in the respective final equations for g_n^m and g_{-n}^m for any value of n are very nearly in the ratio of $n+1$ to n.

These results might be expected, since we have seen in Section V.,

above, that in the case of a spherical surface all except the principal terms vanish, since

$$\Sigma[(X_n^m X_{n_1}^m)\,w]+\Sigma[(Y_n^m Y_{n_1}^m)\,w]+\Sigma[(Z_n^m Z_{n_1}^m)\,w]=0,$$

and

$$\Sigma[(X_n^m X_{-n}^m)\,w]+\Sigma[(Y_n^m Y_{-n}^m)\,w]+\Sigma[(Z_n^m Z_{-n}^m)\,w]=0.$$

We have also seen that on a spherical surface

$$\Sigma[(X_{-n}^m)^2 w]+\Sigma[(Y_{-n}^m)^2 w]+\Sigma[(Z_{-n}^m)^2 w]=\frac{n}{n+1}\{\Sigma[(X_n^m)^2 w]+\Sigma[(Y_n^m)^2 w]+\Sigma[(Z_n^m)^2 w]\}$$
$$=n\,(2n+1)\,\Sigma[(H_n^m)^2 w].$$

18. The solution of the final equations, stopping at equation (e) inclusive and taking the equations corresponding to the polar segments with radius 22° 30′, will give values of the magnetic constants in terms of α, α', β, β', γ, γ' which are simple functions of the forces at the poles. These values of the magnetic constants for a given period, when equated to the values of the same constants given in the table on p. 605 for the same period, will give the values of α, β, γ, &c., from which, in the absence of direct observations, we may derive a, b, c, &c., the forces at the poles. The accuracy of the work may be tested by the close agreement of the values of each of the quantities α, β, γ, &c., derived from the several final equations which give the magnetic constants. The values of these constants and the corresponding values of α, β, γ, &c., for the periods 1845 and 1880 are here given.

For 1845.

$g_1^0 = \;\;6{\cdot}166 \;+{\cdot}0613\;\gamma = \;\;6{\cdot}9808$, which gives $\gamma = 13{\cdot}29$
$g_3^0 = -2{\cdot}5075 \;+{\cdot}14925\gamma = -\;{\cdot}52399$, „ „ $\gamma = 13{\cdot}29$
$g_5^0 = -5{\cdot}5700 \;+{\cdot}4230\;\gamma = \;\;{\cdot}0513465$, „ „ $\gamma = 13{\cdot}29$

$g_2^0 = \;\;{\cdot}0545 \;+{\cdot}0828\;\gamma' = -\;{\cdot}0275845$, „ „ $\gamma' = -{\cdot}991$
$g_4^0 = -\;{\cdot}4454 \;+{\cdot}2302\;\gamma' = -\;{\cdot}67352$, „ „ $\gamma' = -{\cdot}991$
$g_6^0 = \;\;{\cdot}3596 \;+{\cdot}6657\;\gamma' = -\;{\cdot}30013$, „ „ $\gamma' = -{\cdot}991$

$g_1^1 = \;\;{\cdot}58317 - {\cdot}02269\alpha = \;\;{\cdot}602567$, „ „ $\alpha = -{\cdot}855$
$g_3^1 = \;\;{\cdot}6034 \;-{\cdot}0882\;\alpha = \;\;{\cdot}678817$, „ „ $\alpha = -{\cdot}855$
$g_5^1 = -1{\cdot}0459 \;-{\cdot}3059\;\alpha = -\;{\cdot}784390$, „ „ $\alpha = -{\cdot}855$

$g_2^1 = -\;{\cdot}9900 \;-{\cdot}0483\;\alpha' = -1{\cdot}065495$, „ „ $\alpha' = 1{\cdot}563$
$g_4^1 = -\;{\cdot}4403 \;-{\cdot}1742\;\alpha' = -\;{\cdot}712584$, „ „ $\alpha' = 1{\cdot}563$
$g_6^1 = \;\;{\cdot}6625 \;-{\cdot}5980\;\alpha' = -\;{\cdot}272348$, „ „ $\alpha' = 1{\cdot}563$

$$
\begin{array}{lllll}
h_1^1 = -1{\cdot}2356 & -{\cdot}02269\beta = -1{\cdot}254179, & \text{which gives} & \beta = & {\cdot}8188 \\
h_3^1 = \quad {\cdot}3698 & -{\cdot}0882\ \beta = \quad {\cdot}297611, & ,, \quad ,, & \beta = & {\cdot}8185 \\
h_5^1 = \quad {\cdot}7794 & -{\cdot}3059\ \beta = \quad {\cdot}5291705, & ,, \quad ,, & \beta = & {\cdot}8180 \\
\\
h_2^1 = \quad {\cdot}0423 & -{\cdot}0483\ \beta' = \quad {\cdot}0391731, & ,, \quad ,, & \beta' = & {\cdot}06468 \\
h_4^1 = -{\cdot}10817 & -{\cdot}1742\ \beta' = -{\cdot}119440, & ,, \quad ,, & \beta' = & {\cdot}0647 \\
h_6^1 = -{\cdot}10092 & -{\cdot}5980\ \beta' = -{\cdot}139605, & ,, \quad ,, & \beta' = & {\cdot}0647
\end{array}
$$

For 1880.

$$
\begin{array}{lllll}
g_1^0 = \quad 6{\cdot}06496 & +{\cdot}06130\gamma = \quad 6{\cdot}87176, & \text{which gives} & \gamma = & 13{\cdot}16 \\
g_3^0 = -2{\cdot}5445 & +{\cdot}14924\gamma = -{\cdot}58113, & ,, \quad ,, & \gamma = & 13{\cdot}16 \\
g_5^0 = -5{\cdot}2842 & +{\cdot}42295\gamma = \quad {\cdot}27987, & ,, \quad ,, & \gamma = & 13{\cdot}16 \\
\\
g_2^0 = \quad {\cdot}20582 & +{\cdot}08284\gamma' = \quad {\cdot}158464, & ,, \quad ,, & \gamma' = & -{\cdot}5717 \\
g_4^0 = -{\cdot}60033 & +{\cdot}23025\gamma' = -{\cdot}73195, & ,, \quad ,, & \gamma' = & -{\cdot}57165 \\
g_6^0 = \quad {\cdot}30149 & +{\cdot}66566\gamma' = -{\cdot}07904, & ,, \quad ,, & \gamma' = & -{\cdot}5717 \\
\\
g_1^1 = \quad {\cdot}50301 & -{\cdot}02269\alpha = \quad {\cdot}52644, & ,, \quad ,, & \alpha = & -1{\cdot}0326 \\
g_3^1 = \quad {\cdot}81926 & -{\cdot}08822\alpha = \quad {\cdot}91030, & ,, \quad ,, & \alpha = & -1{\cdot}0320 \\
g_5^1 = -{\cdot}93186 & -{\cdot}30589\alpha = -{\cdot}61614, & ,, \quad ,, & \alpha = & -1{\cdot}0325 \\
\\
g_2^1 = -1{\cdot}029775 & -{\cdot}04828\alpha' = -1{\cdot}11386, & ,, \quad ,, & \alpha' = & 1{\cdot}7416 \\
g_4^1 = -{\cdot}49541 & -{\cdot}17419\alpha' = -{\cdot}79880, & ,, \quad ,, & \alpha' = & 1{\cdot}7417 \\
g_6^1 = \quad {\cdot}45071 & -{\cdot}59793\alpha' = -{\cdot}59068, & ,, \quad ,, & \alpha' = & 1{\cdot}7417 \\
\\
h_1^1 = -1{\cdot}28596 & -{\cdot}02269\beta = -1{\cdot}30780, & ,, \quad ,, & \beta = & {\cdot}9625 \\
h_3^1 = \quad {\cdot}24714 & -{\cdot}08822\beta = \quad {\cdot}16224, & ,, \quad ,, & \beta = & {\cdot}9624 \\
h_5^1 = \quad {\cdot}87552 & -{\cdot}30589\beta = \quad {\cdot}58114, & ,, \quad ,, & \beta = & {\cdot}9624 \\
\\
h_2^1 = \quad {\cdot}273996 & -{\cdot}04828\beta' = \quad {\cdot}28051, & ,, \quad ,, & \beta' = & -{\cdot}1349 \\
h_4^1 = -{\cdot}25373 & -{\cdot}17419\beta' = -{\cdot}23026, & ,, \quad ,, & \beta' = & -{\cdot}1347 \\
h_6^1 = -{\cdot}142155 & -{\cdot}59793\beta' = -{\cdot}06162, & ,, \quad ,, & \beta' = & -{\cdot}1346
\end{array}
$$

Replacing α, β, γ, &c., in these equations by their values in terms of a, b, c, &c., we get the values of a, a', b, b', c and c', which are the mean values of the horizontal and vertical magnetic forces for the north and south polar areas bounded by a circle of $22°\frac{1}{2}$ radius. The values of these forces in British units for the north and south polar caps for the periods 1845 and 1880, as derived from these investigations, are here given.

For the period 1845

$$\frac{c-c'}{2} = 13{\cdot}29 \;, \quad \frac{c+c'}{2} = -\,{\cdot}991 \;, \quad \text{hence } c = 12{\cdot}30 \text{ and } c' = -14{\cdot}28 \;,$$

$$\frac{a-a'}{2} = -\,{\cdot}855 \;, \quad \frac{a+a'}{2} = 1{\cdot}563 \;, \quad \text{hence } a = {\cdot}708 \text{ and } a' = 2{\cdot}418 \;,$$

$$\frac{b-b'}{2} = {\cdot}8184, \quad \frac{b+b'}{2} = {\cdot}0647, \quad \text{hence } b = {\cdot}883 \text{ and } b' = -\,{\cdot}7537,$$

whence we derive for the horizontal force, at the north pole 1·13 and at the south pole 2·53 units, and for the total force, at the north pole 12·35 and at the south pole −14·50 British units.

For the period 1880

$$\frac{c-c'}{2} = 13{\cdot}16 \;, \quad \frac{c+c'}{2} = -\,{\cdot}5717, \quad \text{hence } c = 12{\cdot}59 \text{ and } c' = -13{\cdot}73 \;,$$

$$\frac{a-a'}{2} = -\,1{\cdot}0324, \quad \frac{a+a'}{2} = 1{\cdot}7417, \quad \text{hence } a = {\cdot}7093 \text{ and } a' = 2{\cdot}7741,$$

$$\frac{b-b'}{2} = {\cdot}9624, \quad \frac{b+b'}{2} = -\,{\cdot}1347, \quad \text{hence } b = {\cdot}8277 \text{ and } b' = -\,1{\cdot}0971,$$

whence we derive for the horizontal force, at the north pole 1·09 and at the south pole 2·98 units, and for the total force, at the north pole 12·64 and at the south pole −14·05 British units.

19. On referring to the Charts for Sabine's Arctic and Antarctic Magnetic Surveys (1840—1845) and to the Admiralty Charts constructed by Captain Creak for 1880 we find a very close agreement between the above calculated values of the horizontal and vertical magnetic forces for the polar areas and the values of those forces derived from the observations and recorded in the Charts.

Thus the values given by Captain Creak for 1880 are:

for the horizontal force at the north pole 1 British unit and at the south pole 2·75 British units;

for c, the vertical force at the north pole, 12·6 units and for c', the vertical force at the south pole, −13·7 British units.

In the north and south polar charts given at the end of this volume the *black* curved lines indicate the values of the *Total* force as given in Sabine's Arctic and Antarctic Magnetic Surveys (1840—1845), and the *red* lines indicate the values of the *Vertical* and the *Horizontal* forces recorded by Captain Creak for the period 1880.

20. If we take into account the terms depending upon magnetic forces outside the Earth, the *equations of condition* for the polar element will include the following terms in addition to those which have been already given on p. 614:

$$-[9{\cdot}9985458]\,g_{-2}{}^1-[9{\cdot}7525993]\,g_{-4}{}^1-[9{\cdot}3773050]\,g_{-6}{}^1-[8{\cdot}9416945]\,g_{-8}{}^1-[8{\cdot}4709763]\,g_{-10}{}^1=\tfrac{1}{2}(a+a'),$$
$$-g_{-1}{}^1-[9{\cdot}9001816]\,g_{-3}{}^1-[9{\cdot}5750539]\,g_{-5}{}^1-[9{\cdot}1649975]\,g_{-7}{}^1-[8{\cdot}7097914]\,g_{-9}{}^1=\tfrac{1}{2}(a-a'),$$
$$-[9{\cdot}9985458]\,h_{-2}{}^1-[9{\cdot}7525993]\,h_{-4}{}^1-[9{\cdot}3773050]\,h_{-6}{}^1-[8{\cdot}9416945]\,h_{-8}{}^1-[8{\cdot}4709763]\,h_{-10}{}^1=\tfrac{1}{2}(b+b'),$$
$$-h_{-1}{}^1-[9{\cdot}9001816]\,h_{-3}{}^1-[9{\cdot}5750539]\,h_{-5}{}^1-[9{\cdot}1649975]\,h_{-7}{}^1-[8{\cdot}7097914]\,h_{-9}{}^1=\tfrac{1}{2}(b-b'),$$
$$-[\ {\cdot}1234845]\,g_{-2}{}^0-[9{\cdot}9567193]\,g_{-4}{}^0-[9{\cdot}6113883]\,g_{-6}{}^0-[9{\cdot}1915720]\,g_{-8}{}^0-[8{\cdot}7306136]\,g_{-10}{}^0=\tfrac{1}{2}(c+c'),$$
$$-g_{-1}{}^0-[\ {\cdot}0762729]\,g_{-3}{}^0-[9{\cdot}7969026]\,g_{-5}{}^0-[9{\cdot}4080355]\,g_{-7}{}^0-[8{\cdot}9650639]\,g_{-9}{}^0=\tfrac{1}{2}(c-c').$$

These terms have been added to make the equations complete, but they have not as yet been employed in the determination of the magnetic constants.

Description of the following Tables.

The following Tables give the several *final equations* which have been employed in the determination of the magnetic constants for the periods 1845 and 1880; they are derived from the equations given on pp. 554—587 by combining the equations for X, for Y and for Z from the several belts between $67°\tfrac{1}{2}$ N. and $67°\tfrac{1}{2}$ S. latitude, and are of the type given as equation (4) of this section.

Some of the more important *final equations* for the magnetic constants with a negative suffix (g_{-n}^m) have been added to the tables, and their solution would give the values of these constants for the same periods of time. In some other cases the numerical values of the coefficients only of the magnetic constants have been given in the tables, so that by supplying the terms depending on the observed values of the magnetic elements for any given period of time the values of the constants for that period may be determined.

Formation of the Final Equations for $m=0$ for all values of n from 1 to 10, taking into account the equations of condition stopping at equation (e) inclusive (*i.e.* to latitude $67^\circ \tfrac{1}{2}$), and also for the equations (P) corresponding to the polar segment with radius $22^\circ\,30'$. From the equations for X and Z and those equations combined.

(1) When n is odd.

		g_1^0	g_{-1}^0	g_3^0	g_{-3}^0	g_5^0	g_{-5}^0
g_1^0	(X)...	7·6331952	7·5889390	− ·1138565	− ·1226364	− ·0886747	− ·0855784
	(Z)...	12·0636234	− 6·0303603	− 2·1413469	1·5861301	− ·7000106	·5652811
		19·6968186	1·5585787	− 2·2552034	1·4634937	− ·7886853	·4797027
	(P)...	3·5814808	− 1·7728420	2·8844368	− 2·1132110	1·3827704	− 1·1106382
g_{-1}^0	(X)...	7·5889390	7·5452019	− ·1411597	− ·1496043	− ·0860695	− ·0829876
	(Z)...	− 6·0303603	3·1122949	1·0892643	− ·8063501	·3882760	− ·3138450
		1·5585787	10·6574968	·9481046	− ·9559544	·3022065	− ·3968326
	(P)...	− 1·7728420	·8775612	− 1·4278033	1·0460448	− ·6844748	·5497688
g_3^0	(X)...	− ·1138565	− ·1411597	2·8880838	2·8579799	− ·1765114	− ·1777867
	(Z)...	− 2·1413469	1·0892643	2·7856531	− 2·0764461	− ·4744250	·3901540
		− 2·2552034	·9481046	5·6737369	·7815338	− ·6509364	·2123673
	(P)...	2·8844368	− 1·4278033	2·3230544	− 1·7019286	1·1136496	− ·8944808
g_{-3}^0	(X)...	− ·1226364	− ·1496043	2·8579799	2·8284066	− ·1834238	− ·1845693
	(Z)...	1·5861301	− ·8063501	− 2·0764461	1·5478839	+ ·3590274	− ·2952704
		1·4634937	− ·9559544	·7815338	4·3762905	·1756036	− ·4798397
	(P)...	− 2·1132110	1·0460448	− 1·7019286	1·2468758	− ·8158878	·6553193
g_5^0	(X)...	− ·0886747	− ·0860695	− ·1765114	− ·1834238	·3955108	·3902625
	(Z)...	− ·7000106	·3882760	− ·4744250	·3590274	·4394974	− ·3618590
		− ·7886853	·3022065	− ·6509364	·1756036	·8350082	·0284035
	(P)...	1·3827704	− ·6844748	1·1136496	− ·8158878	·5338724	− ·4288052
g_{-5}^0	(X)...	− ·0855784	− ·0829876	− ·1777867	− ·1845693	·3902625	·3851367
	(Z)...	·5652811	− ·3138450	·3901540	− ·2952704	− ·3618590	·2979727
		·4797027	− ·3968326	·2123673	− ·4798397	·0284035	·6831094
	(P)...	− 1·1106382	·5497688	− ·8944808	·6553193	− ·4288052	·3444154
g_7^0	(X)...	− ·0331758	− ·0322894	− ·0709077	− ·0689827	− ·0436160	− ·0441804
	(Z)...	− ·1322591	·0690024	− ·1021726	·0752283	− ·0426039	·0363377
		− ·1654349	·0367130	− ·1730803	·0062456	− ·0862199	− ·0078427
	(P)...	·5451432	− ·2698472	·4390448	− ·3216553	·2104738	− ·1690521
g_{-7}^0	(X)...	− ·0316740	− ·0308277	− ·0677063	− ·0658325	− ·0431758	− ·0437308
	(Z)...	·1108633	− ·0579588	·0855746	− ·0627336	·0371720	− ·0316886
		·0791893	− ·0887865	·0178683	− ·1285661	− ·0060038	− ·0754194
	(P)...	− ·4536338	·2245500	− ·3653456	·2676614	− ·1751431	·1406746

Type of these equations—

$$\{\Sigma[(X'^m_n)^2 w]+\Sigma[(Z'^m_n)^2 w]\}\, g^m_n+\{\Sigma[(X'^m_n X'^m_{n_1})\, w]+\Sigma[(Z'^m_n Z'^m_{n_1})\, w]\}\, g^m_{n_1}+\&c.$$
$$=\Sigma[(X'^m_n x_m')\, w]+\Sigma[(Z'^m_n z_m')\, w].$$

g_7^0	g_{-7}^0	g_9^0	g_{-9}^0	1845	1880
− ·0331758 − ·1322591 − ·1654349 ·5451432	− ·0316740 ·1108633 ·0791893 − ·4536338	− ·0083280 − ·0053853 − ·0137133 ·1936964	− ·0079213 ·0031617 − ·0047596 − ·1635814	53·575026 85·065860 138·640886 1·7728420 γ	52·47937 83·96225 136·44162 1·77284 γ
− ·0322894 ·0690024 ·0367130 − ·2698472	− ·0308277 − ·0579588 − ·0887865 ·2245499	− ·0081309 − ·0013989 − ·0095298 − ·0958801	− ·0077348 ·0019608 − ·0057740 ·0809732	53·280527 − 43·896763 9·383764 − ·8775612 γ	52·19188 − 41·97950 10·21238 − ·87756 γ
− ·0709077 − ·1021726 − ·1730803 ·4390448	− ·0677063 ·0855746 ·0178683 − ·3653456	− ·0181915 − ·0095500 − ·0277415 ·1559982	− ·0171865 ·0091263 − ·0080602 − ·1317444	− 2·456861 − 16·292667 − 18·749528 1·4278032 γ	− 2·49548 − 16·48110 − 18·97658 1·42780 γ
− ·0689827 ·0752283 ·0062456 − ·3216553	− ·0658325 − ·0627336 − ·1285661 ·2676614	− ·0172844 ·0069433 − ·0103411 − ·1142882	− ·0167079 − ·0066536 − ·0233615 ·0965193	− 2·502657 12·074223 9·571566 − 1·046045 γ	− 2·54028 12·21777 9·67749 − 1·046045 γ
− ·0436160 − ·0426039 − ·0862199 ·2104738	− ·0431758 ·0371720 − ·0060038 − ·1751431	− ·0121489 − ·0079890 − ·0201379 ·0747840	− ·0114552 ·0066715 − ·0047837 − ·0631570	− ·4538875 − 4·6678164 − 5·1217039 ·6844748 γ	− ·41034 − 4·39735 − 4·80769 ·684475 γ
− ·0441804 ·0363377 − ·0078427 − ·1690521	− ·0437308 − ·0316886 − ·0754194 ·1406746	− ·0117219 ·0063960 − ·0053259 − ·0600634	− ·0110452 − ·0053359 − ·0163811 ·0507275	− 0·4265168 3·7264380 3·2999212 − ·54977 γ	− 0·38422 3·54478 3·16056 − ·54977 γ
·0429919 ·0545269 ·0975188 ·0829772	·0422967 − ·0468285 − ·0045318 − ·0690484	− ·0048407 − ·0039784 − ·0088191 ·0294828	− ·0047965 ·0035077 − ·0012888 − ·0248990	− ·2036836 − ·8617810 − 1·0654646 ·2698472 γ	− ·19362 − ·86153 − 1·05515 ·26985 γ
·0422967 − ·0468285 − ·0045318 − ·0690484	·0416203 ·0402263 ·0818466 ·0574577	− ·0049062 ·0035991 − ·0013071 − ·0245338	− ·0048630 − ·0031720 − ·0080350 ·0207194	− ·1947500 ·7223115 ·5275615 − ·224550 γ	− ·18518 ·72255 ·53737 − ·22455 γ

(2) When n is even.

		g_2^0	g_{-2}^0	g_4^0	g_{-4}^0	g_6^0	g_{-6}^0
g_2^0	(X)...	5·9353003	5·8869711	− ·2100908	− ·2152736	− ·1084497	− ·1041086
	(Z)...	6·4063186	− 4·2536687	− 1·2109824	·9557641	− ·3156105	·2605933
		12·3416189	1·6333024	− 1·4210732	·7404905	− ·4240602	·1564847
	(P)...	3·6055460	− 2·3637887	2·0741556	− 1·6100652	·8858576	− ·7269633
g_{-2}^0	(X)...	5·8869711	5·8393445	− ·2270148	− ·2319257	− ·1058105	− ·1015158
	(Z)...	− 4·2536687	2·8244633	·8140945	− ·6425696	·2072157	− ·1710558
		1·6333024	8·6638078	·5870797	− ·8744953	·1014052	− ·2725716
	(P)...	− 2·3637887	1·5496957	− 1·3598128	·1055556	− ·5807651	·4765958
g_4^0	(X)...	− ·2100908	− ·2270148	1·1321535	1·1183960	− ·0999649	− ·0994945
	(Z)...	− 1·2109824	·8140945	1·1328694	− ·8984056	− ·1498668	·1271930
		− 1·4210732	·5870797	2·2650229	·2199904	− ·2498317	·0276985
	(P)...	2·0741556	− 1·3598128	1·1931960	− ·9262194	·5096056	− ·4181990
g_{-4}^0	(X)...	− ·2152736	− ·2319257	1·1183960	1·1049212	− ·1021352	− ·1016243
	(Z)...	·9557641	− ·6425696	− ·8984056	·6125267	·1215705	− ·1031697
		·7404905	− ·8744953	·2199904	1·7174479	·0194353	− ·2047940
	(P)...	− 1·6100652	·1055556	− ·9262194	·7189787	− ·3955821	·3246272
g_6^0	(X)...	− ·1084497	− ·1058105	− ·0999649	− ·1021352	·1311547	·1292581
	(Z)...	− ·3156105	·2072157	− ·1498668	·1215705	·1604164	− ·1353768
		− ·4240602	·1014052	− ·2498317	·0194353	·2915711	− ·0061187
	(P)...	·8858576	− ·5807651	·5096056	− ·3955821	·2176492	− ·1786098
g_{-6}^0	(X)...	− ·1041086	− ·1015158	− ·0994945	− ·1016243	·1292581	·1274087
	(Z)...	·2605933	− ·1710558	·1271930	− ·1031697	− ·1353768	·1142659
		·1564847	− ·2725716	·0276985	− ·2047940	− ·0061187	·2416746
	(P)...	− ·7269633	·4765958	− ·4181990	·3246272	− ·1786098	·1465730
g_8^0	(X)...	− ·0335070	− ·0326737	− ·0329997	− ·0319625	− ·0156604	− ·0158347
	(Z)...	− ·0434978	·0284817	− ·0282696	·0219325	− ·0107012	·0109209
		− ·0770048	− ·0041920	− ·0612693	− ·0100300	− ·0263616	− ·0049138
	(P)...	·3292836	− ·2158777	·1894264	− ·1470424	·0809028	− ·0663914
g_{-8}^0	(X)...	− ·0318180	− ·0310269	− ·0313367	− ·0303335	− ·0154763	− ·0156478
	(Z)...	·0368033	− ·0240987	·0238825	− ·0185145	·0110148	− ·0098406
		·0049853	− ·0551256	− ·0074542	− ·0488480	− ·0044615	− ·0254884
	(P)...	− ·2765007	·1812733	− ·1590620	·1234721	− ·0679343	·0557491

g_8^0	g_{-8}^0	g_{10}^0	g_{-10}^0	1845	1880
− ·03350696	− ·0318180	− ·0067837	− ·0063725	·088152	1·11998
− ·04349786	·0368033	·0031556	− ·0027079	·655802	1·90941
− ·0770048	·0049853	− ·0036281	− ·0090804	·743954	3·02939
·3292836	− ·2765007	·1128936	− ·0956616	1·7787882 γ'	1·77879 γ'
− ·0326737	− ·0310269	− ·0066102	− ·0062096	·0982751	1·12141
·0284817	− ·0240987	− ·0021097	·0018102	− ·4424197	− 1·27410
− ·0041920	− ·0551256	− ·0087199	− ·0043994	− ·3441446	− 0·15269
− ·2158777	·1812733	− ·0740127	·0627156	− 1·16617 γ'	− 1·16617 γ'
− ·0329997	− ·0313367	− ·0069775	− ·0065508	− ·657740	− ·93417
− ·0282696	·0238825	− ·0019579	·0016583	− ·753613	− ·92917
− ·0612693	− ·0074542	− ·0089354	− ·0048925	− 1·411353	− 1·86334
·1894264	− ·1590620	·0649440	− ·0550311	1·0232802 γ'	1·02328 γ'
− ·0319625	− ·0303335	− ·0067491	− ·0063364	− ·6491774	− ·92383
·0219325	− ·0185145	·0014869	− ·0012884	·5973065	+ ·73604
− ·0100300	− ·0488480	− ·0052622	− ·0076248	− ·0518709	− ·18779
− ·1470424	·1234721	− ·0504128	·0427179	− ·794322 γ'	− ·79432 γ'
− ·0156604	− ·0154763	− ·0037478	− ·0035171	·041058	·05543
− ·0107012	·0110148	− ·0026558	·0022607	·051397	·03719
− ·0263616	− ·0044615	− ·0064036	− ·0012564	·092455	·09262
·0809028	− ·0679343	·0277372	− ·0235034	·4370358 γ'	·437036 γ'
− ·0158347	− ·0156478	− ·0035909	− ·0033645	·041080	·05588
·0109209	− ·0098406	·0021719	− ·0018457	− ·079677	− ·03305
− ·0049138	− ·0254884	− ·0014190	− ·0052102	− ·038597	·02283
− ·0663914	·0557491	− ·0227620	·0192876	− ·358646 γ'	− ·35865 γ'
·0141275	·0138606	− ·0013474	− ·0013438	·0210180	·02105
·0173116	− ·0150678	− ·0013909	·0012652	·0148518	·00329
·0314391	− ·0012072	− ·0027383	− ·0000786	·0358698	·02434
·0300724	− ·0252520	·0103104	− ·0087365	·1624514 γ'	·16245 γ'
·0138606	·0136022	− ·0013778	− ·0013730	·0198806	·01671
− ·0150678	·0131186	·0012693	− ·0011534	− ·0125449	− ·00378
− ·0012072	·0267208	− ·0001085	− ·0025264	·0073359	·01293
− ·0252520	·0212042			− ·136411 γ'	− ·13641 γ'

Final equations for $m=1$, 2, and 3 for all values of n from 1 to 10. From the equations for X, Y and Z respectively and those equations combined.

(1) When n is odd.

		g_1^1 or h_1^1	g_{-1}^1 or h_{-1}^1	g_3^1 or h_3^1	g_{-3}^1 or h_{-3}^1	g_5^1 or h_5^1	g_{-5}^1 or h_{-5}^1
g_1^1 or h_1^1	(X)...	2·9862372	3·0002743	− 2·0046256	− 1·9767768	− ·4049405	− ·3919431
	(Y)...	10·6203049	10·5899389	·9080995	·8860421	·0375997	·0356222
	(Z)...	30·4174073	− 15·1493187	− ·3256928	·2543667	− ·2100608	·1692676
		44·0239494	− 1·5591055	− 1·4222189	− ·8363680	− ·5774016	− ·1870533
	(P)...	·8953702	·8864210	·7211092	·7044036	·3456926	·3331915
g_{-1}^1 or h_{-1}^1	(X)...	3·0002743	3·0147177	− 2·0276952	− 1·0690535	− ·4010352	− ·1882371
	(Y)...	10·5899389	10·5597278	·8986170	·8766950	·0370508	·0350977
	(Z)...	− 15·1493187	7·5452128	− ·1784060	·1224843	·1031542	− ·0830781
		− 1·5591055	21·1196593	− 1·3074842	− ·0698742	− ·2608302	− ·2358175
	(P)...	·8864210	·8775612	·7139016	·6973632	·3422374	·3298612
g_3^1 or h_3^1	(X)...	− 2·0046256	− 2·0276952	2·7010559	2·6809220	− ·3038483	− ·3021935
	(Y)...	·9080995	·8986170	·7688186	·7583899	·0479263	·0457609
	(Z)...	− ·3256928	− ·1784060	5·1262964	− 3·8109739	− ·2832645	·2368500
		− 1·4222189	− 1·3074842	8·5961709	− ·3716620	− ·5391865	− ·0195826
	(P)...	·7211092	·7139016	·5807636	·5673095	·2784124	·2683442
g_{-3}^1 or h_{-3}^1	(X)...	− 1·9767768	− 1·0690535	2·6809220	2·6611426	− ·3110444	− ·3092685
	(Y)...	·8860421	·8766950	·7583899	·7481307	·0465404	·0444172
	(Z)...	·2543667	·1224843	− 3·8109739	2·8333066	·2177050	− ·1838957
		− ·8363680	− ·0698742	− ·3716620	6·2425799	− ·0467990	− ·4487470
	(P)...	·7044036	·6973632	·5673095	·5541669	·2719626	·2621277
g_5^1 or h_5^1	(X)...	− ·4049405	− ·4010352	− ·3038483	− ·3110444	·4729736	·4663226
	(Y)...	·0375997	·0370508	·0479263	·0465404	·0357094	·0349857
	(Z)...	− ·2100608	·1031542	− ·2832645	·2177050	·5690702	− ·4683139
		− ·5774016	− ·2608302	− ·5391865	− ·0467990	1·0777532	·0329944
	(P)...	·3456926	·3422374	·2784124	·2719626	·1334681	·1286416
g_{-5}^1 or h_{-5}^1	(X)...	− ·3919431	− ·1882371	− ·3021935	− ·3092685	·4663226	·4598053
	(Y)...	·0356222	·0350977	·0457609	·0444172	·0349857	·0342804
	(Z)...	·1692676	− ·0830781	·2368500	− ·1838957	− ·4683139	·3854417
		− ·1870533	− ·2358175	− ·0195826	− ·4487470	·0329944	·8795274
	(P)...	·3331915	·3298612	·2683442	·2621277	·1286416	·1239896
g_7^1 or h_7^1	(X)...	− ·0547088	− ·0540479	− ·0540704	− ·0524302	− ·0297014	− ·0310557
	(Y)...	− ·0161300	− ·0160152	− ·0082538	− ·0081395	− ·0002910	− ·0003256
	(Z)...	− ·0751122	·0368732	− ·1077065	·0761847	− ·0598334	·0505280
		− ·1459510	− ·0331899	− ·1700307	·0156150	− ·0898258	·0191467
	(P)...	·1362858	·1349236	·1097612	·1072184	·0526184	·0507156

Type of these equations—

$$\{\Sigma[(X'^m_n)^2 w] + \Sigma[(Y'^m_n)^2 w] + \Sigma[(Z'^m_n)^2 w]\} g^m_n$$
$$+ \{\Sigma[(X'^m_n X'^m_{n_1}) w] + \Sigma[(Y'^m_n Y'^m_{n_1}) w] + \Sigma[(Z'^m_n Z'^m_{n_1}) w]\} g^m_{n_1} + \&c.$$
$$= \Sigma[(X'^m_n x'_m) w] + \Sigma[(Y'^m_n y'_m) w] + \Sigma[(Z'^m_n z'_m) w].$$

g_7^1 or h_7^1	g_{-7}^1 or h_{-7}^1	g_9^1 or h_9^1	g_{-9}^1 or h_{-9}^1	For g		For h	
				1845	1880	1845	1880
− ·0547088 − ·0161300 − ·0751122 − ·1459510 ·1362858	− ·0522847 − ·0155632 ·0628502 − ·0049977 ·1296097	·0021806 − ·0058117 − ·0183987 − ·0220298 ·0484241	·0020722 − ·0055082 ·0168543 ·0134183 ·0454393	·6460745 7·0375320 18·3312516 26·0148581 − ·886421 α	·06856 6·03140 16·13701 22·23697 − ·88642 α	− 4·7000693 − 13·3237434 − 37·9189145 − 55·9427272 − ·886421 β	− 4·53794 − 13·25900 − 40·34372 − 58·14066 − ·88642 β
− ·0540479 − ·0160152 ·0368732 − ·0331899 ·1349236	− ·0516638 − ·0154522 − ·0308536 − ·0979696 ·1283142			·6652797 7·0132906 − 9·1115250 − 1·4329547 − ·8775612 α	·05491 6·00857 − 8·01121 − 1·94773 − ·87756 α	− 4·7225008 − 13·2877883 18·8923360 0·8820469 − ·8775612 β	− 4·55663 − 13·22337 20·09631 2·31631 − ·87756 β
− ·0540704 − ·0082538 − ·1077065 − ·1700307 ·1097612	− ·0513246 − ·0080101 ·0901317 ·0307970 ·1043845	− ·0047340 − ·0037514 − ·0268697 − ·0353551 ·0389995	− ·0044417 − ·0035601 ·0228663 ·0148645 ·0365957	·9192013 1·0138712 3·4681067 5·4011792 − ·7139016 α	1·34058 1·07387 4·99417 7·40862 − ·71390 α	3·1891606 − ·9252286 1·7927788 4·0567108 − ·7139016 β	2·78482 − ·91569 1·07219 2·94123 − ·71390 β
− ·0524302 − ·0081395 ·0761847 ·0156150 ·1072184	− ·0500019 − ·0078979 − ·0659662 − ·1238660 ·1019663						
− ·0297014 − ·0002910 − ·0598334 − ·0898258 ·0526184	− ·0300582 − ·0003379 ·0518208 − ·0822169 ·0500409	− ·0066442 − ·0010670 − ·0161685 − ·0238797 ·0186960	− ·0062526 − ·0010173 ·0137568 ·0064869 ·0175436	− ·8469248 ·0168797 − ·7292658 − 1·5593109 − ·3422374 α	− ·72421 ·01964 − ·75427 − 1·45884 − ·34224 α	·6884089 − ·0166866 ·4622901 1·1340124 − ·3422374 β	·79895 − ·01151 ·50652 1·29396 − ·34224 β
− ·0310557 − ·0003256 ·0505280 ·0191467 ·0507156	− ·0313732 − ·0003708 − ·0437589 − ·0755029 ·0482313						
·0551723 ·0015791 ·0561205 ·1128719 ·0207443	·0541060 ·0015444 − ·0483389 ·0073115 ·0197281	− ·0038989 − ·0000597 − ·0061454 − ·0101040 ·0073707	− ·0039218 − ·0000606 ·0054783 ·0014959 ·0069164	− ·0248367 − ·0148057 − ·0692747 − ·1089171 − ·1349236 α	− ·03763 − ·01435 − ·08540 − ·13738 − ·13492 α	·0550275 ·0179512 ·0384398 ·1114185 − ·1349236 β	·06486 ·01784 ·06718 ·14988 − ·13492 β

(2) When n is even.

		g_2^1 or h_2^1	g_{-2}^1 or h_{-2}^1	g_4^1 or h_4^1	g_{-4}^1 or h_{-4}^1	g_6^1 or h_6^1	g_{-6}^1 or h_{-6}^1
g_2^1	(X)...	4·7252731	4·7060898	− ·8834794	− ·8735948	− ·1654593	− ·1590115
	(Y)...	3·0362275	3·0102976	·2553293	·2469362	− ·0068366	− ·0069194
	(Z)...	13·3230622	− 8·8298252	− ·3986114	·3245072	− ·1888655	·1556849
		21·0845628	− 1·1134378	− 1·0267615	− ·3021514	− ·3611614	− ·0102460
	(P)...	·9013865	·8864210	·5185389	·5031453	·2214644	·2120309
g_{-2}^1	(X)...	4·7060898	4·6872742	− ·8972124	− ·8871820	− ·1626641	− ·1562616
	(Y)...	3·0102976	2·9846303	·2506420	·2423593	− ·0068698	− ·0067840
	(Z)...	− 8·8298252	5·8524019	·2836142	− ·2304424	·1234607	− ·1017250
		− 1·1134378	15·5243064	− ·3629562	− ·8752651	− ·0460732	− ·2647706
	(P)...	·8864210	·8717040	·5099298	·4947918	·2177875	·2085106
g_4^1	(X)...	− ·8834794	− ·8972124	1·2083344	1·1952134	− ·0940175	− ·0943619
	(Y)...	·2553293	·2506420	·1729017	·1698536	·0051341	·0047370
	(Z)...	− ·3986114	·2836142	1·7670285	− 1·3981366	− ·1458594	·1244065
		− 1·0267615	− ·3629562	3·1482646	− ·0330696	− ·2347428	·0347816
	(P)...	·5185389	·5099298	·2982990	·2894436	·1274014	·1219747
g_{-4}^1	(X)...	− ·8735948	− ·8871820	1·1952134	1·1823646	− ·0973812	− ·0974470
	(Y)...	·2469362	·2423593	·1698536	·1668714	·0048574	·0052711
	(Z)...	·3245072	− ·2304424	− 1·3981366	1·1063548	·1191653	− ·1015789
		− ·3021514	− ·8752651	− ·0330696	2·4555908	·0266415	− ·1937548
	(P)...	·5031453	·4947918	·2894436	·2808510	·1236193	·1183537
g_6^1	(X)...	− ·1654593	− ·1626641	− ·0940175	− ·0973812	·1679758	·1651158
	(Y)...	− ·0068366	− ·0068698	·0051341	·0048574	·0072219	·0070683
	(Z)...	− ·1888655	·1234607	− ·1458594	·1191653	·1783999	− ·1508073
		− ·3611614	− ·0460732	− ·2347428	·0266415	·3535976	·0213768
	(P)...	·2214644	·2177875	·1274014	·1236193	·0544123	·0520945
g_{-6}^1	(X)...	− ·1590115	− ·1562616	− ·0943619	− ·0974470	·1651158	·1623380
	(Y)...	− ·0069194	− ·0067840	·0047370	·0052711	·0070683	·0069188
	(Z)...	·1556849	− ·1017250	·1244065	− ·1015789	− ·1508073	·1275009
		− ·0102460	− ·2647706	·0347816	− ·1937548	·0213768	·2967577
	(P)...	·2120309	·2085106	·1219747	·1183537	·0520945	·0498755

g_8^1 or h_8^1	g_{-8}^1 or h_{-8}^1	g_{10}^1 or h_{10}^1	g_{-10}^1 or h_{-10}^1	For g		For h	
				1845	1880	1845	1880
− ·0166647 − ·0098160 − ·0562815 − ·0827622 ·0823209	− ·0158014 − ·0093844 ·0475754 ·0223896 ·0777658	·0028091 − ·0024763 − ·0111427 − ·0108099 ·0282234	·0026536 − ·0023302 ·0095270 ·0098504 0263069	− 4·6983874 − 3·2648495 − 13·6722508 − 21·6354877 − ·889394 α'	− 4·57101 − 3·43537 − 14·44531 − 22·45169 − ·88939 α'	·5335877 − ·0587768 ·5241924 ·9990033 − ·889394 β'	1·40581 ·61340 4·15380 6·17301 − ·88939 β'
				− 4·6665148 − 3·2352472 9·0478425 1·1460805 − ·8746276 α'	− 4·54028 − 3·40450 9·55827 1·61349 − ·87463 α'	·5335511 − ·0587976 − ·3483488 ·1264047 − ·8746276 β'	1·40379 ·60928 − 2·75709 − ·74402 − ·87463 β'
− ·0175987 − ·0035432 − ·0462972 − ·0674391 ·0473566	− ·0166724 − ·0034025 ·0391347 ·0190598 ·0447362	− ·0021205 − ·0010833 − ·0095729 − ·0127767 ·0162360	− ·0019751 − ·0010211 ·0081801 ·0051839 ·0151336	·1065771 − ·3723795 − ·8196584 − 1·0854608 − ·51164 α'	·12159 − ·38209 − ·97200 − 1·23250 − ·51164 α'	− ·2536825 − ·0291936 − ·1006035 − ·3834796 − ·51164 β'	− ·48670 − ·01651 − ·49527 − ·99848 − ·51164 β'
− ·0103865 − ·0002741 − ·0205582 − ·0312188 ·0202257	− ·0105226 − ·0002771 ·0180593 ·0072596 ·0191065	− ·0027303 − ·0002542 − ·0048024 − ·0077869 ·0069343	− ·0025638 − ·0002412 ·0041017 ·0012967 ·0064634	·2040272 ·0005837 ·2511771 ·4557880 − ·2185179 α'	·18532 ·00658 ·18903 ·38093 − ·21852 α'	− ·0081682 ·0000802 − ·0273859 − ·0354739 − ·2185179 β'	− ·07374 − ·00470 ·00940 − ·06904 − ·21852 β'

		g_2^2 or h_2^2	g_{-2}^2 or h_{-2}^2	g_4^2 or h_4^2	g_{-4}^2 or h_{-4}^2	g_6^2 or h_6^2	g_{-6}^2 or h_{-6}^2
g_2^2 or h_2^2	(X)...	5·8560865	5·8736892	− 1·9023717	− 1·8760836	− ·2761720	− ·2650029
	(Y)...	30·4176397	30·3185676	1·5914424	1·5438057	·1239966	·1183499
	(Z)...	55·1758930	− 36·6376841	− ·0103389	·0370318	− ·0462303	·0380836
		91·4496192	− 0·4454273	− ·3212682	− ·2952461	− ·1984057	− ·1085694
g_4^2 or h_4^2	(X)...	− 1·9023717	− 1·9268351	1·7384537	1·7255506	− ·2190847	− ·2167639
	(Y)...	1·5914424	1·5648733	1·3699973	1·3487733	·1158533	·1106619
	(Z)...	− ·0103389	·0525484	4·1222073	− 3·2602014	− ·0474292	·0456663
		− ·3212682	− ·3094134	7·2306583	− ·1858775	− ·1506606	− ·0604357
g_6^2 or h_6^2	(X)...	− ·2761720	− ·2717991	− ·2190847	− ·2226879	·2174322	·2144368
	(Y)...	·1239966	·1217870	·1158533	·1122404	·0714518	·0696389
	(Z)...	− ·0462303	·1302408	− ·0474292	·0459700	·3574373	− ·3011288
		− ·1984057	− ·0197713	− ·1506606	− ·0644775	·6463213	− ·0170531

		g_3^2 or h_3^2	g_{-3}^2 or h_{-3}^2	g_5^2 or h_5^2	g_{-5}^2 or h_{-5}^2	g_7^2 or h_7^2	g_{-7}^2 or h_{-7}^2
g_3^2 or h_3^2	(X)...	4·1941269	4·1770132	− ·6716362	− ·6631724	− ·1103617	− ·1052833
	(Y)...	5·9425073	5·8855425	·4638440	·4466743	·0291874	·0283133
	(Z)...	14·0034282	− 10·4217632	− ·0501784	·0554246	− ·0402190	·0336243
		24·1400624	− ·3592075	− ·2579706	− ·1610735	− ·1213933	− ·0433457
g_5^2 or h_5^2	(X)...	− ·6716362	− ·6811298	·6335524	·6268705	− ·0665022	− ·0660358
	(Y)...	·4638440	·4528531	·3185157	·3118832	·0246492	·0235123
	(Z)...	− ·0501784	·0586707	1·2264016	− 1·0070993	− ·0301289	·0278199
		− ·2579706	− ·1696060	2·1784697	− ·0683456	− ·0719819	− ·0147036
g_7^2 or h_7^2	(X)...	− ·1103617	− ·1078222	− ·0665022	− ·0678847	·0718631	·0706311
	(Y)...	·0291874	·0283966	·0246492	·0236636	·0152717	·0148315
	(Z)...	− ·0402190	·0293679	− ·0301289	·0277380	·1015832	− ·0872007
		− ·1213933	− ·0500577	− ·0719819	− ·0164831	·1887180	− ·0017381

		g_3^3 or h_3^3	g_{-3}^3 or h_{-3}^3	g_5^3 or h_5^3	g_{-5}^3 or h_{-5}^3	g_7^3 or h_7^3	g_{-7}^3 or h_{-7}^3
g_3^3 or h_3^3	(X)...	7·7831711	7·8043450	− 1·7807839	− 1·7570094	− ·1688230	− ·1607705
	(Y)...	55·1586439	54·9743153	1·7600996	1·6966403	·1288671	·1224368
	(Z)...	84·1499066	− 62·8652751	·0598019	− ·0080192	− ·0087517	·0073628
		147·0917216	− ·0866148	·0391176	− ·0683883	− ·0487076	− ·0309709
g_5^3 or h_5^3	(X)...	− 1·7807839	− 1·8354689	1·2550739	1·2452721	− ·1411364	− ·1396354
	(Y)...	1·7600996	1·9860540	1·5452837	1·5200845	·1177833	·1119510
	(Z)...	·0598019	·0150036	3·4195822	− 2·8133805	− ·0021999	·0071231
		·0391176	·1655887	6·2199398	·0480239	− ·0255530	− ·0205613
g_7^3 or h_7^3	(X)...	− ·1688230	− ·1650357	− ·1411364	− ·1435309	·1186787	·1170206
	(Y)...	·1288671	·1258253	·1177833	·1134852	·0748286	·0727383
	(Z)...	− ·0087517	·0064123	− ·0021999	·0087933	·2313050	− ·1983397
		− ·0487076	− ·0327981	− ·0255530	− ·0212524	·4248123	− ·0085808

g_8^2 or h_8^2	g_{-8}^2 or h_{-8}^2	For g		For h	
		1845	1880	1845	1880
- ·0448692	- ·0425640	·5584839	- ·01140	- 1·7810965	- 1·78354
·0017479	·0015650	- ·4994256	- 3·42615	- 7·2429571	- 7·85075
- ·0201855	·0170368	- ·7117042	- 6·98186	- 14·3351455	- 15·93574
- ·0633068	- ·0239622	- ·6526459	- 10·41941	- 23·3591991	- 25·57003
- ·0374467	- ·0355132	- ·7068864	- ·54015	·7651172	·78835
·0031731	·0028676	- ·4300292	- ·72765	- ·1387209	- ·27529
- ·0254972	·0215329	- 1·3017542	- 1·75082	1·0175575	·42191
- ·0597708	- ·0111127	- 2·4386698	- 3·01862	1·6439538	·93497
- ·0189448	- ·0189261	·1694613	·14794	·0404818	·02076
·0042949	·0040039	- ·0129446	- ·04427	- ·0246750	- ·04368
- ·0150805	·0137159	·0891637	·08486	- ·0423906	- ·05420
- ·0297304	- ·0012063	·2456804	·18853	- ·0265838	- ·07712

g_9^2 or h_9^2	g_{-9}^2 or h_{-9}^2	For g		For h	
		1845	1880	1845	1880
- ·0167968	- ·0158386	- 2·1013078	- 2·08403	- ·5872382	- ·28199
- ·0035370	- ·0034086	- 3·7947663	- 3·77384	- ·3900929	- ·02187
- ·0150774	·0128141	- 8·6025393	- 8·76508	- 1·1571948	·49497
- ·0354112	- ·0064331	- 14·4986134	- 14·62295	- 2·1345259	·19111
- ·0112581	- ·0106082	- ·0088187	·04153	·1131649	·03071
- ·0008544	- ·0008528	- ·5127447	- ·46504	- ·0288192	·01060
- ·0128794	·0109511	- ·7624143	- ·68693	- ·1163853	·09710
- ·0249919	- ·0005099	- 1·2839777	- 1·11044	- ·0320396	·13841
- ·0052179	- ·0052569	·1118711	·09353	- ·0070686	·00047
·0005244	·0004721	- ·0357963	- ·02746	·0005182	·00816
- ·0063181	·0057192	·0511579	- ·00822	- ·0182992	- ·03715
- ·0110116	·0009344	·1272327	·05785	- ·0248496	- ·02852

g_9^3 or h_9^3	g_{-9}^3 or h_{-9}^3	For g		For h	
		1845	1880	1845	1880
- ·0230096	- ·0216543	- ·1188299	- ·56925	- 1·2685284	- ·90664
·0085247	·0079409	- 2·1839724	- 4·33651	- 8·1899166	- 6·47443
- ·0042155	·0037113	- 4·3163385	- 4·32315	- 12·1609314	- 8·35188
- ·0187004	- ·0100021	- 6·6191408	- 14·22891	- 21·6193764	- 15·73295
- ·0197814	- ·0186187	·0239183	·08573	·3538548	·28086
·0086185	·0080416	- ·1026141	- ·17179	- ·2683575	- ·19301
- ·0051281	·0043542	- ·0693361	·01250	- ·0316070	·08529
- ·0162910	- ·0062229	- ·1480319	- ·07356	·0538903	·17314
- ·0111551	- ·0110869	- ·0052944	·02706	·0037425	- ·00263
·0062529	·0058460	- ·0086410	- ·02560	- ·0342865	- ·03285
- ·0027065	·0028960	·0176366	- ·00872	- ·0004055	- ·05898
- ·0076087	- ·0023449	·0037012	- ·00726	- ·0309495	- ·09446

Final equations for $m=3$, 4, and 5. From the equations for X, Y and Z respectively and those equations combined.

		g_4^3 or h_4^3	g_{-4}^3 or h_{-4}^3	g_6^3 or h_6^3	g_{-6}^3 or h_{-6}^3	For g 1845	For g 1880	For h 1845	For h 1880
g_4^3 or h_4^3	(X)...	3·7596608	3·7436908	− ·4980032	− ·4919583	·1974902	·24841	·2184196	·29184
	(Y)...	7·8947196	7·8163481	·4633943	·4436183	·8760844	1·21499	·6758584	·77386
	(Z)...	14·6737803	−11·6395368	·0113225	·0055937	·9384537	1·71960	1·3353052	1·40443
		26·3281607	− 0·0794979	− ·0232864	− ·0427463	2·0120283	3·18300	2·2295832	2·47013
g_6^3 or h_6^3	(X)...	− ·4980032	− ·5056255	·3916522	·3873632	·0125871	− ·00630	− ·0367976	·02747
	(Y)...	·4633943	·4496436	·3350340	·3275434	·1450888	·17778	·0530271	·06721
	(Z)...	·0113225	·0118681	·8750271	− ·7375755	·2274858	·42424	·1803857	·08664
		− ·0232864	− ·0441138	1·6017133	− ·0226689	·3851617	·59572	·1966152	·18132

		g_4^4 or h_4^4	g_{-4}^4 or h_{-4}^4	g_6^4 or h_6^4	g_{-6}^4 or h_{-6}^4	For g 1845	For g 1880	For h 1845	For h 1880
g_4^4 or h_4^4	(X)...	9·2748125	9·3002568	− 1·6990760	− 1·6766981	− ·3406078	− ·30798	− ·0059673	·29056
	(Y)...	84·1188388	83·8363769	1·7532997	1·6791849	·2739375	− 1·34317	1·9757506	2·54297
	(Z)...	116·8795672	−93·1437554	·0792532	− ·0156921	·7018231	− 3·27198	2·6181162	1·43816
		210·2732185	− ·0071217	·1334769	− ·0132052	·6351528	− 4·92313	4·5878995	4·27169
g_6^4 or h_6^4	(X)...	− 1·6990760	− 1·7229646	·9502502	·9426434	·0587329	·08075	·0035777	− ·02571
	(Y)...	1·7532997	1·7018389	1·5604387	1·5345601	·0796647	·10038	·1218758	·16325
	(Z)...	·0792532	·0072411	2·9414055	− 2·4871751	·2144875	·23458	·2070971	·08239
		·1334769	− ·0138846	5·4520944	− ·0099716	·3528851	·41571	·3325506	·21993

		g_5^4 or h_5^4	g_{-5}^4 or h_{-5}^4	g_7^4 or h_7^4	g_{-7}^4 or h_{-7}^4	For g 1845	For g 1880	For h 1845	For h 1880
g_5^4 or h_5^4	(X)...	3·3954751	3·3804662	− ·3947004	− ·3900007	·2402352	·07945	− ·0156344	·10605
	(Y)...	9·4062063	9·3119869	·4029091	·3832829	·0569385	·28656	·3404510	·54629
	(Z)...	15·3847031	− 12·7067159	·0202085	− ·0023698	·4674347	− ·24398	·1822892	·04925
		28·1863845	− ·0142628	·0284172	− ·0090876	·7646084	·12203	·5071058	·70159
g_7^4 or h_7^4	(X)...	− ·3947004	− ·4007976	·2608021	·2578464	·0195493	·02170	·0142034	·00737
	(Y)...	·4029091	·3884603	·3039539	·2969660	·0441174	·04341	·0363896	·03616
	(Z)...	·0202085	·0030701	·6517791	− ·5601749	·0071250	·00258	·0880327	·10346
		·0284172	− ·0092672	1·2165351	− ·0053625	·0707917	·06769	·1386257	·14699

		g_5^5 or h_5^5	g_{-5}^5 or h_{-5}^5	g_7^5 or h_7^5	g_{-7}^5 or h_{-7}^5	For g 1845	For g 1880	For h 1845	For h 1880
g_5^5 or h_5^5	(X)...	10·5483242	10·5780175	− 1·6284259	− 1·6070178	− ·3435114	− ·13196	− ·0503192	− ·11806
	(Y)...	116·8370545	116·4443446	1·7037606	1·6211956	− 1·2591969	− 1·50018	− 1·1416378	− ·69460
	(Z)...	152·9977736	− 127·0146070	·0887097	− ·0163579	− 2·6366137	− 2·39064	− 1·2307555	− ·34785
		280·3831523	·0077551	·1640444	− ·0021801	− 4·2393220	− 4·02278	− 2·4227125	− 1·16051
g_7^5 or h_7^5	(X)...	− 1·6284259	− 1·6512329	·7468552	·7408538	·0779101	·02878	·0421939	− ·00806
	(Y)...	1·7037606	1·6430569	1·5288367	1·5033592	·0069740	·03815	·0225418	·00728
	(Z)...	·0887097	·0056878	2·6033553	− 2·2460357	·1209663	·04535	− ·0905983	− ·04790
		·1640444	− ·0024882	4·8790472	− ·0018227	·2058504	·11228	− ·0258626	− ·04868

For $m=5, 6, 7, 8$.

						For g		For h	
		g_6^5 or h_6^5	g_{-6}^5 or h_{-6}^5	g_8^5 or h_8^5	g_{-8}^5 or h_{-8}^5	1845	1880	1845	1880
g_6^5 or h_6^5	(X)...	3·1058721	3·0929438	− ·3258660	− ·3219633	− ·0284830	·01298	− ·1127118	·02028
	(Y)...	10·6967502	10·5893721	·3421787	·3234222	− ·1177225	− ·35194	− ·4579321	− ·41870
	(Z)...	16·1124421	− 13·6845781	·0203162	− ·0031223	− ·1388805	− ·30552	− ·9024802	− ·47522
		29·9150644	− ·0022622	·0366289	− ·0016634	− ·2850860	− ·64448	− 1·4731241	− ·87364

						For g		For h	
		g_6^6 or h_6^6	g_{-6}^6 or h_{-6}^6	g_8^6 or h_8^6	g_{-8}^6 or h_{-8}^6	1845	1880	1845	1880
g_6^6 or h_6^6	(X)...	11·6854670	11·7191204	− 1·5624578	− 1·5418887	− ·1779289	− ·07856	·0073801	·00799
	(Y)...	152·9453749	152·4311814	1·6474470	1·5575217	− ·2220763	·26143	− 1·0303846	·55783
	(Z)...	192·2170999	− 164·1402540	·0958928	− ·0158521	1·5178139	− ·35184	− ·9982489	·82618
		356·8479418	·0100478	·1808820	− ·0002191	1·1178087	− ·16897	− 2·0212534	1·39200

						For g		For h	
		g_7^6 or h_7^6	g_{-7}^6 or h_{-7}^6	g_9^6 or h_9^6	g_{-9}^6 or h_{-9}^6	1845	1880	1845	1880
g_7^6 or h_7^6	(X)...	2·8776859	2·8659442	− ·2755131	− ·2721708	·1575352	− ·02877	·0043533	·06096
	(Y)...	11·8491124	11·7300956	·2925225	·2747123	·0421333	·07857	− ·1551986	·09813
	(Z)...	16·8377550	− 14·5960771	·0192181	− ·0028115	− ·1048524	·04519	·0860001	·04268
		31·5645533	− ·0000373	·0362275	− ·0002700	·0948161	·09499	− ·0648452	·20177

						For g		For h	
		g_7^7 or h_7^7	g_{-7}^7 or h_{-7}^7	g_9^7 or h_9^7	g_{-9}^7 or h_{-9}^7	1845	1880	1845	1880
g_7^7 or h_7^7	(X)...	12·7240775	12·7613903	− 1·5011444	− 1·4813468	·0790212	− ·07122	·1116204	·04136
	(Y)...	192·1558537	191·5097356	1·5931800	1·4965793	− 1·0079847	− ·79713	− ·0396364	1·11264
	(Z)...	234·3103054	− 204·2611132	·1022515	− ·0152097	− 1·9821143	·17830	·4994436	− ·02976
		439·1902366	·0100127	·1942871	·0000228	− 2·9110778	− ·69005	·5714276	1·12424

						For g		For h	
		g_8^7 or h_8^7	g_{-8}^7 or h_{-8}^7	g_{10}^7 or h_{10}^7	g_{-10}^7 or h_{-10}^7	1845	1880	1845	1880
g_8^7 or h_8^7	(X)...	2·6929069	2·6821095	− ·2367956	− ·2338881	·0156549	·02181	− ·1082369	·00010
	(Y)...	12·9016251	12·7720042	·2571514	·2362881	− ·0435159	·12575	·0155932	·19953
	(Z)...	17·5507768	− 15·4538077	·0181124	− ·0024316	·1074283	− ·35573	− ·1023934	·06012
		33·1453088	·0003060	·0384682	− ·0000316	·0795673	− ·20817	− ·1950371	·25975

						For g		For h	
		g_8^8 or h_8^8	g_{-8}^8 or h_{-8}^8	g_{10}^8 or h_{10}^8	g_{-10}^8 or h_{-10}^8	1845	1880	1845	1880
g_8^8 or h_8^8	(X)...	13·6861471	13·7268560	− 1·4447678	− 1·4256831	− ·0567578	− ·01289	− ·2362381	·03769
	(Y)...	234·2411668	233·4586971	1·5431857	1·4404066	− ·1321388	·08507	− ·1354841	·33660
	(Z)...	279·0932359	− 247·1700388	·1081721	− ·0146233	·8033517	·30497	− 1·0552595	− ·05027
		527·0205498	·0155143	·2065900	·0001002	·6144551	·37715	− 1·4269817	·32402

For $m=8,\ 9$.

		g_9^8 or h_9^8	g_{-9}^8 or h_{-9}^8	For g 1845	For g 1880	For h 1845	For h 1880
g_9^8 or h_9^8	(X)...	2·5396427	2·5296081	·0089086	·01975	·0382869	·01143
	(Y)...	13·8765774	13·7371378	·0340929	·08455	− ·0776123	·09258
	(Z)...	18·2472160	− 16·2664099	·3052325	− ·04093	·0694911	·10303
		34·6634361	·0003360	·3482340	·06337	·0301657	·20704

		g_9^9 or h_9^9	g_{-9}^9 or h_{-9}^9	For g 1845	For g 1880	For h 1845	For h 1880
g_9^9 or h_9^9	(X)...	14·5862998	14·6301805	− ·0514447	·00749	·0426089	·00103
	(Y)...	279·0168466	278·0797898	− ·0315765	·42969	− ·5884055	·26948
	(Z)...	326·4133488	− 292·6991621	− 1·5247793	·49846	·1020889	·10606
		620·0164952	·0108082	− 1·6078005	·93564	− ·4437077	·37657

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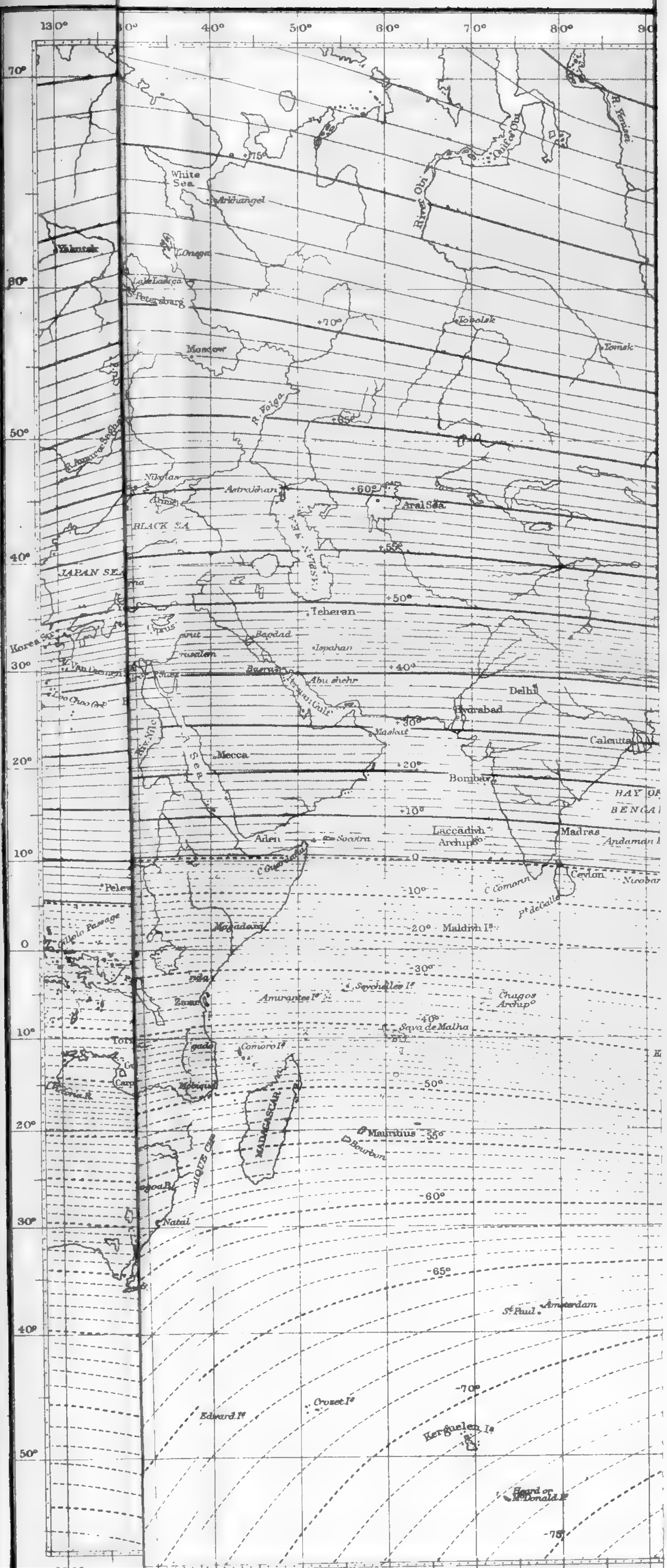
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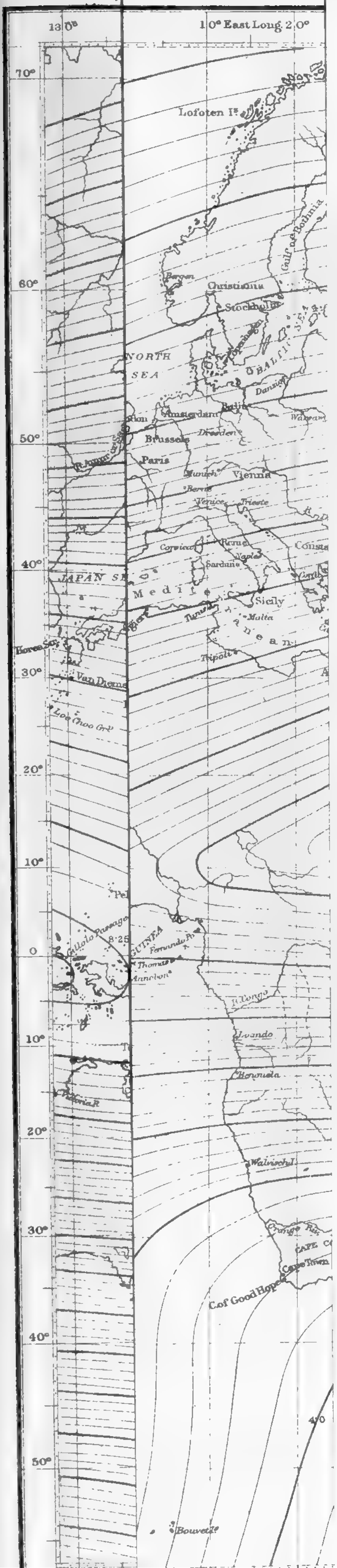

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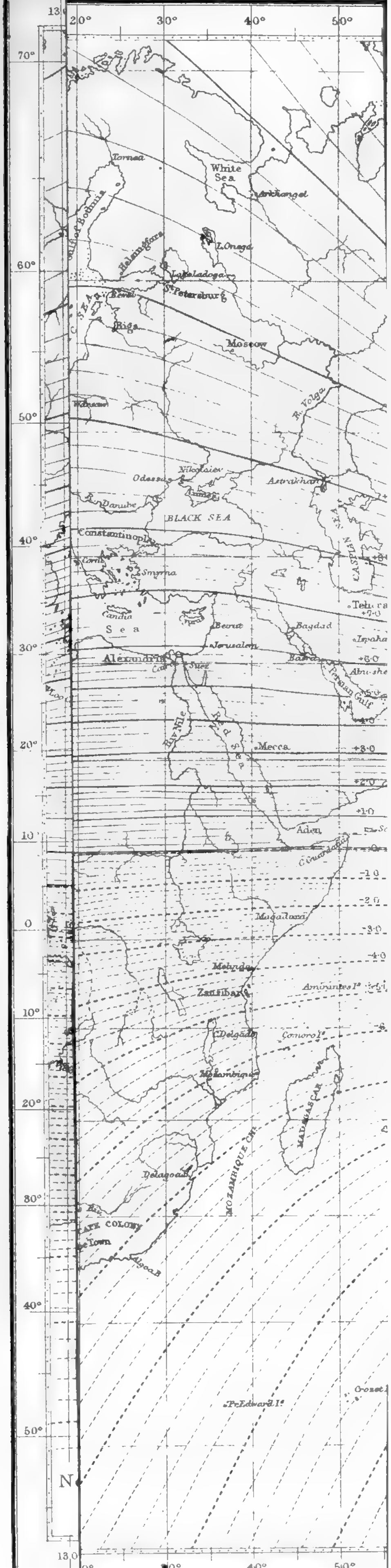
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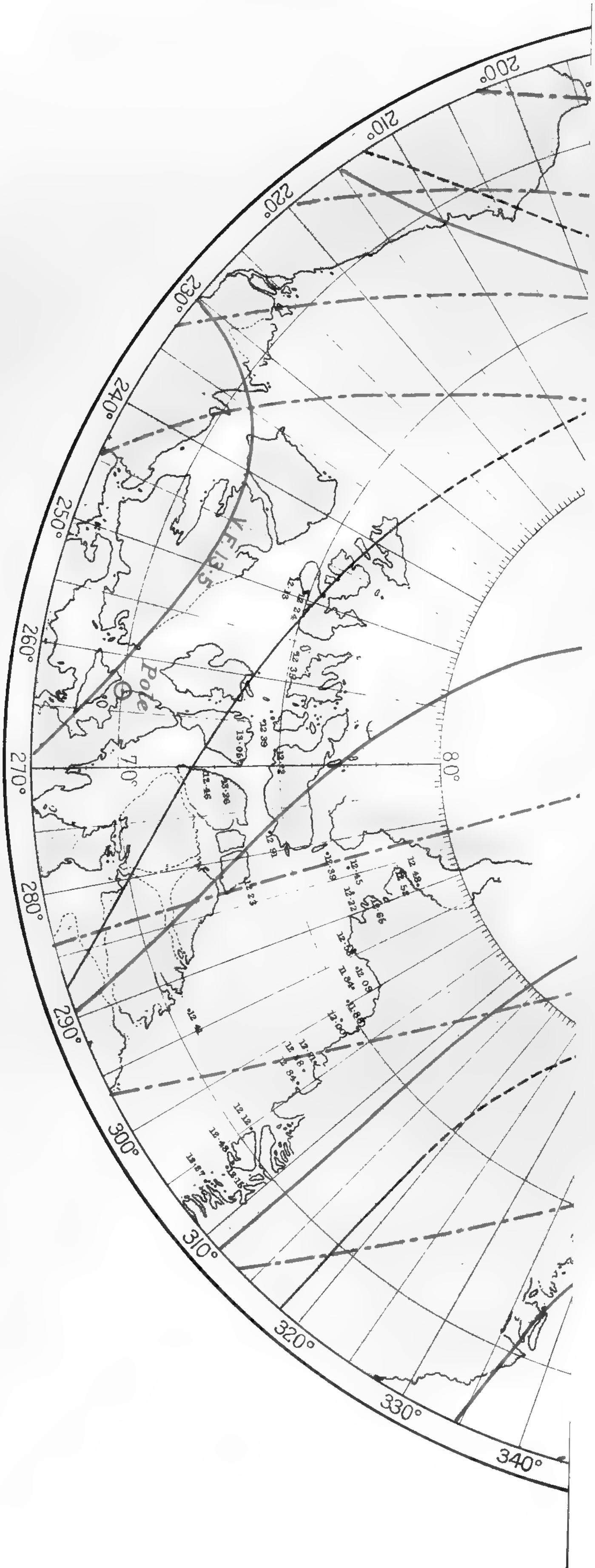
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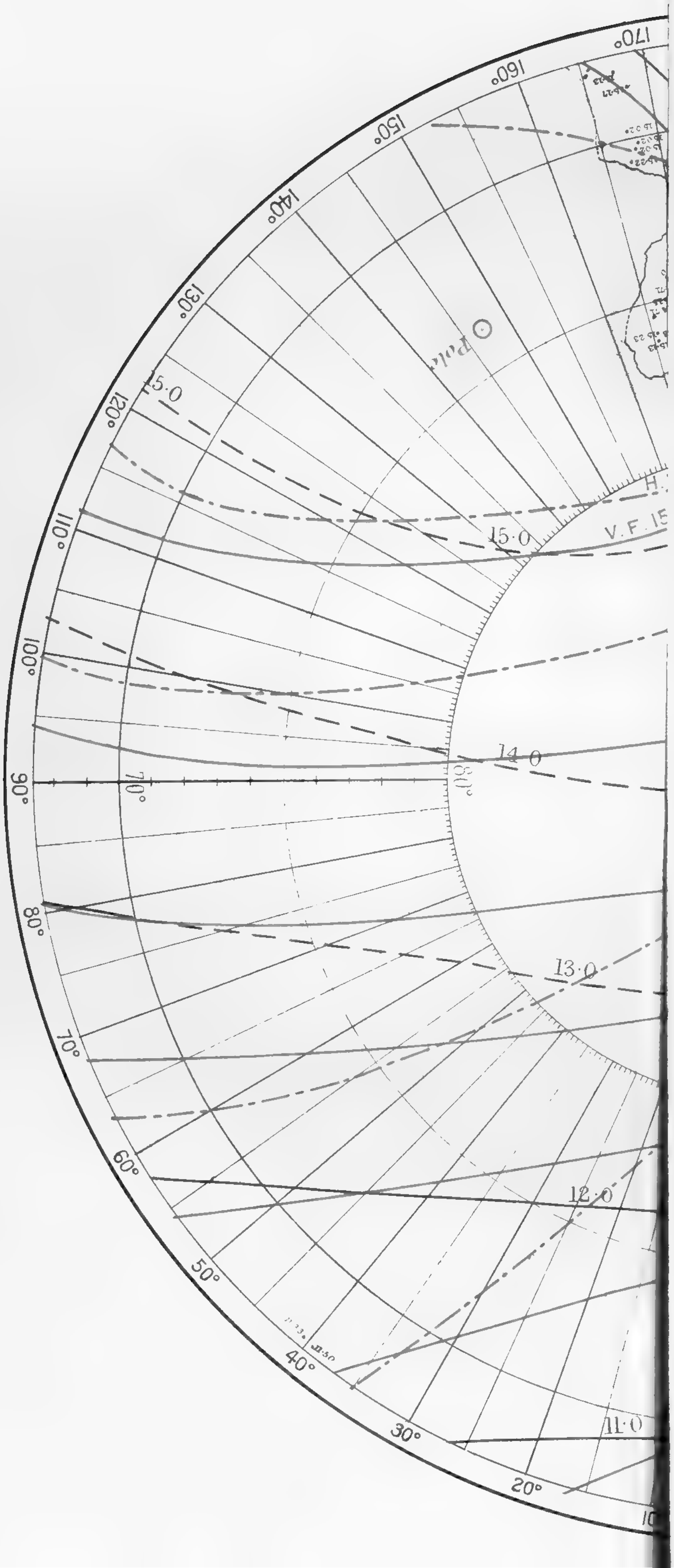
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